Michio Masujima

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To my granddaughter, Honoka

Contents

Applied Mathematical Methods in Theoretical Physics, Second Edition. Michio Masujima

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Preface

This book on integral equations and the calculus of variations is intended for use by senior undergraduate students and first-year graduate students in science and engineering. Basic familiarity with theories of linear algebra, calculus, differential equations, and complex analysis on the mathematics side, and classical mechanics, classical electrodynamics, quantum mechanics including the second quantization, and quantum statistical mechanics on the physics side is assumed. Another prerequisite on the mathematics side for this book is a sound understanding of local analysis and global analysis.

This book grew out of the course notes for the last of the three-semester sequence of Methods of Applied Mathematics I (Local Analysis), II (Global Analysis) and III (Integral Equations and Calculus of Variations) taught in the Department of Mathematics at MIT. About two thirds of the course is devoted to integral equations and the remaining one third to the calculus of variations. Professor Hung Cheng taught the course on integral equations and the calculus of variations every other year from the mid-1960s through the mid-1980s at MIT. Since then, younger faculty have been teaching the course in turn. The course notes evolved in the intervening years. This book is the culmination of these joint efforts.

There will be a natural question: Why now another book on integral equations and the calculus of variations? There exist many excellent books on the theory of integral equations. No existing book, however, discusses the singular integral equations in detail, in particular, Wiener–Hopf integral equations and Wiener–Hopf sum equations with the notion of the Wiener–Hopf index. In this book, the notion of the Wiener–Hopf index is discussed in detail.

This book is organized as follows. In Chapter 1, we discuss the notion of function space, the linear operator, the Fredholm alternative, and Green's functions, preparing the reader for the further development of the material. In Chapter 2, we discuss a few examples of integral equations and Green's functions. In Chapter 3, we discuss integral equations of the Volterra type. In Chapter 4, we discuss integral equations of the Fredholm type. In Chapter 5, we discuss the Hilbert–Schmidt theories of symmetric kernel. In Chapter 6, we discuss singular integral equations of the Cauchy type. In Chapter 7, we discuss the Wiener–Hopf method for the mixed boundary-value problem in classical electrodynamics, Wiener–Hopf integral equations, and Wiener–Hopf sum equations, the latter two topics being

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discussed in terms of the notion of index. In Chapter 8, we discuss nonlinear integral equations of the Volterra type, Fredholm type, and Hammerstein type. In Chapter 9, we discuss calculus of variations, covering the topics on the second variations, Legendre test, Jacobi test, and relationship with the theory of integral equations. In Chapter 10, we discuss the Hamilton–Jacobi equation and quantum mechanics, Feynman's action principle, Schwinger's action principle, system of Schwinger–Dyson equation in quantum theory, Feynman's variational principle and polaron, Poincaré transformation and spin, conservation law and Noether's theorem, Weyl's gauge principle, the path integral quantization of non-Abelian gauge fields, renormalization of non-Abelian gauge fields, asymptotic disaster (asymptotic freedom) of Abelian gauge field (non-Abelian gauge field) interacting with fermions with tri-Γ approximation, renormalization group equation, standard model, lattice gauge field theory, WKB method, and Hartree–Fock equation.

Chapter 10 is taken from my book, titled *Path Integrals and Stochastic Processes in Theoretical Physics*, Feshbach Publishing, Minnesota.

Reasonable understanding of Chapter 10 requires the reader to have a basic understanding of classical mechanics, classical field theory, classical electrodynamics, quantum mechanics including the second quantization, and quantum statistical mechanics. For this reason, Chapter 10 can be read as a side reference on theoretical physics.

The examples are mostly taken from classical mechanics, classical field theory, classical electrodynamics, quantum mechanics, quantum statistical mechanics, and quantum field theory. Most of them are worked out in detail to illustrate the methods of the solutions. Those examples which are not worked out in detail are either intended to illustrate the general methods of the solutions or left to the reader to complete the solutions.

At the end of each chapter with the exception of Chapter 1, problem sets are given for sound understanding of the contents of the main text. The reader is recommended to solve all the problems at the end of each chapter. Many of the problems were created by Professor Hung Cheng during the past three decades. The problems due to him are designated with the note (due to H. C.). Some of the problems are those encountered by Professor Hung Cheng in the course of his own research activity.

Most of the problems can be solved with the direct application of the method illustrated in the main text. Difficult problems are accompanied with the citation of the original references. The problems for Chapter 10 are mostly taken from classical mechanics, classical electrodynamics, quantum mechanics, quantum statistical mechanics, and quantum field theory.

Bibliography is provided at the end of the book for the in-depth study of the background materials in physics besides the standard references on the theory of integral equations and the calculus of variations.

The instructor can cover Chapters 1 through 9 in one semester or two quarters with a choice of the topics of his or her own taste from Chapter 10.

I would like to express many heart-felt thanks to Professor Hung Cheng at MIT, who appointed me as his teaching assistant for the course when I was a

graduate student in the Department of Mathematics at MIT for his permission to publish this book under my single authorship and for his criticism and constant encouragement without which this book would not have materialized.

I would like to thank the late Professor Francis E. Low and Professor Kerson Huang at MIT, who taught me many topics of theoretical physics. I would like to thank Professor Roberto D. Peccei at Stanford University, now at UCLA, who taught me quantum field theory and dispersion theory.

I would like to thank Professor Richard M. Dudley at MIT, who taught me real analysis and theories of probability and stochastic processes. I would like to thank Professor Herman Chernoff at Harvard University, who taught me many topics in mathematical statistics starting from multivariate normal analysis for his supervision of my Ph.D. thesis at MIT.

I would like to thank Dr. Ali Nadim, for supplying his version of the course notes and Dr. Dionisios Margetis at MIT, for supplying examples and problems of integral equations from his courses at Harvard University and MIT. The problems due to him are designated with the note (due to D. M.). I would like to thank Dr. George Fikioris at National Technical University of Athens, for supplying the references on the Yagi–Uda semi-infinite arrays.

I would like to thank my parents, Mikio and Hanako Masujima, who made my undergraduate study at MIT possible, for their financial support during my undergraduate student days at MIT. I would like to thank my wife, Mari, and my son, Masachika, for their encouragement during the period of writing of this book.

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Tokyo, Japan *Michio Masujima*
May, 2009

Introduction

Many problems of theoretical physics are frequently formulated in terms of ordinary differential equations or partial differential equations. We can frequently convert them into integral equations with boundary conditions or initial conditions built in. We can formally develop the perturbation series by iterations. A good example is the Born series for the potential scattering problem in quantum mechanics. In some cases, the resulting equations are nonlinear integro-differential equations. A good example is the Schwinger–Dyson equation in quantum field theory and quantum statistical mechanics. It is the nonlinear integro-differential equation, and is exact and closed. It provides the starting point of Feynman–Dyson-type perturbation theory in configuration space and in momentum space. In some singular cases, the resulting equations are Wiener–Hopf integral equations. They originate from research on the radiative equilibrium on the surface of a star. In the two-dimensional Ising model and the analysis of the Yagi–Uda semi-infinite arrays of antennas, among others, we have the Wiener–Hopf sum equations.

The theory of integral equations is best illustrated with the notion of functionals defined on some function space. If the functionals involved are quadratic in the function, the integral equations are said to be linear integral equations, and if they are higher than quadratic in the function, the integral equations are said to be nonlinear integral equations. Depending on the form of the functionals, the resulting integral equations are said to be of the first kind, of the second kind, and of the third kind. If the kernels of the integral equations are square-integrable, the integral equations are said to be nonsingular, and if the kernels of the integral equations are not square-integrable, the integral equations are said to be singular. Furthermore, depending on whether the end points of the kernel are fixed constants or not, the integral equations are said to be of the Fredholm type, Volterra type, Cauchy type, or Wiener–Hopf types, etc. Through discussion of the variational derivative of the quadratic functional, we can also establish the relationship between the theory of integral equations and the calculus of variations. The integro-differential equations can be best formulated in this manner. Analogies of the theory of integral equations with the system of linear algebraic equations are also useful.

The integral equation of the Cauchy type has an interesting application to classical electrodynamics, namely, dispersion relations. Dispersion relations were derived

Applied Mathematical Methods in Theoretical Physics, Second Edition. Michio Masujima

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by Kramers in 1927 and Kronig in 1926, for X-ray dispersion and optical dispersion, respectively. Kramers–Kronig dispersion relations are of very general validity which only depends on the assumption of the causality. The requirement of the causality alone determines the region of analyticity of dielectric constants. In the mid-1950s, these dispersion relations were also derived from quantum field theory and applied to strong interaction physics. The application of the covariant perturbation theory to strong interaction physics was hopeless due to the large coupling constant. From mid-1950s to 1960s, the dispersion theoretic approach to strong interaction physics was the only realistic approach that provided many sum rules. To cite a few, we have the Goldberger–Treiman relation, the Goldberger–Miyazawa–Oehme formula and the Adler–Weisberger sum rule. In dispersion theoretic approach to strong interaction physics, experimentally observed data were directly used in the sum rules. The situation changed dramatically in the early 1970s when quantum chromodynamics, the relativistic quantum field theory of strong interaction physics, was invented with the use of asymptotically free non-Abelian gauge field theory.

The region of analyticity of the scattering amplitude in the upper-half k-plane in quantum field theory when expressed in terms of Fourier transform is immediate since quantum field theory has the microscopic causality. But, the region of analyticity of the scattering amplitude in the upper-half k-plane in quantum mechanics when expressed in terms of Fourier transform is not immediate since quantum mechanics does not have the microscopic causality. We shall invoke the generalized triangular inequality to derive the region of analyticity of the scattering amplitude in the upper-half k-plane in quantum mechanics. The region of analyticity of the scattering amplitudes in the upper-half k-plane in quantum mechanics and quantum field theory strongly depends on the fact that the scattering amplitudes are expressed in terms of Fourier transform. When the other expansion basis is chosen, like Fourier–Bessel series, the region of analyticity drastically changes its domain.

In the standard application of the calculus of variations to the variety of problems of theoretical physics, we simply write the Euler equation and are rarely concerned with the second variations, the Legendre test and the Jacobi test. Examination of the second variations and the application of the Legendre test and the Jacobi test become necessary in some cases of the application of the calculus of variations to the problems of theoretical physics. In order to bring the development of theoretical physics and the calculus of variations much closer, some historical comments are in order here.

Euler formulated Newtonian mechanics by the variational principle, the Euler equation. Lagrange started the whole field of calculus of variations. He also introduced the notion of generalized coordinates into classical mechanics and completely reduced the problem to that of differential equations, which are presently known as Lagrange equations of motion, with the Lagrangian appropriately written in terms of kinetic energy and potential energy. He successfully converted classical mechanics into analytical mechanics with the variational principle. Legendre constructed the transformation methods for thermodynamics which are presently known as the Legendre transformations. Hamilton succeeded in transforming the Lagrange equations of motion, which are of the second order, into a set of first-order differential equations with twice as many variables. He did this by introducing the canonical momenta which are conjugate to the generalized coordinates. His equations are known as Hamilton's canonical equations of motion. He successfully formulated classical mechanics in terms of the principle of least action. The variational principles formulated by Euler and Lagrange apply only to the conservative system. Hamilton recognized that the principle of least action in classical mechanics and Fermat's principle of shortest time in geometrical optics are strikingly analogous, permitting the interpretation of optical phenomena in mechanical terms and vice versa. Jacobi quickly realized the importance of the work of Hamilton. He noted that Hamilton was using just one particular set of the variables to describe the mechanical system and formulated the canonical transformation theory with the Legendre transformation. He duly arrived at what is presently known as the Hamilton–Jacobi equation. He formulated his version of the principle of least action for the time-independent case.

From what we discussed, we may be led to the conclusion that calculus of variations is the finished subject by the end of the 19th century. We shall note that, from the 1940s to 1950s, we encountered the resurgence of the action principle for the systemization of quantum field theory. The subject matters are Feynman's action principle and Schwinger's action principle.

Path integral quantization procedure invented by Feynman in 1942 in the Lagrangian formalism is justified by the Hamiltonian formalism of quantum theory in the standard treatment. We can deduce the canonical formalism of quantum theory from the path integral formalism. The path integral quantization procedure originated from the old paper published by Dirac in 1934 on the Lagrangian in quantum mechanics. This quantization procedure is called Feynman's action principle.

Schwinger's action principle proposed in the early 1950s is the differential formalism of action principle to be compared with Feynman's action principle which is the integral formalism of action principle. These two quantum action principles are equivalent to each other and are essential in carrying out the computation of electrodynamic level shifts of the atomic energy level.

Schwinger's action principle is a convenient device to develop Schwinger theory of Green's functions. When it is applied to the two-point "full" Green's functions with the use of the proper self energy parts and the vertex operator of Dyson, we obtain Schwinger–Dyson equation for quantum field theory and quantum statistical mechanics. When it is applied to the four-point "full" Green's functions, we obtain Bethe–Salpeter equation. We focus on Bethe–Salpeter equation for the bound state problem. This equation is highly nonlinear and does not permit the exact solution except for the Wick–Cutkosky model. In all the rests of the models proposed, we employ the certain type of the approximation in the interaction kernel of Bethe–Salpeter equation. Frequently, we employ the ladder approximation in high energy physics.

Feynman's variational principle in quantum statistical mechanics can be derived by the analytic continuation in time from a real time to an imaginary time of

Feynman's action principle in quantum mechanics. The polaron problem can be discussed with Feynman's variational principle.

There exists a close relationship between a global continuous symmetry of the Lagrangian $L(q_r(t), \dot{q}_r(t), t)$ (the Lagrangian density $\mathcal{L}(\psi_a(x), \partial_\mu \psi_a(x))$) and the current conservation law, commonly known as Noether's theorem. When the global continuous symmetry of the Lagrangian (the Lagrangian density) exists, the conserved current results at classical level, and hence the conserved charge. A conserved current need not be a vector current. It can be a tensor current with the conservation index. It may be an energy–momentum tensor whose conserved charge is the energy–momentum four-vector. At quantum level, however, the otherwise conserved classical current frequently develops the anomaly and the current conservation fails to hold at quantum level any more. An axial current is a good example.

When we extend the global symmetry of the field theory to the local symmetry, Weyl's gauge principle naturally comes in. With Weyl'sgauge principle, electrodynamics of James Clark Maxwell can be deduced.

Weyl's gauge principle still attracts considerable attention due to the fact that all forces in nature can be unified with the extension of Weyl's gauge principle with the appropriate choice of the grand unifying Lie groups as the gauge group.

Based on the tri-Γ approximation to the set of completely renormalized Schwinger–Dyson equations for non-Abelian gauge field in interaction with the fermion field, which is free from the overlapping divergence, we can demonstrate asymptotic freedom, as stipulated above, nonperturbatively. This property arises from the non-Abelian nature of the gauge group and such property is not present for Abelian gauge field like *QED*. Actually, no quantum field theory is asymptotically free without non-Abelian gauge field.

With the tri-Γ approximation, we can demonstrate asymptotic disaster of Abelian gauge field in interaction with the fermion field. Asymptotic disaster of Abelian gauge field was discovered in mid-1950s by Gell–Mann and Low and independently by Landau, Abrikosov, Galanin, and Khalatnikov. Soon after this discovery was made, quantum field theory was once abandoned for a decade, and dispersion theory became fashionable.

There exist the Gell–Mann–Low renormalization group equation, which originates from the perturbative calculation of the massless *QED* with the use of the mathematical theory of the regular variations. There also exist the renormalization group equation, called the Callan–Symanzik equation, which is slightly different from the former. The relationship between the two approaches is established with some effort. We note that the method of the renormalization group essentially consists of separating the field components into the rapidly varying field components ($k^2 > \Lambda^2$) and the slowly varying field components ($k^2 < \Lambda^2$), path-integrating out the rapidly varying field components ($k^2 > \Lambda^2$) in the generating functional of Green's functions, and focusing our attention to the slowly varying field components ($k^2 < \Lambda^2$) to analyze the low energy phenomena at $k^2 < \Lambda^2$. We remark that the scale of Λ depends on the kind of physics we analyze and, to some extent, is arbitrary. The Gell–Mann–Low analysis exhibited the astonishing result; *QED* is not asymptotically free. It becomes the strong coupling theory at short distances. At the same time, the Gell–Mann–Low renormalization group equation and the Callan–Symanzik equation address themselves to the deep Euclidean momentum space. High energy experimental physicists also focus their attention to the deep Euclidean region. So the analysis based on the renormalization group equations is the standard procedure for high energy experimental physicists. Popov employed the same method to path-integrate out the rapidly varying field components ($k^2 > \Lambda^2$) and focus his attention to the slowly varying field components ($k^2 < \Lambda^2$) in the detailed analysis of the superconductivity. Wilson introduced the renormalization group equation to analyze the critical exponents in solid state physics. He employed the method of the block spin and coarse-graining in his formulation of the renormalization group equation. The Wilson approach looks quite dissimilar to the Gell–Mann–Low approach and the Callan–Symanzik approach. In the end, the Wilson approach is identical to the former two approaches.

Renormalization group equation can be regarded as one application of calculus of variations which attempts to maintain the renormalizability of quantum theory under variations of some physical parameters.

Electro-weak unification of Glashow, Weinberg, and Salam is based on the gauge group,

$$SU(2)_{\text{weak isospin}} \times U(1)_{\text{weak hypercharge}},$$

while maintaining Gell-Mann-Nishijima relation in the lepton sector. It suffers from the problem of the nonrenormalizability due to the triangular anomaly in the lepton sector. In the early 1970s, it is discovered that non-Abelian gauge field theory is asymptotically free at short distance, i.e., it behaves like a free field at short distance. Then the relativistic quantum field theory of the strong interaction based on the gauge group $SU(3)_{\text{color}}$ is invented, and is called quantum chromodynamics.

Standard model with the gauge group,

$$SU(3)_{\text{color}} \times SU(2)_{\text{weak isospin}} \times U(1)_{\text{weak hypercharge}},$$

which describes the weal interaction, the electromagnetic interaction, and the strong interaction is free from the triangular anomaly. It suffers, however, from a serious defect; the existence of the classical instanton solution to the field equation in the Euclidean metric for the $SU(2)$ gauge field theory. In the $SU(2)$ gauge field theory, we have the Belavin–Polyakov–Schwartz–Tyupkin instanton solution which is a classical solution to the field equation in the Euclidean metric. A proper account for the instanton solution requires the addition of the strong CP-violating term to the QCD Lagrangian density in the path integral formalism. The Peccei–Quinn axion and the invisible axion scenario resolve this strong CP-violation problem. In the grand unified theories, we assume that the subgroup of the grand unifying gauge group is the gauge group $SU(3)_{\text{color}} \times SU(2)_{\text{weak isospin}} \times U(1)_{\text{weak hypercharge}}$. We now attempt to unify the weak interaction, the electromagnetic interaction, and the strong interaction by starting from the much larger gauge group G, which is reduced to $SU(3)_{\text{color}} \times SU(2)_{\text{weak isospin}} \times U(1)_{\text{weak hypercharge}}$ and further down to $SU(3)_{\text{color}} \times U(1)_{\text{E.M.}}$.

Lattice gauge field theory can explain consistently the phenomena of the quark confinement. The discretized space–time spacing of the lattice gauge field theory plays the role of the momentum cutoff of the continuum theory.

A customary WKB method in quantum mechanics is the short wavelength approximation to wave mechanics. The WKB method in quantum theory in path integral formalism consists of the replacement of general Lagrangian (density) with a quadratic Lagrangian (density).

The Hartree–Fock program is the one of the classic variational problems in quantum mechanics of a system of A identical fermions. One-body and two-body density matrices are introduced. But extremization of the energy functional with respect to density matrices is difficult to implement. We thus introduce a Slater determinant for the wavefunction of A identical fermions. After extremizing the energy functional under variation of the parameters in orbitals of the Slater determinant, however, there remain two important questions. One question has to do with the stability of the iterative solutions, i.e., do they provide the true minimum? The second variation of the energy functional with respect to the variation parameters in orbitals should be examined. Another question has to do with the degeneracy of the Hartree–Fock solution.

Weyl's gauge principle, Feynman's action principle, Schwinger's action principle, Feynman's variational principle as applied to the polaron problem, and the method of the renormalization group equations are the modern applications of calculus of variations. Thus, the calculus of variations is well and alive in theoretical physics to this day, contrary to a common brief that the calculus of variations is a dead subject.

In this book, we address ourselves to theory of integral equations and the calculus of variations, and their application to the modern development of theoretical physics, while referring the reader to other sources for theory of ordinary differential equations and partial differential equations.

1
Function Spaces, Linear Operators, and Green's Functions

1.1
Function Spaces

Consider the set of all complex-valued functions of the real variable x, denoted by $f(x), g(x), \ldots$, and defined on the interval (a, b). We shall restrict ourselves to those functions which are *square-integrable*. Define the *inner product* of any two of the latter functions by

$$(f, g) \equiv \int_a^b f^*(x)\, g(x)\, dx, \tag{1.1.1}$$

in which $f^*(x)$ is the complex conjugate of $f(x)$. The following properties of the inner product follow from definition (1.1.1):

$$\begin{cases} (f, g)^* &= (g, f), \\ (f, g + h) &= (f, g) + (f, h), \\ (f, \alpha g) &= \alpha(f, g), \\ (\alpha f, g) &= \alpha^*(f, g), \end{cases} \tag{1.1.2}$$

with α a complex scalar.

While the inner product of any two functions is in general a complex number, the inner product of a function with itself is a real number and is nonnegative. This prompts us to define the *norm of a function* by

$$\|f\| \equiv \sqrt{(f, f)} = \left[\int_a^b f^*(x) f(x) dx\right]^{\frac{1}{2}}, \tag{1.1.3}$$

provided that f is square-integrable , i.e., $\|f\| < \infty$. Equation (1.1.3) constitutes a proper definition for a norm since it satisfies the following conditions:

$$\begin{cases} (i) & \text{scalar multiplication} & \|\alpha f\| = |\alpha| \cdot \|f\|, & \text{for all comlex } \alpha, \\ (ii) & \text{positivity} & \|f\| > 0, & \text{for all } f \neq 0, \\ & & \|f\| = 0, & \text{if and only if } f = 0, \\ (iii) & \text{triangular inequality} & \|f + g\| \leq \|f\| + \|g\|. & \end{cases} \tag{1.1.4}$$

Applied Mathematical Methods in Theoretical Physics, Second Edition. Michio Masujima

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A very important inequality satisfied by the inner product (1.1.1) is the so-called *Schwarz inequality* which says

$$\left|(f, g)\right| \leq \|f\| \cdot \|g\| . \tag{1.1.5}$$

To prove the latter, start with the trivial inequality $\|(f + \alpha g)\|^2 \geq 0$, which holds for any $f(x)$ and $g(x)$ and for any complex number α. With a little algebra, the left-hand side of this inequality may be expanded to yield

$$(f,f) + \alpha^*(g,f) + \alpha(f, g) + \alpha\alpha^*(g, g) \geq 0. \tag{1.1.6}$$

The latter inequality is true for any α, and is true for the value of α which minimizes the left-hand side. This value can be found by writing α as $a + ib$ and minimizing the left-hand side of Eq. (1.1.6) with respect to the real variables a and b. A quicker way would be to treat α and α^* as independent variables and requiring $\partial/\partial\alpha$ and $\partial/\partial\alpha^*$ of the left-hand side of Eq. (1.1.6) to vanish. This immediately yields $\alpha = -(g,f)/(g, g)$ as the value of α at which the minimum occurs. Evaluating the left-hand side of Eq. (1.1.6) at this minimum then yields

$$\|f\|^2 \geq \left|(f, g)\right|^2 / \|g\|^2 , \tag{1.1.7}$$

which proves the Schwarz inequality (1.1.5).

Once the Schwarz inequality has been established, it is relatively easy to prove the *triangular inequality* (1.1.4.*iii*). To do this, we simply begin from the definition

$$\|f + g\|^2 = (f + g, f + g) = (f,f) + (f, g) + (g,f) + (g, g). \tag{1.1.8}$$

Now the right-hand side of Eq. (1.1.8) is a sum of complex numbers. Applying the usual triangular inequality for complex numbers to the right-hand side of Eq. (1.1.8) yields

$$\begin{aligned}\left|\text{Right-hand side of Eq. (1.1.8)}\right| &\leq \|f\|^2 + \left|(f, g)\right| + \left|(g,f)\right| + \|g\|^2 \\ &= \left(\|f\| + \|g\|\right)^2 .\end{aligned} \tag{1.1.9}$$

Combining Eqs. (1.1.8) and (1.1.9) finally proves the triangular inequality (1.1.4.*iii*).

We finally remark that the set of functions $f(x), g(x), \ldots$, is an example of a *linear vector space*, equipped with an inner product and a norm based on that inner product. A similar set of properties, including the Schwarz and triangular inequalities, can be established for other linear vector spaces. For instance, consider the set of all complex column vectors $\vec{u}, \vec{v}, \vec{w}, \ldots$, of finite dimension n. If we define the inner product

$$(\vec{u}, \vec{v}) \equiv (\vec{u}^*)^{\mathrm{T}}\vec{v} = \sum_{k=1}^{n} u_k^* v_k, \tag{1.1.10}$$

and the related norm

$$\|\vec{u}\| \equiv \sqrt{(\vec{u}, \vec{u})}, \tag{1.1.11}$$

then the corresponding Schwarz and triangular inequalities can be proven in an identical manner yielding

$$|(\vec{u}, \vec{v})| \leq \|\vec{u}\| \, \|\vec{v}\| , \tag{1.1.12}$$

and

$$\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\| . \tag{1.1.13}$$

1.2 Orthonormal System of Functions

Two functions $f(x)$ and $g(x)$ are said to be *orthogonal* if their inner product vanishes, i.e.,

$$(f, g) = \int_a^b f^*(x) g(x) dx = 0. \tag{1.2.1}$$

A function is said to be *normalized* if its norm is equal to unity, i.e.,

$$\|f\| = \sqrt{(f, f)} = 1. \tag{1.2.2}$$

Consider a set of normalized functions $\{\phi_1(x), \phi_2(x), \phi_3(x), \ldots\}$ which are mutually orthogonal. This type of set is called an *orthonormal set of functions*, satisfying the orthonormality condition

$$(\phi_i, \phi_j) = \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise,} \end{cases} \tag{1.2.3}$$

where δ_{ij} is the *Kronecker delta symbol* itself defined by Eq. (1.2.3).

An orthonormal set of functions $\{\phi_n(x)\}$ is said to form a *basis for a function space*, or to be *complete*, if any function $f(x)$ in that space can be expanded in a series of the form

$$f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x). \tag{1.2.4}$$

(This is not the exact definition of a complete set but it will do for our purposes.) To find the coefficients of the expansion in Eq. (1.2.4), we take the inner product of both sides with $\phi_m(x)$ from the left to obtain

$$\begin{aligned}(\phi_m, f) &= \sum_{n=1}^{\infty} (\phi_m, a_n\phi_n) \\ &= \sum_{n=1}^{\infty} a_n (\phi_m, \phi_n) \\ &= \sum_{n=1}^{\infty} a_n\delta_{mn} = a_m. \end{aligned} \tag{1.2.5}$$

In other words, for any n,

$$a_n = (\phi_n, f) = \int_a^b \phi_n^* (x) f (x)\, dx. \tag{1.2.6}$$

An example of an orthonormal system of functions on the interval $(-l, l)$ is the infinite set

$$\phi_n (x) = \frac{1}{\sqrt{2l}} \exp[in\pi x/l], \quad n = 0, \pm 1, \pm 2, \dots, \tag{1.2.7}$$

with which the expansion of a square-integrable function $f(x)$ on $(-l, l)$ takes the form

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp[in\pi x/l], \tag{1.2.8a}$$

with

$$c_n = \frac{1}{2l} \int_{-l}^{+l} f(x) \exp[-in\pi x/l], \tag{1.2.8b}$$

which is the familiar complex form of the *Fourier series* of $f(x)$.

Finally, the *Dirac delta function* $\delta(x - x')$, defined with x and x' in (a, b), can be expanded in terms of a complete set of orthonormal functions $\phi_n(x)$ in the form

$$\delta (x - x') = \sum_n a_n\phi_n(x)$$

with

$$a_n = \int_a^b \phi_n^*(x)\delta(x - x')dx = \phi_n^* (x').$$

That is,

$$\delta(x - x') = \sum_n \phi_n^*(x')\phi_n(x). \tag{1.2.9}$$

Expression (1.2.9) is sometimes taken as the statement which implies the *completeness of an orthonormal system of functions*.

1.3
Linear Operators

An *operator* can be thought of as a mapping or a transformation which acts on a member of the function space (a function) to produce another member of that space (another function). The operator, typically denoted by a symbol like L, is said to be *linear* if it satisfies

$$L(\alpha f + \beta g) = \alpha Lf + \beta Lg, \tag{1.3.1}$$

where α and β are complex numbers, and f and g are members of that function space. Some trivial examples of linear operators L are

(i) multiplication by a constant scalar,

$$L\phi = a\phi,$$

(ii) taking the third derivative of a function, which is a differential operator

$$L\phi = \frac{d^3}{dx^3}\phi \quad \text{or} \quad L = \frac{d^3}{dx^3},$$

(iii) multiplying a function by the kernel, $K(x, x')$, and integrating over (a, b) with respect to x', which is an integral operator,

$$L\phi(x) = \int_a^b K(x, x')\phi(x')dx'.$$

An important concept in the theory of linear operators is that of *adjoint* of the operator which is defined as follows. Given the operator L, together with an inner product defined on a vector space, the adjoint L^{adj} of the operator L is that operator for which

$$(\psi, L\phi) = (L^{\text{adj}}\psi, \phi) \tag{1.3.2}$$

is an identity for any two members ϕ and ψ of the vector space. Actually, as we shall see later, in the case of the differential operators, we frequently need to worry to some extent about the boundary conditions associated with the original and the adjoint problems. Indeed, there often arise additional terms on the right-hand side of Eq. (1.3.2) which involve the boundary points, and a prudent choice of the adjoint boundary conditions will need to be made in order to avoid unnecessary difficulties. These issues will be raised in connection with Green's functions for differential equations.

As our first example of the adjoint operator, consider the liner vector space of n-dimensional complex column vectors $\vec{u}$, $\vec{v}$,. . ., with their inner product (1.1.10). In this space, $n\times n$ square matrices A, B, . . ., with complex entries are linear operators when multiplied with the n-dimensional complex column vectors according to the usual rules of matrix multiplication. We now consider the problem of finding the adjoint matrix A^{adj} of the matrix A. According to definition (1.3.2) of the adjoint operator, we search for the matrix A^{adj} satisfying

$$\left(\vec{u}, A\vec{v}\right) = (A^{\text{adj}}\vec{u}, \vec{v}). \tag{1.3.3}$$

Now, from the definition of the inner product (1.1.10), we must have

$$\vec{u}^{*\mathrm{T}}(A^{\text{adj}})^{*\mathrm{T}}\vec{v} = \vec{u}^{*\mathrm{T}}A\vec{v},$$

i.e.,

$$(A^{\text{adj}})^{*\mathrm{T}} = A \quad \text{or} \quad A^{\text{adj}} = A^{*\mathrm{T}}. \tag{1.3.4}$$

That is, the adjoint A^{adj} of a matrix A is equal to the complex conjugate of its transpose, which is also known as its *Hermitian transpose*,

$$A^{\text{adj}} = A^{*\mathrm{T}} \equiv A^{\mathrm{H}}. \tag{1.3.5}$$

As a second example, consider the problem of finding the adjoint of the linear integral operator

$$L = \int_a^b dx' K(x, x'), \tag{1.3.6}$$

on our function space. By definition, the adjoint L^{adj} of L is the operator which satisfies Eq. (1.3.2). Upon expressing the left-hand side of Eq. (1.3.2) explicitly with the operator L given by Eq. (1.3.6), we find

$$\begin{aligned}(\psi, L\phi) &= \int_a^b dx \psi^*(x) L\phi(x) \\ &= \int_a^b dx' \left[\int_a^b dx K(x, x')\psi^*(x)\right]\phi(x').\end{aligned} \tag{1.3.7}$$

Requiring Eq. (1.3.7) to be equal to

$$(L^{\text{adj}}\psi, \phi) = \int_a^b dx (L^{\text{adj}}\psi(x))^* \phi(x)$$

necessitates defining

$$L^{\text{adj}}\psi(x) = \int_a^b d\xi\, K^*(\xi, x)\psi(\xi).$$

Hence the adjoint of integral operator (1.3.6) is found to be

$$L^{\mathrm{adj}} = \int_a^b dx' K^*(x', x). \tag{1.3.8}$$

Note that aside from the complex conjugation of the kernel $K(x, x')$, the integration in Eq. (1.3.6) is carried out with respect to the second argument of $K(x, x')$ while that in Eq. (1.3.8) is carried out with respect to the first argument of $K^*(x', x)$. Also be careful of which of the variables throughout the above is the dummy variable of integration.

Before we end this section, let us define what is meant by a *self-adjoint* operator. An operator L is said to be self-adjoint (or *Hermitian*) if it is equal to its own adjoint L^{adj}. Hermitian operators have very nice properties which will be discussed in Section 1.6. Not the least of these is that their eigenvalues are real. (Eigenvalue problems are discussed in the next section.)

Examples of self-adjoint operators are Hermitian matrices, i.e., matrices which satisfy

$$A = A^{\mathrm{H}},$$

and linear integral operators of the type (1.3.6) whose kernel satisfies

$$K(x, x') = K^*(x', x),$$

each of them on their respective linear spaces and with their respective inner products.

1.4 Eigenvalues and Eigenfunctions

Given a linear operator L on a linear vector space, we can set up the following eigenvalue problem:

$$L\phi_n = \lambda_n \phi_n \quad (n = 1, 2, 3, \ldots). \tag{1.4.1}$$

Obviously the trivial solution $\phi(x) = 0$ always satisfies this equation, but it also turns out that for some particular values of λ (called the *eigenvalues* and denoted by λ_n), nontrivial solutions to Eq. (1.4.1) also exist. Note that for the case of the differential operators on bounded domains, we must also specify an appropriate homogeneous boundary condition (such that $\phi = 0$ satisfies those boundary conditions) for the *eigenfunctions* $\phi_n(x)$. We have affixed the subscript n to the eigenvalues and the eigenfunctions under the assumption that the eigenvalues are discrete and they can be counted (i.e., with $n = 1, 2, 3, \ldots$). This is not always the case. The conditions which guarantee the existence of a discrete (and complete) set of eigenfunctions are beyond the scope of this introductory chapter and will not be discussed. So, for the moment, let us tacitly assume that the eigenvalues λ_n of Eq. (1.4.1) are discrete and their eigenfunctions ϕ_n form a basis for their space.

Similarly the adjoint L^{adj} of the operator L possesses a set of eigenvalues and eigenfunctions satisfying

$$L^{\text{adj}}\psi_m = \mu_m\psi_m \quad (m = 1, 2, 3, \ldots). \tag{1.4.2}$$

It can be shown that the eigenvalues μ_m of the adjoint problem are equal to complex conjugates of the eigenvalues λ_n of the original problem. If λ_n is an eigenvalue of L, λ_n^* is an eigenvalue of L^{adj}. We rewrite Eq. (1.4.2) as

$$L^{\text{adj}}\psi_m = \lambda_m^*\psi_m \quad (m = 1, 2, 3, \ldots). \tag{1.4.3}$$

It is then a trivial matter to show that the eigenfunctions of the adjoint and original operators are all orthogonal, except those corresponding to the same index ($n = m$). To do this, take the inner product of Eq. (1.4.1) with ψ_m from the left, and the inner product of Eq. (1.4.3) with ϕ_n from the right to find

$$(\psi_m, L\phi_n) = (\psi_m, \lambda_n\phi_n) = \lambda_n(\psi_m, \phi_n) \tag{1.4.4}$$

and

$$(L^{\text{adj}}\psi_m, \phi_n) = (\lambda_m^*\psi_m, \phi_n) = \lambda_m(\psi_m, \phi_n). \tag{1.4.5}$$

Subtract the latter two equations and get

$$0 = (\lambda_n - \lambda_m)(\psi_m, \phi_n). \tag{1.4.6}$$

This implies

$$(\psi_m, \phi_n) = 0 \quad \text{if} \quad \lambda_n \neq \lambda_m, \tag{1.4.7}$$

which proves the desired result. Also, since each of ϕ_n and ψ_m is determined to within a multiplicative constant (e.g., if ϕ_n satisfies Eq. (1.4.1) so does $\alpha\phi_n$), the normalization for the latter can be chosen such that

$$(\psi_m, \phi_n) = \delta_{mn} = \begin{cases} 1, & \text{for } n = m, \\ 0, & \text{otherwise} \end{cases} \tag{1.4.8}$$

Now, if the set of eigenfunctions ϕ_n ($n = 1, 2, \ldots$) forms a complete set, any arbitrary function $f(x)$ in the space may be expanded as

$$f(x) = \sum_n a_n\phi_n(x), \tag{1.4.9}$$

and to find the coefficients a_n, we simply take the inner product of both sides with ψ_k to get

$$(\psi_k, f) = \sum_n (\psi_k, a_n\phi_n) = \sum_n a_n(\psi_k, \phi_n) = \sum_n a_n\delta_{kn} = a_k,$$

i.e.,

$$a_n = (\psi_n, f) \quad (n = 1, 2, 3, \ldots). \tag{1.4.10}$$

Note the difference between Eqs. (1.4.9) and (1.4.10) and Eqs. (1.2.4) and (1.2.6) for an orthonormal system of functions. In the present case, neither $\{\phi_n\}$ nor $\{\psi_n\}$ form an orthonormal system, but they are orthogonal to one another.

Above we claimed that the eigenvalues of the adjoint of an operator are complex conjugates of those of the original operator. Here we show this for the matrix case. The eigenvalues of a matrix A are given by $\det(A - \lambda I) = 0$. The eigenvalues of A^{adj} are determined by setting $\det(A^{\text{adj}} - \mu I) = 0$. Since the determinant of a matrix is equal to that of its transpose, we easily conclude that the eigenvalues of A^{adj} are the complex conjugates of λ_n.

1.5 The Fredholm Alternative

The *Fredholm Alternative,* which is alternatively called the *Fredholm solvability condition,* is concerned with the existence of the solution $y(x)$ of the inhomogeneous problem

$$Ly(x) = f(x), \tag{1.5.1}$$

where L is a given linear operator and $f(x)$ a known forcing term. As usual, if L is a differential operator, additional boundary or initial conditions are also to be specified.

The Fredholm Alternative states that the unknown function $y(x)$ can be determined uniquely if the corresponding homogeneous problem

$$L\phi_H(x) = 0 \tag{1.5.2}$$

with homogeneous boundary conditions has no nontrivial solutions. On the other hand, if the homogeneous problem *(1.5.2)* does possess a nontrivial solution, then the inhomogeneous problem *(1.5.1)* has either no solution or infinitely many solutions. What determines the latter is the homogeneous solution ψ_H to the adjoint problem

$$L^{\text{adj}}\psi_H = 0. \tag{1.5.3}$$

Taking the inner product of Eq. (1.5.1) with ψ_H from the left,

$$(\psi_H, Ly) = (\psi_H, f).$$

Then, by the definition of the adjoint operator (excluding the case wherein L is a differential operator to be discussed in Section 1.7), we have

$$(L^{\text{adj}}\psi_H, y) = (\psi_H, f).$$

The left-hand side of the above equation is zero by the definition of ψ_H, Eq. (1.5.3). Thus the *criteria for the solvability of the inhomogeneous problem (1.5.1) are given by*

$$(\psi_H, f) = 0.$$

If these criteria are satisfied, there will be an infinity of solutions to Eq. (1.5.1); otherwise Eq. (1.5.1) will have no solution.

To understand the above claims, let us suppose that L and L^{adj} possess complete sets of eigenfunctions satisfying

$$L\phi_n = \lambda_n\phi_n \quad (n = 0, 1, 2, \ldots), \tag{1.5.4a}$$

$$L^{\text{adj}}\psi_n = \lambda_n^*\psi_n \quad (n = 0, 1, 2, \ldots), \tag{1.5.4b}$$

$$(\psi_m, \phi_n) = \delta_{mn}. \tag{1.5.4c}$$

The existence of a nontrivial homogeneous solution $\phi_H(x)$ to Eq. (1.5.2), as well as $\psi_H(x)$ to Eq. (1.5.3), is the same as having one of the eigenvalues λ_n in Eqs. (1.5.4a) and (1.5.4b) be zero. If this is the case, i.e., if zero is an eigenvalue of Eq. (1.5.4a) and hence Eq. (1.5.4b), we shall choose the subscript $n = 0$ to signify that eigenvalue ($\lambda_0 = 0$), and in that case ϕ_0 and ψ_0 are the same as ϕ_H and ψ_H. The two circumstances in the Fredholm Alternative correspond to cases where zero is an eigenvalue of Eqs. (1.5.4a) and (1.5.4b) and where it is not.

Let us proceed with the problem of solving the inhomogeneous problem (1.5.1). Since the set of eigenfunctions ϕ_n of Eq. (1.5.4a) is assumed to be complete, both the known function $f(x)$ and the unknown function $y(x)$ in Eq. (1.5.1) can presumably be expanded in terms of $\phi_n(x)$:

$$f(x) = \sum_{n=0}^{\infty} \alpha_n\phi_n(x), \tag{1.5.5}$$

$$y(x) = \sum_{n=0}^{\infty} \beta_n\phi_n(x), \tag{1.5.6}$$

where the α_n's are known (since $f(x)$ is known), i.e., according to Eq. (1.4.10)

$$\alpha_n = (\psi_n, f), \tag{1.5.7}$$

while the β_n's are unknown. Thus, if all the β_n's can be determined, then the solution $y(x)$ to Eq. (1.5.1) is regarded as having been found.

To try to determine the β_n's, substitute both Eqs. (1.5.5) and (1.5.6) into Eq. (1.5.1) to find

$$\sum_{n=0}^{\infty} \lambda_n \beta_n \phi_n = \sum_{k=0}^{\infty} \alpha_k \phi_k. \tag{1.5.8}$$

Here different summation indices have been used on the two sides to remind the reader that the latter are dummy indices of summation. Next take the inner product of both sides with ψ_m(with an index which has to be different from the above two) to get

$$\sum_{n=0}^{\infty} \lambda_n \beta_n (\psi_m, \phi_n) = \sum_{k=0}^{\infty} \alpha_k (\psi_m, \phi_k), \quad \text{or} \quad \sum_{n=0}^{\infty} \lambda_n \beta_n \delta_{mn} = \sum_{k=0}^{\infty} \alpha_k \delta_{mk},$$

i.e.,

$$\lambda_m \beta_m = \alpha_m. \tag{1.5.9}$$

Thus, for any $m = 0, 1, 2, \ldots$, we can solve Eq. (1.5.9) for the unknowns β_m to get

$$\beta_n = \alpha_n / \lambda_n \quad (n = 0, 1, 2, \ldots), \tag{1.5.10}$$

provided that λ_n is not equal to zero. Obviously the only possible difficulty occurs if one of the eigenvalues (which we take to be λ_0) is equal to zero. In that case, Eq. (1.5.9) with $m = 0$ reads

$$\lambda_0 \beta_0 = \alpha_0 \quad (\lambda_0 = 0). \tag{1.5.11}$$

Now if $\alpha_0 \neq 0$, then we cannot solve for β_0 and thus the problem $Ly = f$ has no solution. On the other hand if $\alpha_0 = 0$, i.e., if

$$(\psi_0, f) = (\psi_H, f) = 0, \tag{1.5.12}$$

meaning that f is orthogonal to the homogeneous solution to the adjoint problem, then Eq. (1.5.11) is satisfied by any choice of β_0. All the other β_n's ($n = 1, 2, \ldots$) are uniquely determined but there are infinitely many solutions $y(x)$ to Eq. (1.5.1) corresponding to the infinitely many values possible for β_0. The reader must make certain that he or she understands the equivalence of the above with the original statement of the Fredholm Alternative.

1.6 Self-Adjoint Operators

Operators which are self-adjoint or Hermitian form a very useful class of operators. They possess a number of special properties, some of which are described in this section.

The first important property of self-adjoint operators under consideration is that their *eigenvalues are real*. To prove this, begin with

$$\begin{cases} L\phi_n = \lambda_n \phi_n, \\ L\phi_m = \lambda_m \phi_m, \end{cases} \tag{1.6.1}$$

and take the inner product of both sides of the former with ϕ_m from the left, and the latter with ϕ_n from the right to obtain

$$\begin{cases} (\phi_m, L\phi_n) = \lambda_n(\phi_m, \phi_n), \\ (L\phi_m, \phi_n) = \lambda_m^*(\phi_m, \phi_n). \end{cases} \tag{1.6.2}$$

For a self-adjoint operator $L = L^{\text{adj}}$, the two left-hand sides of Eq. (1.6.2) are equal and hence, upon subtraction of the latter from the former, we find

$$0 = (\lambda_n - \lambda_m^*)(\phi_m, \phi_n). \tag{1.6.3}$$

Now, if $m = n$, the inner product $(\phi_n, \phi_n) = \|\phi_n\|^2$ is nonzero and Eq. (1.6.3) implies

$$\lambda_n = \lambda_n^*, \tag{1.6.4}$$

proving that all the eigenvalues are real. Thus Eq. (1.6.3) can be rewritten as

$$0 = (\lambda_n - \lambda_m)(\phi_m, \phi_n), \tag{1.6.5}$$

indicating that if $\lambda_n \neq \lambda_m$, then the eigenfunctions ϕ_m and ϕ_n are orthogonal. Thus, upon normalizing each ϕ_n, we verify a second important property of self-adjoint operators that (upon normalization) the *eigenfunctions of a self-adjoint operator form an orthonormal set*.

The Fredholm Alternative can also be restated for a self-adjoint operator L in the following form: the inhomogeneous problem $Ly = f$ (with L self-adjoint) is solvable for y, if f is orthogonal to all eigenfunctions ϕ_0 of L with eigenvalue zero (if any indeed exist). If zero is not an eigenvalue of L, the solution is unique. Otherwise, there is no solution if $(\phi_0, f) \neq 0$, and an infinite number of solutions if $(\phi_0, f) = 0$.

Diagonalization of Self-Adjoint Operators: Any linear operator can be expanded in terms of any orthonormal basis set. To elaborate on this, suppose that the orthonormal system $\{e_i(x)\}_i$, with $(e_i, e_j) = \delta_{ij}$, forms a complete set. Any function $f(x)$ can be expanded as

$$f(x) = \sum_{j=1}^{\infty} \alpha_j e_j(x), \quad \alpha_j = (e_j, f). \tag{1.6.6}$$

Thus the function $f(x)$ can be thought of as an infinite-dimensional vector with components α_j. Now consider the action of an arbitrary linear operator L on the function $f(x)$. Obviously

$$Lf(x) = \sum_{j=1}^{\infty} \alpha_j Le_j(x). \tag{1.6.7}$$

But L acting on $e_j(x)$ is itself a function of x which can be expanded in the orthonormal basis $\{e_i(x)\}_i$. Thus we write

$$Le_j(x) = \sum_{i=1}^{\infty} l_{ij} e_i(x), \tag{1.6.8}$$

wherein the coefficients l_{ij} of the expansion are found to be $l_{ij} = (e_i, Le_j)$. Substitution of Eq. (1.6.8) into Eq. (1.6.7) then shows

$$Lf(x) = \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} l_{ij} \alpha_j \right) e_i(x). \tag{1.6.9}$$

We discover that just as we can think of $f(x)$ as the infinite-dimensional vector with components α_j, we can consider L to be equivalent to an infinite-dimensional matrix with components l_{ij}, and we can regard Eq. (1.6.9) as a regular multiplication of the matrix L (components l_{ij}) with the vector f (components α_j). However, this equivalence of the operator L with the matrix whose components are l_{ij}, i.e., $L \Leftrightarrow l_{ij}$, depends on the choice of the orthonormal set.

For a self-adjoint operator $L = L^{\text{adj}}$, the natural choice of the basis set is the set of eigenfunctions of L. Denoting these by $\{\phi_i(x)\}_i$, the components of the equivalent matrix for L take the form

$$l_{ij} = (\phi_i, L\phi_j) = (\phi_i, \lambda_j \phi_j) = \lambda_j (\phi_i, \phi_j) = \lambda_j \delta_{ij}. \tag{1.6.10}$$

1.7 Green's Functions for Differential Equations

In this section, we describe the conceptual basis of the theory of *Green's functions*. We do this by first outlining the abstract themes involved and then by presenting a simple example. More complicated examples will appear in later chapters.

Prior to discussing Green's functions, recall some of elementary properties of the so-called Dirac delta function $\delta(x-x')$. In particular, remember that if x' is inside the domain of integration (a,b), for any well-behaved function $f(x)$, we have

$$\int_a^b \delta(x-x')f(x)dx = f(x'), \tag{1.7.1}$$

which can be written as

$$(\delta(x-x'), f(x)) = f(x'), \tag{1.7.2}$$

with the inner product taken with respect to x. Also remember that $\delta(x-x')$ is equal to zero for any $x \neq x'$.

Suppose now that we wish to solve a differential equation

$$Lu(x) = f(x), \tag{1.7.3}$$

on the domain $x \in (a,b)$ and subject to given boundary conditions, with L a differential operator. Consider what happens when a function $g(x,x')$ (which is as yet unknown but will end up being Green's function) is multiplied on both sides of Eq. (1.7.3) followed by integration of both sides with respect to x from a to b. That is, consider taking the inner product of both sides of Eq. (1.7.3) with $g(x,x')$ with respect to x. (We suppose everything is real in this section so that no complex conjugation is necessary.) This yields

$$(g(x,x'), Lu(x)) = (g(x,x'), f(x)). \tag{1.7.4}$$

Now by definition of the adjoint L^{adj} of L, the left-hand side of Eq. (1.7.4) can be written as

$$(g(x,x'), Lu(x)) = (L^{\text{adj}}g(x,x'), u(x)) + \text{boundary terms}. \tag{1.7.5}$$

In this expression, we explicitly recognize the terms involving the boundary points which arise when L is a differential operator. The *boundary terms* on the right-hand side of Eq. (1.7.5) emerge when we integrate by parts. It is difficult to be more specific than this when we work in the abstract, but our example should clarify what we mean shortly. If Eq. (1.7.5) is substituted back into Eq. (1.7.4), it provides

$$(L^{\text{adj}}g(x,x'), u(x)) = (g(x,x'), f(x)) + \text{boundary terms}. \tag{1.7.6}$$

So far we have not discussed what function $g(x, x')$ to choose. Suppose we choose that $g(x, x')$ which satisfies

$$L^{\text{adj}} g(x, x') = \delta(x - x'), \tag{1.7.7}$$

subject to appropriately selected boundary conditions which eliminate all the unknown terms within the boundary terms. This function $g(x, x')$ is known as Green's function. Substituting Eq. (1.7.7) into Eq. (1.7.6) and using property (1.7.2) then yields

$$u(x') = (g(x, x'), f(x)) + \text{known boundary terms}, \tag{1.7.8}$$

which is the solution to the differential equation since everything on the right-hand side is known once $g(x, x')$ has been found. More properly, if we change x' to x in the above and use a different dummy variable ξ of integration in the inner product, we have

$$u(x) = \int_a^b g(\xi, x) f(\xi) d\xi + \text{known boundary terms}. \tag{1.7.9}$$

In summary, to solve the linear inhomogeneous differential equation

$$Lu(x) = f(x)$$

using Green's function, we first solve the equation

$$L^{\text{adj}} g(x, x') = \delta(x - x')$$

for Green's function $g(x, x')$, subject to the appropriately selected boundary conditions, and immediately obtain the solution given by Eq. (1.7.9) to our differential equation.

The above we hope will become more clear in the context of the following simple example.

❑ **Example 1.1.** Consider the problem of finding the displacement $u(x)$ of a taut string under the *distributed load* $f(x)$ as in Figure 1.1.

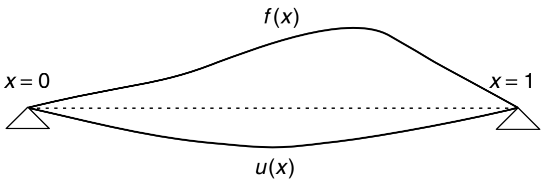

Fig. 1.1 Displacement $u(x)$ of a taut string under the distributed load $f(x)$ with $x \in (0, 1)$.

Solution. The governing ordinary differential equation for the vertical displacement $u(x)$ has the form

$$\frac{d^2u}{dx^2} = f(x) \quad \text{for} \quad x \in (0,1) \tag{1.7.10}$$

subject to boundary conditions

$$u(0) = 0 \quad \text{and} \quad u(1) = 0. \tag{1.7.11}$$

To proceed formally, we multiply both sides of Eq. (1.7.10) by $g(x, x')$ and integrate from 0 to 1 with respect to x to find

$$\int_0^1 g(x,x')\frac{d^2u}{dx^2}dx = \int_0^1 g(x,x')f(x)dx.$$

Integrate the left-hand side by parts twice to obtain

$$\begin{aligned}&\int_0^1 \frac{d^2}{dx^2}g(x,x')u(x)dx\\ &+\left[g(1,x')\frac{du}{dx}\,|_{x=1} - g(0,x')\frac{du}{dx}\,|_{x=0} - u(1)\frac{dg(1,x')}{dx} + u(0)\frac{dg(0,x')}{dx}\right]\\ &= \int_0^1 g(x,x')f(x)dx.\end{aligned} \tag{1.7.12}$$

The terms contained within the square brackets on the left-hand side of (1.7.12) are the boundary terms. Because of the boundary conditions (1.7.11), the last two terms vanish. Hence a prudent choice of boundary conditions for $g(x, x')$ would be to set

$$g(0,x') = 0 \quad \text{and} \quad g(1,x') = 0. \tag{1.7.13}$$

With that choice, all the boundary terms vanish (this does not necessarily happen for other problems). Now suppose that $g(x, x')$ satisfies

$$\frac{d^2g(x,x')}{dx^2} = \delta(x-x'), \tag{1.7.14}$$

subject to the boundary conditions (1.7.13). Use of Eqs. (1.7.14) and (1.7.13) in Eq. (1.7.12) yields

$$u(x') = \int_0^1 g(x,x')f(x)dx, \tag{1.7.15}$$

as our solution, once $g(x, x')$ has been obtained. Remark that if the original differential operator d^2/dx^2 is denoted by L, its adjoint L^{adj} is also d^2/dx^2 as found by twice integrating by parts. Hence the latter operator is indeed self-adjoint.

The last step involves the actual solution of (1.7.14) subject to (1.7.13). The variable x' plays the role of a parameter throughout. With x' somewhere between 0 and 1, Eq. (1.7.14) can actually be solved separately in each domain $0 < x < x'$ and $x' < x < 1$. For each of these, we have

$$\frac{d^2 g(x,x')}{dx^2} = 0 \quad \text{for} \quad 0 < x < x', \tag{1.7.16a}$$

$$\frac{d^2 g(x,x')}{dx^2} = 0 \quad \text{for} \quad x' < x < 1. \tag{1.7.16b}$$

The general solution in each subdomain is easily written down as

$$g(x,x') = Ax + B \quad \text{for} \quad 0 < x < x', \tag{1.7.17a}$$

$$g(x,x') = Cx + D \quad \text{for} \quad x' < x < 1. \tag{1.7.17b}$$

The general solution involves the four unknown constants A, B, C, and D. Two relations for the constants are found using the two boundary conditions (1.7.13). In particular, we have

$$g(0,x') = 0 \to B = 0; \quad g(1,x') = 0 \to C + D = 0. \tag{1.7.18}$$

To provide two more relations which are needed to permit all four of the constants to be determined, we return to the governing equation (1.7.14). Integrate both sides of the latter with respect to x from $x' - \varepsilon$ to $x' + \varepsilon$ and take the limit as $\varepsilon \to 0$ to find

$$\lim_{\varepsilon\to 0}\int_{x'-\varepsilon}^{x'+\varepsilon} \frac{d^2 g(x,x')}{dx^2}dx = \lim_{\varepsilon\to 0}\int_{x'-\varepsilon}^{x'+\varepsilon} \delta(x-x')dx,$$

from which, we obtain

$$\frac{dg(x,x')}{dx}\Big|_{x=x'+} - \frac{dg(x,x')}{dx}\Big|_{x=x'-} = 1. \tag{1.7.19}$$

Thus the first derivative of $g(x,x')$ undergoes a jump discontinuity as x passes through x'. But we can expect $g(x,x')$ itself to be continuous across x', i.e.,

$$g(x,x')\,|_{x=x'+} = g(x,x')\,|_{x=x'-}\,. \tag{1.7.20}$$

In the above, x'^{+} and x'^{-} denote points infinitesimally to the right and the left of x', respectively. Using solutions (1.7.17a) and (1.7.17b) for $g(x,x')$ in each subdomain, we find that Eqs. (1.7.19) and (1.7.20), respectively, imply

$$C - A = 1, \quad Cx' + D = Ax' + B. \tag{1.7.21}$$

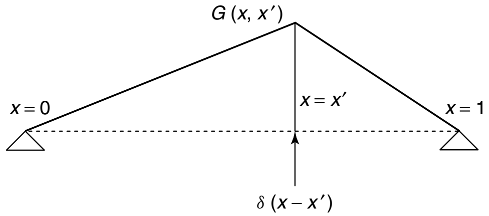

Fig. 1.2 Displacement $u(x)$ of a taut string under the concentrated load $\delta(x - x')$ at $x = x'$.

Equations (1.7.18) and (1.7.21) can be used to solve for the four constants A, B, C, and D to yield

$$A = x' - 1, \quad B = 0, \quad C = x', \quad D = -x',$$

from whence our solution (1.7.17) takes the form

$$g(x, x') = \begin{cases} (x' - 1)x & \text{for} \quad x < x', \\ x'(x - 1) & \text{for} \quad x > x', \end{cases} \tag{1.7.22a}$$

$$= x_<(x_> - 1) \text{ for } \begin{cases} x_< & = & ((x + x')/2) - |x - x'|/2, \\ x_> & = & ((x + x')/2) + |x - x'|/2. \end{cases} \tag{1.7.22b}$$

Physically Green's function (1.7.22) represents the displacement of the string subject to a *concentrated load* $\delta(x - x')$ at $x = x'$ as in Figure 1.2. For this reason, it is also called the *influence function*.

Since we have the influence function above for a concentrated load, the solution with any given distributed load $f(x)$ is given by Eq. (1.7.15) as

$$\begin{aligned} u(x) &= \int_0^1 g(\xi, x) f(\xi) d\xi \\ &= \int_0^x (x - 1)\xi f(\xi) d\xi + \int_x^1 x(\xi - 1) f(\xi) d\xi \\ &= (x - 1) \int_0^x \xi f(\xi) d\xi + x \int_x^1 (\xi - 1) f(\xi) d\xi. \end{aligned} \tag{1.7.23}$$

Although this example has been rather elementary, we hope that it has provided the reader with a basic understanding of what Green's function is. More complex and hence more interesting examples are encountered in later chapters.

1.8 Review of Complex Analysis

Let us review some important results from complex analysis.

Cauchy Integral Formula: Let $f(z)$ be analytic on and inside the closed, positively oriented contour C. Then we have

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z} d\zeta. \tag{1.8.1}$$

Differentiate this formula with respect to z to obtain

$$\frac{d}{dz} f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z)^2} d\zeta \quad \text{and} \quad \left(\frac{d}{dz}\right)^n f(z) = \frac{n!}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta. \tag{1.8.2}$$

Liouville's theorem: The only entire functions which are bounded (at infinity) are constants.

Proof: Suppose that $f(z)$ is entire. Then it can be represented by the Taylor series,

$$f(z) = f(0) + f^{(1)}(0)z + \frac{1}{2!} f^{(2)}(0)z^2 + \cdots.$$

Now consider $f^{(n)}(0)$. By the Cauchy Integral Formula, we have

$$f^{(n)}(0) = \frac{n!}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta^{n+1}} d\zeta.$$

Since $f(\zeta)$ is bounded, we have

$$|f(\zeta)| \le M.$$

Consider C to be a circle of radius R, centered at the origin. Then we have

$$|f^{(n)}(0)| \le \frac{n!}{2\pi} \cdot \frac{2\pi RM}{R^{n+1}} = n! \cdot \frac{M}{R^n} \to 0 \quad \text{as} \quad R \to \infty.$$

Thus

$$f^{(n)}(0) = 0 \quad \text{for} \quad n = 1, 2, 3, \ldots.$$

Hence

$$f(z) = \text{constant},$$

□

More generally,

(i) Suppose that $f(z)$ is entire and we know $|f(z)| \le |z|^a$ as $R \to \infty$, with $0 < a < 1$. We still find $f(z) = \text{constant}$.

(ii) Suppose that $f(z)$ is entire and we know $|f(z)| \le |z|^a$ as $R \to \infty$, with $n - 1 \le a < n$. Then $f(z)$ is at most a polynomial of degree $n - 1$.

Discontinuity theorem: Suppose that $f(z)$ has a branch cut on the real axis from a to b. It has no other singularities and it vanishes at infinity. If we know the difference

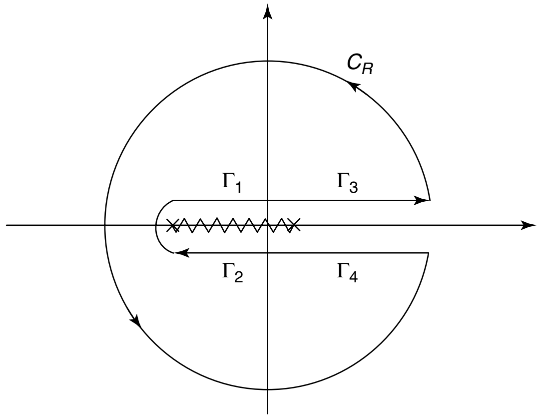

Fig. 1.3 The contours of the integration for $f(z)$. C_R is the circle of radius R centered at the origin.

between the value of $f(z)$ above and below the cut,

$$D(x) \equiv f(x+i\varepsilon) - f(x-i\varepsilon) \quad (a \le x \le b), \tag{1.8.3}$$

with ε positive infinitesimal, then

$$f(z) = \frac{1}{2\pi i}\int_a^b (D(x)/(x-z))dx. \tag{1.8.4}$$

Proof: By the Cauchy Integral Formula, we know

$$f(z) = \frac{1}{2\pi i}\oint_\Gamma \frac{f(\zeta)}{\zeta - z}d\zeta,$$

where Γ consists of the following pieces (see Figure 1.3), $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 + C_R$.

The contribution from C_R vanishes since $|f(z)| \to 0$ as $R \to \infty$, while the contributions from Γ_3 and Γ_4 cancel each other. Hence we have

$$f(z) = \frac{1}{2\pi i}\left(\int_{\Gamma_1} + \int_{\Gamma_2}\right)\frac{f(\zeta)}{\zeta - z}d\zeta.$$

On Γ_1, we have

$$\zeta = x + i\varepsilon \quad \text{with} \quad x : a \to b, \quad f(\zeta) = f(x+i\varepsilon),$$

$$\int_{\Gamma_1}\frac{f(\zeta)}{\zeta - z}d\zeta = \int_a^b \frac{f(x+i\varepsilon)}{x - z + i\varepsilon}dx \to \int_a^b \frac{f(x+i\varepsilon)}{x-z}dx \quad \text{as} \quad \varepsilon \to 0^+.$$

On Γ_2, we have

$$\zeta = x - i\varepsilon \quad \text{with} \quad x : b \to a, \quad f(\zeta) = f(x - i\varepsilon),$$

$$\int_{\Gamma_2} \frac{f(\zeta)}{\zeta - z} d\zeta = \int_b^a \frac{f(x - i\varepsilon)}{x - z - i\varepsilon} dx \to -\int_a^b \frac{f(x - i\varepsilon)}{x - z} dx \quad \text{as} \quad \varepsilon \to 0^+.$$

Thus we obtain

$$f(z) = \frac{1}{2\pi i} \int_a^b \frac{f(x + i\varepsilon) - f(x - i\varepsilon)}{x - z} dx = \frac{1}{2\pi i} \int_a^b (D(x)/(x - z)) dx.$$

□

If, in addition, $f(z)$ is known to have other singularities elsewhere, or may possibly be nonzero as $|z| \to \infty$, then it is of the form

$$f(z) = \frac{1}{2\pi i} \int_a^b (D(x)/(x - z)) dx + g(z), \tag{1.8.5}$$

with $g(z)$ free of cut on $[a, b]$. This is a very important result. Memorizing it will give a better understanding of the subsequent sections. □

Behavior near the end points: Consider the case when z is in the vicinity of the end point a. The behavior of $f(z)$ as $z \to a$ is related to the form of $D(x)$ as $x \to a$. Suppose that $D(x)$ is finite at $x = a$, say $D(a)$. Then we have

$$f(z) = \frac{1}{2\pi i} \int_a^b \frac{D(a) + D(x) - D(a)}{x - z} dx$$

$$= \frac{D(a)}{2\pi i} \ln\left(\frac{b - z}{a - z}\right) + \frac{1}{2\pi i} \int_a^b \frac{D(x) - D(a)}{x - z} dx. \tag{1.8.6}$$

The second integral above converges as $z \to a$ as long as $D(x)$ satisfies a *Holder condition* (which is implicitly assumed) requiring

$$|D(x) - D(a)| < A\,|x - a|^{\mu}, \quad A, \mu > 0. \tag{1.8.7}$$

Thus the end point behavior of $f(z)$ as $z \to a$ is of the form

$$f(z) = O\left(\ln(a - z)\right) \quad \text{as} \quad z \to a, \tag{1.8.8}$$

if

$$D(x) \quad \text{finite} \quad \text{as} \quad x \to a. \tag{1.8.9}$$

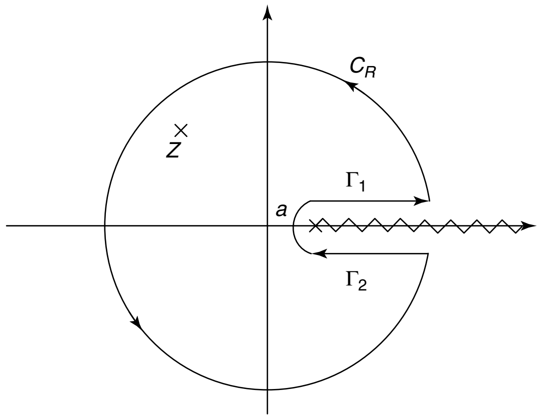

Fig. 1.4 The contour Γ of the integration for $1/(z-a)^\alpha$.

Another possibility is for $D(x)$ to be of the form

$$D(x) \to 1/(x-a)^\alpha \quad \text{with} \quad \alpha < 1 \quad \text{as} \quad x \to a, \tag{1.8.10}$$

since even with such a singularity in $D(x)$, the integral defining $f(z)$ is well defined. We claim that in that case, $f(z)$ also behaves like

$$f(z) = O\left(1/(z-a)^\alpha\right) \quad \text{as} \quad z \to a, \quad \text{with} \quad \alpha < 1, \tag{1.8.11}$$

that is, $f(z)$ is less singular than a simple pole.

Proof of the claim: Using the Cauchy Integral Formula, we have

$$1/\,(z-a)^\alpha = \frac{1}{2\pi i}\int_\Gamma \frac{d\zeta}{(\zeta-a)^\alpha\,(\zeta-z)},$$

where Γ consists of the following paths (see Figure 1.4) $\Gamma = \Gamma_1 + \Gamma_2 + C_R$. The contribution from C_R vanishes as $R \to \infty$.

On Γ_1, we set

$$\zeta - a = r \quad \text{and} \quad (\zeta-a)^\alpha = r^\alpha,$$

$$\frac{1}{2\pi i}\int_{\Gamma_1} \frac{d\zeta}{(\zeta-a)^\alpha(\zeta-z)} = \frac{1}{2\pi i}\int_0^{+\infty} \frac{dr}{r^\alpha(r+a-z)}.$$

On Γ_2, we set

$$\zeta - a = re^{2\pi i} \quad \text{and} \quad (\zeta-a)^\alpha = r^\alpha e^{2\pi i\alpha},$$

$$\frac{1}{2\pi i}\int_{\Gamma_2} \frac{d\zeta}{(\zeta-a)^\alpha(\zeta-z)} = \frac{e^{-2\pi i\alpha}}{2\pi i}\int_{+\infty}^{0} \frac{dr}{r^\alpha(r+a-z)}.$$

Thus we obtain

$$1/(z-a)^{\alpha} = \frac{1-e^{-2\pi i\alpha}}{2\pi i}\int_a^{+\infty}\frac{dx}{(x-a)^{\alpha}(x-z)},$$

which may be written as

$$1/(z-a)^{\alpha} = \frac{1-e^{-2\pi i\alpha}}{2\pi i}\left[\int_a^{b}\frac{dx}{(x-a)^{\alpha}(x-z)} + \int_b^{+\infty}\frac{dx}{(x-a)^{\alpha}(x-z)}\right].$$

The second integral above is convergent for z close to a. Obviously then, we have

$$\frac{1}{2\pi i}\int_a^{b}\frac{dx}{(x-a)^{\alpha}(x-z)} = O\left(\frac{1}{(z-a)^{\alpha}}\right) \quad \text{as} \quad z \to a.$$

A similar analysis can be done as $z \to b$. □

Summary of behavior near the end points

$$f(z) = \frac{1}{2\pi i}\int_a^{b}\frac{D(x)dx}{x-z},$$

$$\begin{cases} \text{if } D(x\to a) = D(a), & \text{then} \quad f(z) = O(\ln(a-z)), \\ \text{if } D(x\to a) = 1/(x-a)^{\alpha} \quad (0<\alpha<1), & \text{then} \quad f(z) = O(1/(z-a)^{\alpha}), \end{cases} \tag{1.8.12a}$$

$$\begin{cases} \text{if } D(x\to b) = D(b), & \text{then} \quad f(z) = O(\ln(b-z)), \\ \text{if } D(x\to b) = 1/(x-b)^{\beta} \quad (0<\beta<1), & \text{then} \quad f(z) = O(1/(z-b)^{\beta}). \end{cases} \tag{1.8.12b}$$

Principal Value Integrals: We define the principal value integral by

$$\mathrm{P}\int_a^{b}\frac{f(x)}{x-y}dx \equiv \lim_{\varepsilon\to 0^+}\left[\int_a^{y-\varepsilon}\frac{f(x)}{x-y}dx + \int_{y+\varepsilon}^{b}\frac{f(x)}{x-y}dx\right]. \tag{1.8.13}$$

Graphically expressed, the principal value integral contour is as in Figure 1.5. As such, to evaluate a principal value integral by doing complex integration, we usually make use of either of the two contours as in Figure 1.6.

Now, the contour integrals on the right of Figure 1.6 usually can be done and hence the principal value integral can be evaluated. Also, the contributions from the lower semicircle C_- and the upper semicircle C_+ take the forms

$$\int_{C_-}\frac{f(z)}{z-y}dz = i\pi f(y), \quad \int_{C_+}\frac{f(z)}{z-y}dz = -i\pi f(y),$$

as $\varepsilon \to 0^+$, as long as $f(z)$ is not singular at y.

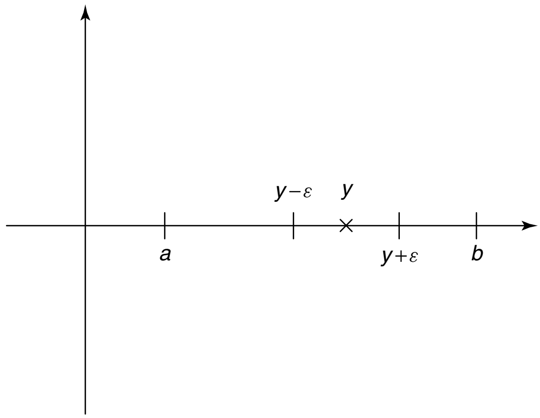

Fig. 1.5 The principal value integral contour.

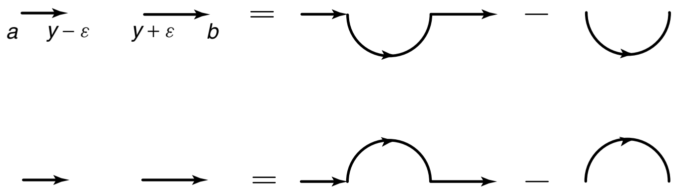

Fig. 1.6 Two contours for the principal value integral (1.8.13).

Mathematically expressed, the principal value integral is given by either of the following formulas, known as the *Plemelj formula*:

$$\frac{1}{2\pi i}\mathrm{P}\int_a^b \frac{f(x)}{x-y}dx = \lim_{\varepsilon\to 0^+}\frac{1}{2\pi i}\int_a^b \frac{f(x)}{x-y\mp i\varepsilon}dx \mp \frac{1}{2}f(y), \tag{1.8.14}$$

This is customarily written as

$$\lim_{\varepsilon\to 0^+} 1\diagup\left(x-y\mp i\varepsilon\right) = \mathrm{P}(1\diagup(x-y)) \pm i\pi\delta(x-y), \tag{1.8.15a}$$

or equivalently written as

$$\mathrm{P}(1\diagup(x-y)) = \lim_{\varepsilon\to 0^+} 1\diagup\left(x-y\mp i\varepsilon\right) \mp i\pi\delta(x-y). \tag{1.8.15b}$$

Then we interchange the order of the limit $\varepsilon \to 0^+$ and the integration over x. The principal value integrand seems to diverge at $x = y$, but it is actually finite at $x = y$ as long as $f(x)$ is not singular at $x = y$. This comes about as follows:

$$\begin{aligned}\frac{1}{x-y\mp i\varepsilon} &= \frac{(x-y)\pm i\varepsilon}{(x-y)^2+\varepsilon^2} = \frac{(x-y)}{(x-y)^2+\varepsilon^2} \pm i\pi\cdot\frac{1}{\pi}\frac{\varepsilon}{(x-y)^2+\varepsilon^2}\\ &= \frac{(x-y)}{(x-y)^2+\varepsilon^2} \pm i\pi\delta_\varepsilon(x-y),\end{aligned} \tag{1.8.16}$$

where $\delta_\varepsilon(x-y)$ is defined by

$$\delta_\varepsilon(x-y) \equiv \frac{1}{\pi}\frac{\varepsilon}{(x-y)^2+\varepsilon^2}, \tag{1.8.17}$$

with the following properties:

$$\delta_\varepsilon(x \neq y) \to 0^+ \quad \text{as} \quad \varepsilon \to 0^+; \quad \delta_\varepsilon(x=y) = \frac{1}{\pi}\frac{1}{\varepsilon} \to +\infty \quad \text{as} \quad \varepsilon \to 0^+,$$

$$\int_{-\infty}^{+\infty} \delta_\varepsilon(x-y)dx = 1.$$

The first term on the right-hand side of Eq. (1.8.16) vanishes at $x = y$ before we take the limit $\varepsilon \to 0^+$, while the second term $\delta_\varepsilon(x-y)$ approaches the Dirac delta function, $\delta(x-y)$, as $\varepsilon \to 0^+$. This is the content of Eq. (1.8.15a).

1.9 Review of Fourier Transform

The Fourier transform of a function $f(x)$, where $-\infty < x < \infty$, is defined as

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx \exp[-ikx] f(x). \tag{1.9.1}$$

There are two distinct theories of the Fourier transforms.

(I) Fourier transform of square-integrable functions.
It is assumed that

$$\int_{-\infty}^{\infty} dx \left|f(x)\right|^2 < \infty. \tag{1.9.2}$$

The inverse Fourier transform is given by

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \exp[ikx]\tilde{f}(k). \tag{1.9.3}$$

We note that in this case $\tilde{f}(k)$ is defined for real k. Accordingly, the inversion path in Eq. (1.9.3) coincides with the entire real axis. It should be borne in mind that Eq. (1.9.1) is meaningful in the sense of the convergence in the mean, namely, Eq. (1.9.1) means that there exists $\tilde{f}(k)$ for all real k such that

$$\lim_{R\to\infty} \int_{-\infty}^{\infty} dk \left| \tilde{f}(k) - \int_{-R}^{R} dx \exp[-ikx] f(x) \right|^2 = 0. \tag{1.9.4}$$

Symbolically we write

$$\tilde{f}(k) = \lim_{R\to\infty} \int_{-R}^{R} dx \exp[-ikx] f(x). \tag{1.9.5}$$

Similarly in Eq. (1.9.3), we mean that, given $\tilde{f}(k)$, there exists an $f(x)$ such that

$$\lim_{R\to\infty} \int_{-\infty}^{\infty} dx \left| f(f) - \int_{-R}^{R} \frac{dk}{2\pi} \exp[ikx] \tilde{f}(k) \right|^2 = 0. \tag{1.9.6}$$

We can then prove that

$$\int_{-\infty}^{\infty} dk \left| \tilde{f}(k) \right|^2 = 2\pi \int_{-\infty}^{\infty} dx \left| f(x) \right|^2 . \tag{1.9.7}$$

This is Parseval's identity for the square-integrable functions. We see that the pair $(f(x), \tilde{f}(k))$ defined this way consists of two functions with very similar properties. We shall find that this situation may change drastically if condition (1.9.2) is relaxed.

(II) Fourier transform of integrable functions.
We relax the condition on the function $f(x)$ as

$$\int_{-\infty}^{\infty} dx \left| f(x) \right| < \infty. \tag{1.9.8}$$

Then we can still define $\tilde{f}(k)$ for real k. Indeed, from Eq. (1.9.1), we obtain

$$\begin{aligned} \left| \tilde{f}(k\text{: real}) \right| &= \left| \int_{-\infty}^{\infty} dx \exp[-ikx] f(x) \right| \\ &\leq \int_{-\infty}^{\infty} dx \left| \exp[-ikx] f(x) \right| = \int_{-\infty}^{\infty} dx \left| f(x) \right| < \infty. \end{aligned} \tag{1.9.9}$$

We can further show that the function defined by

$$\tilde{f}_{+}(k) = \int_{-\infty}^{0} dx \exp[-ikx] f(x) \tag{1.9.10}$$

is analytic in the *upper* half-plane of the complex k plane, and

$$\tilde{f}_{+}(k) \to 0 \quad \text{as} \quad |k| \to \infty \quad \text{with} \quad \operatorname{Im} k > 0. \tag{1.9.11}$$

Similarly, we can show that the function defined by

$$\tilde{f}_{-}(k) = \int_{0}^{\infty} dx \exp[-ikx] f(x) \tag{1.9.12}$$

is analytic in the *lower* half-plane of the complex k plane, and

$$\tilde{f}_-(k) \to 0 \quad \text{as} \quad |k| \to \infty \quad \text{with} \quad \operatorname{Im} k < 0. \tag{1.9.13}$$

Clearly we have

$$\tilde{f}(k) = \tilde{f}_+(k) + \tilde{f}_-(k), \quad k\text{: real.} \tag{1.9.14}$$

We can show that

$$\tilde{f}(k) \to 0 \quad \text{as} \quad k \to \pm\infty, \quad k\text{: real.} \tag{1.9.15}$$

This is a property in common with the Fourier transform of the square-integrable functions.

❑ **Example 1.2.** Find the Fourier transform of the following function:

$$f(x) = \frac{\sin(ax)}{x}, \quad a > 0, \quad -\infty < x < \infty. \tag{1.9.16}$$

Solution. The Fourier transform $\tilde{f}(k)$ is given by

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx \exp[ikx] \frac{\sin(ax)}{x} = \int_{-\infty}^{\infty} dx \exp[ikx] \frac{\exp[iax] - \exp[-iax]}{2ix}$$

$$= \int_{-\infty}^{\infty} dx \frac{\exp[i(k+a)x] - \exp[i(k-a)x]}{2ix} = I(k+a) - I(k-a),$$

where we define the integral $I(b)$ by

$$I(b) \equiv \int_{-\infty}^{\infty} dx \frac{\exp[ibx]}{2ix} = \int_{\Gamma} dx \frac{\exp[ibx]}{2ix}.$$

The contour Γ extends from $x = -\infty$ to $x = \infty$ with the infinitesimal indent below the real x-axis at the pole $x = 0$. Noting that $x = \operatorname{Re} x + i \operatorname{Im} x$ for the complex x, we have

$$I(b) = \begin{cases} 2\pi i \cdot \operatorname{Res}\left[\frac{\exp[ibx]}{2ix}\right]_{x=0} = \pi, & b > 0, \\ 0, & b < 0. \end{cases}$$

Thus we have

$$\tilde{f}(k) = I(k+a) - I(k-a) = \int_{-\infty}^{\infty} dx \exp[ikx] \frac{\sin(ax)}{x} = \begin{cases} \pi & \text{for} \quad |k| < a, \\ 0 & \text{for} \quad |k| > a, \end{cases} \tag{1.9.17}$$

while at $k = \pm a$, we have

$$\tilde{f}(k = \pm a) = \frac{\pi}{2},$$

which is equal to

$$\frac{1}{2}[\tilde{f}(k = \pm a^+) + \tilde{f}(k = \pm a^-)].$$

❑ **Example 1.3.** Find the Fourier transform of the following function:

$$f(x) = \frac{\sin(ax)}{x(x^2 + b^2)}, \quad a, b > 0, \quad -\infty < x < \infty. \tag{1.9.18}$$

Solution. The Fourier transform $\tilde{f}(k)$ is given by

$$\tilde{f}(k) = \int_\Gamma dz \frac{\exp[i(k+a)z]}{2iz(z^2 + b^2)} - \int_\Gamma dz \frac{\exp[i(k-a)z]}{2iz(z^2 + b^2)} = I(k+a) - I(k-a), \tag{1.9.19a}$$

where we define the integral $I(c)$ by

$$I(c) \equiv \int_{-\infty}^{\infty} dz \frac{\exp[icz]}{2iz(z^2 + b^2)} = \int_\Gamma dz \frac{\exp[icz]}{2iz(z^2 + b^2)}, \tag{1.9.19b}$$

where the contour Γ is the same as in Example 1.2. The integrand has the simple poles at

$$z = 0 \quad \text{and} \quad z = \pm ib.$$

Noting $z = \operatorname{Re} z + i \operatorname{Im} z$, we have

$$I(c) = \begin{cases} 2\pi i \cdot \operatorname{Res}\left[\frac{\exp[icz]}{2iz(z^2+b^2)}\right]_{z=0} + 2\pi i \cdot \operatorname{Res}\left[\frac{\exp[icz]}{2iz(z^2+b^2)}\right]_{z=ib}, & c > 0, \\ -2\pi i \cdot \operatorname{Res}\left[\frac{\exp[icz]}{2iz(z^2+b^2)}\right]_{z=-ib}, & c < 0, \end{cases}$$

or

$$I(c) = \begin{cases} (\pi/2b^2)(2 - \exp[-bc]), & c > 0, \\ (\pi/2b^2)\exp[bc], & c < 0. \end{cases}$$

Thus we have

$$\tilde{f}(k) = I(k+a) - I(k-a) = \begin{cases} (\pi/b^2)\sinh(ab)\exp[bk], & k < -a, \\ (\pi/b^2)\{1 - \exp[-ab]\cosh(bk)\}, & |k| < a, \\ (\pi/b^2)\sinh(ab)\exp[-bk], & k > a. \end{cases} \tag{1.9.20}$$

We note that $\tilde{f}(k)$ is *step-discontinuous* at $k = \pm a$ in Example 1.2. We also note that $\tilde{f}(k)$ and $\tilde{f}'(k)$ are *continuous* for real k, while $\tilde{f}''(k)$ is *step-discontinuous* at $k = \pm a$ in Example 1.3.

We note that the rate with which

$$f(x) \to 0 \quad \text{as} \quad |x| \to +\infty$$

affects the degree of smoothness of $\tilde{f}(k)$. For the square-integrable functions, we usually have

$$f(x) = O\left(\frac{1}{x}\right) \text{ as } |x| \to +\infty \Rightarrow \tilde{f}(k) \text{ step-discontinuous},$$

$$f(x) = O\left(\frac{1}{x^2}\right) \text{ as } |x| \to +\infty \Rightarrow \left\{ \begin{array}{c} \tilde{f}(k) \text{ continuous}, \\ \tilde{f}'(k) \text{ step-discontinuous}, \end{array} \right.$$

$$f(x) = O\left(\frac{1}{x^3}\right) \text{ as } |x| \to +\infty \Rightarrow \left\{ \begin{array}{c} \tilde{f}(k), \tilde{f}'(k) \text{ continuous}, \\ \tilde{f}''(k) \text{ step-discontinuous}, \end{array} \right.$$

and so on.

Having learned in above the abstract notions relating to linear space, inner product, operator and its adjoint, eigenvalue and eigenfunction, Green's function, and the review of Fourier transform and complex analysis, we are now ready to embark on our study of integral equations. We encourage the reader to make an effort to connect the concrete example that will follow with the abstract idea of linear function space and linear operator. This will not be possible in all circumstances.

The abstract idea of function space is also useful in the discussion of the calculus of variations where a piecewise continuous but nowhere differentiable function and a discontinuous function show up as the solution of the problem.

We present the applications of the calculus of variations to theoretical physics, specifically, classical mechanics, canonical transformation theory, the Hamilton–Jacobi equation, classical electrodynamics, quantum mechanics, quantum field theory and quantum statistical mechanics.

The mathematically oriented reader is referred to the monographs by *R. Kress*, and *I.M. Gelfand* , and *S.V. Fomin* for details of the theories of integral equations and calculus of variations.

2
Integral Equations and Green's Functions

2.1
Introduction to Integral Equations

An integral equation is the equation in which function to be determined appears in an integral. There exist several types of integral equations:

Fredholm integral equation of the second kind:

$$\phi(x) = F(x) + \lambda \int_a^b K(x, y)\phi(y)dy \quad (a \leq x \leq b),$$

Fredholm integral equation of the first kind:

$$F(x) = \int_a^b K(x, y)\phi(y)dy \quad (a \leq x \leq b),$$

Volterra integral equation of the second kind:

$$\phi(x) = F(x) + \lambda \int_0^x K(x, y)\phi(y)dy \quad \text{with} \quad K(x, y) = 0 \quad \text{for} \quad y > x,$$

Volterra integral equation of the first kind:

$$F(x) = \int_0^x K(x, y)\phi(y)dy \quad \text{with} \quad K(x, y) = 0 \quad \text{for} \quad y > x.$$

In the above, $K(x, y)$ is the *kernel* of the integral equation and $\phi(x)$ is the unknown function. If $F(x) = 0$, the equations are said to be *homogeneous*, and if $F(x) \neq 0$, they are said to be *inhomogeneous*.

Now, begin with some simple examples of Fredholm Integral Equations.

❑ **Example 2.1.** Inhomogeneous Fredholm Integral Equation of the second kind.

$$\phi(x) = x + \lambda \int_{-1}^{1} xy\phi(y)dy, \quad -1 \leq x \leq 1. \tag{2.1.1}$$

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Solution. Since $\int_{-1}^{1} y\phi(y)dy$ is some constant, define

$$A = \int_{-1}^{1} y\phi(y)dy. \tag{2.1.2}$$

Then Eq. (2.1.1) takes the form

$$\phi(x) = x(1 + \lambda A). \tag{2.1.3}$$

Substituting Eq. (2.1.3) into the right-hand side of Eq. (2.1.2), we obtain

$$A = \int_{-1}^{1} (1 + \lambda A)y^2 dy = \frac{2}{3}(1 + \lambda A).$$

Solving for A, we obtain

$$\left(1 - \frac{2}{3}\lambda\right) A = \frac{2}{3}.$$

If $\lambda = \frac{3}{2}$, no such A exists. Otherwise A is uniquely determined to be

$$A = \frac{2}{3} \Big/ \left(1 - \frac{2}{3}\lambda\right). \tag{2.1.4}$$

Thus, if $\lambda = \frac{3}{2}$, no solution exists. Otherwise, a unique solution exists and is given by

$$\phi(x) = x \Big/ \left(1 - \frac{2}{3}\lambda\right). \tag{2.1.5}$$

We now consider the homogeneous counter part of the inhomogeneous Fredholm integral equation of the second kind considered in Example 2.1

❑ **Example 2.2.** Homogeneous Fredholm Integral Equation of the second kind.

$$\phi(x) = \lambda \int_{-1}^{1} xy\phi(y)dy, \quad -1 \le x \le 1. \tag{2.1.6}$$

Solution. As in Example 2.1, define

$$A = \int_{-1}^{1} y\phi(y)dy. \tag{2.1.7}$$

Then

$$\phi(x) = \lambda A x. \tag{2.1.8}$$

Substituting Eq. (2.1.8) into Eq. (2.1.7), we obtain

$$A = \int_{-1}^{1} \lambda A y^2 dy = \frac{2}{3}\lambda A. \tag{2.1.9}$$

The solution exists only when $\lambda = \frac{3}{2}$. Thus the nontrivial homogeneous solution exists only for $\lambda = \frac{3}{2}$, whence $\phi(x)$ is given by $\phi(x) = \alpha x$ with α arbitrary. If $\lambda \neq \frac{3}{2}$, no nontrivial homogeneous solution exists.

We observe the following correspondence in Examples 2.1 and 2.2:

	Inhomogeneous case	*Homogeneous case*
$\lambda \neq 3/2$	Unique solution	Trivial solution
$\lambda = 3/2$	No solution	Infinitely many solutions

(2.1.10)

We further note the analogy of an integral equation to a *system of inhomogeneous linear algebraic equations (matrix equations)*:

$$(K - \mu I)\vec{U} = \vec{F} \tag{2.1.11}$$

where K is an $n \times n$ matrix, I is the $n \times n$ identity matrix, $\vec{U}$ and $\vec{F}$ are n-dimensional vectors, and μ is a number. Equation (2.1.11) has the unique solution,

$$\vec{U} = (K - \mu I)^{-1}\vec{F}, \tag{2.1.12}$$

provided that $(K - \mu I)^{-1}$ exists, or equivalently that

$$\det(K - \mu I) \neq 0. \tag{2.1.13}$$

The homogeneous equation corresponding to Eq. (2.1.11) is

$$(K - \mu I)\vec{U} = 0 \quad \text{or} \quad K\vec{U} = \mu\vec{U}, \tag{2.1.14}$$

which is the eigenvalue equation for the matrix K. The solutions to the homogeneous equation (2.1.14) exist for certain values of $\mu = \mu_n$, which are called the eigenvalues. If μ is equal to an eigenvalue μ_n, $(K - \mu I)^{-1}$ fails to exist and Eq. (2.1.11) has generally no finite solution.

❑ **Example 2.3.** Change the inhomogeneous term x of Example 2.1 to 1.

$$\phi(x) = 1 + \lambda \int_{-1}^{1} xy\phi(y)dy, \quad -1 \leq x \leq 1. \tag{2.1.15}$$

Solution. As before, define

$$A = \int_{-1}^{1} y\phi(y)dy. \tag{2.1.16}$$

Then

$$\phi(x) = 1 + \lambda Ax. \tag{2.1.17}$$

Substituting Eq. (2.1.17) into Eq. (2.1.16), we obtain $A = \int_{-1}^{1} y(1 + \lambda Ay)dy = \frac{2}{3}\lambda A$. Thus, for $\lambda \neq \frac{3}{2}$, the unique solution exists with $A = 0$, and $\phi(x) = 1$, while for $\lambda = \frac{3}{2}$, infinitely many solutions exist with A arbitrary and $\phi(x) = 1 + \frac{3}{2}Ax$.

The above three examples illustrate the *Fredholm Alternative*:
For $\lambda = \frac{3}{2}$, the homogeneous problem has a solution, given by

$$\phi_H(x) = \alpha x \quad \text{for any } \alpha.$$

For $\lambda \neq \frac{3}{2}$, the inhomogeneous problem has a unique solution, given by

$$\phi(x) = \begin{cases} x/\left(1 - \frac{2}{3}\lambda\right) & \text{when} \quad F(x) = x, \\ 1 & \text{when} \quad F(x) = 1. \end{cases}$$

For $\lambda = \frac{3}{2}$, the inhomogeneous problem has no solution when $F(x) = x$, while it has infinitely many solutions when $F(x) = 1$. In the former case, $(\phi_H, F) = \int_{-1}^{1} \alpha x \cdot x dx \neq 0$, while in the latter case, $(\phi_H, F) = \int_{-1}^{1} \alpha x \cdot 1 dx = 0$.

It is not surprising that Eq. (2.1.15) has infinitely many solutions when $\lambda = 3/2$. Generally, if ϕ_0 is a solution of an inhomogeneous equation, and ϕ_1 is a solution of the corresponding homogeneous equation, then $\phi_0 + a\phi_1$ is also a solution of the inhomogeneous equation, where a is any constant. Thus, if λ is equal to an eigenvalue, an inhomogeneous equation has infinitely many solutions as long as it has one solution. The nontrivial question is: Under what condition can we expect the latter to happen? In the present example, the relevant condition is $\int_{-1}^{1} y dy = 0$, which means that the inhomogeneous term (which is 1) multiplied by y and integrated from -1 to 1, is zero. There is a counterpart of this condition for matrix equations. It is well known that, under certain circumstances, the inhomogeneous matrix equation (2.1.11) has solutions even if μ is equal to an eigenvalue. Specifically this happens if the inhomogeneous term $\vec{F}$ is a linear superposition of the vectors each of which forms a column of $(K - \mu I)$. There is another way to phrase this. Consider all vectors $\vec{V}$ satisfying

$$(K^{\mathrm{T}} - \mu I)\vec{V} = 0, \tag{2.1.18}$$

where K^{T} is the transpose of K. The equation above says that $\vec{V}$ is an eigenvector of K^{T} with the eigenvalue μ. It also says that $\vec{V}$ is perpendicular to all row vectors of $(K^{\mathrm{T}} - \mu I)$. If $\vec{F}$ is a linear superposition of the column vectors of $(K - \mu I)$ (which are the row vectors of $(K^{\mathrm{T}} - \mu I)$), then $\vec{F}$ is perpendicular to $\vec{V}$. Therefore, the inhomogeneous equation (2.1.11) has solutions when μ is an eigenvalue, if and only if $\vec{F}$ is perpendicular to all eigenvectors of K^{T} with eigenvalue μ. Similarly an inhomogeneous integral equation has solutions even when λ is equal to an eigenvalue, as long as the inhomogeneous term is perpendicular to all of the eigenfunctions of the transposed kernel (the kernel with $x \leftrightarrow y$) of that particular eigenvalue.

As we have seen in Chapter 1, just like a matrix, a kernel and its transpose have the same eigenvalues. Hence, the homogeneous integral equation with the transposed kernel has no solution if λ is not equal to an eigenvalue of the kernel. Hence, if λ is not an eigenvalue, any inhomogeneous term is trivially perpendicular to all solutions of the homogeneous integral equation with the transposed kernel, since all of them are trivial. Together with the result in the preceding paragraph, we arrived at the necessary and sufficient condition for an inhomogeneous integral equation to have a solution: the inhomogeneous term must be perpendicular to all solutions of the homogeneous integral equation with the transposed kernel.

There exists another kind of integral equations in which the unknown appears only in the integrals. Consider one more example of a Fredholm Integral Equation.

❑ **Example 2.4.** Fredholm Integral Equation of the first kind.
Case (A)

$$1 = \int_0^1 xy\phi(y)dy, \quad 0 \le x \le 1. \tag{2.1.19}$$

Case (B)

$$x = \int_0^1 xy\phi(y)dy, \quad 0 \le x \le 1. \tag{2.1.20}$$

Solution. In both the cases, divide both sides of the equations by x to obtain
Case (A)

$$1/x = \int_0^1 y\phi(y)dy. \tag{2.1.21}$$

Case (B)

$$1 = \int_0^1 y\phi(y)dy. \tag{2.1.22}$$

In the case of Eq. (2.1.21), no solution exists, while in the case of Eq. (2.1.22), infinitely many $\phi(x)$ are possible. Any function $\psi(x)$ which satisfies

$$\int_0^1 y\psi(y)dy \neq 0 \quad \text{or} \quad \infty,$$

can be made a solution to Eq. (2.1.20). Indeed,

$$\phi(x) = \psi(x)/\int_0^1 y\psi(y)dy \tag{2.1.23}$$

will do.

Therefore, for the kind of integral equations considered in Example 2.4, we have no solution for some inhomogeneous terms, while we have infinitely many solutions for some other inhomogeneous terms.

Next, we shall consider an example of a Volterra Integral Equation of the second kind with the transformation of an integral equation into an ordinary differential equation.

❑ **Example 2.5.** Volterra Integral Equation of the second kind.

$$\phi(x) = ax + \lambda x \int_0^x \phi(x')dx'. \tag{2.1.24}$$

Solution. Divide both sides of Eq. (2.1.24) by x to obtain

$$\phi(x)/x = a + \lambda \int_0^x \phi(x')\, dx'. \tag{2.1.25}$$

Differentiate both sides of Eq. (2.1.25) with respect to x to obtain

$$\frac{d}{dx}(\phi(x)/x) = \lambda\phi(x). \tag{2.1.26}$$

By setting

$$u(x) = \phi(x)/x,$$

the following differential equation results:

$$du(x)/u(x) = \lambda x dx. \tag{2.1.27}$$

By integrating both sides,

$$\ln u(x) = \frac{1}{2}\lambda x^2 + \text{constant}.$$

Hence the solution is given by

$$u(x) = Ae^{\frac{1}{2}\lambda x^2}, \quad \text{or} \quad \phi(x) = Axe^{\frac{1}{2}\lambda x^2}. \tag{2.1.28}$$

To determine the integration constant A in Eq. (2.1.28), note that as $x \to 0$, based on the integral equation (2.1.24), $\phi(x)$ above behaves as

$$\phi(x) \to ax + O(x^3) \tag{2.1.29}$$

while our solution (2.1.28) behaves as

$$\phi(x) \to Ax + O(x^3). \tag{2.1.30}$$

Hence, from Eqs. (2.1.29) and (2.1.30), we identify

$$A = a.$$

Thus the final form of the solution is

$$\phi(x) = axe^{\frac{1}{2}\lambda x^2}, \tag{2.1.31}$$

which is the unique solution for all λ.

We observe three points:

(1) The integral equation (2.1.24) has a unique solution for all values of λ. It follows that the corresponding homogeneous integral equation, obtained from Eq. (2.1.24) by setting $a = 0$, does not have a nontrivial solution. Indeed, this can be directly verified by setting $a = 0$ in Eq. (2.1.31). This means that the kernel for Eq. (2.1.24) has no eigenvalues. This is true for all square-integrable kernels of the Volterra type.

(2) While the solution to the differential equation (2.1.26) or (2.1.27) contains an arbitrary constant, the solution to the corresponding integral equation (2.1.24) does not. More precisely, Eq. (2.1.24) is equivalent to Eq. (2.1.26) or Eq. (2.1.27) plus an initial condition.

(3) The transformation of Volterra Integral Equation of the second kind to an ordinary differential equation is possible whenever the kernel of the Volterra integral equation is a sum of the factored terms.

In the above example, we solved the integral equation by transforming it into a differential equation. This is not often possible. On the other hand, it is, in general, easy to transform a differential equation into an integral equation. However, lest there be any misunderstanding, let me state that we never solve a differential equation by such a transformation . Indeed, an integral equation is much more difficult to solve than a differential equation in a closed form. It is very rare that this can be done. Therefore, whenever it is possible to transform an integral equation into a differential equation, it is a good idea to do so. On the other hand, there are advantages in transforming a differential equation into an integral equation. This transformation may facilitate the discussion of the existence and uniqueness of the solution, the spectrum of the eigenvalue, and the analyticity of the solution. It also enables us to obtain the perturbative solution of the equation.

2.2 Relationship of Integral Equations with Differential Equations and Green's Functions

To help the sense of bearing of the reader, we shall discuss the transformation of a differential equation to an integral equation. This transformation is accomplished with the use of *Green's functions*.

As an example, consider the one-dimensional Schrödinger equation with potential $U(x)$:

$$\left(\frac{d^2}{dx^2}+k^2\right)\phi(x) = U(x)\phi(x). \tag{2.2.1}$$

It is assumed that $U(x)$ vanishes rapidly as $|x| \to \infty$. Although Eq. (2.2.1) is most usually thought of as an initial value problem, let us suppose that we are given

$$\phi(0) = a \quad \text{and} \quad \phi^{'}(0) = b, \tag{2.2.2}$$

and are interested in the solution for $x > 0$.

Green's function: We first treat the right-hand side of Eq. (2.2.1) as an inhomogeneous term $f(x)$. Namely, consider the following inhomogeneous problem:

$$L\phi(x) = f(x) \quad \text{with} \quad L = \frac{d^2}{dx^2}+k^2, \tag{2.2.3}$$

and the boundary conditions specified by Eq. (2.2.2). Multiply both sides of Eq. (2.2.3) by $g(x, x')$ and integrate with respect to x from 0 to ∞. Then

$$\int_0^\infty g(x,x')L\phi(x) = \int_0^\infty g(x,x')f(x)dx. \tag{2.2.4}$$

Integrate by parts twice on the left-hand side of Eq. (2.2.4) to obtain

$$\begin{aligned}&\int_0^\infty (Lg(x,x'))\phi(x)dx + g(x,x')\phi^{'}(x)\Big|_{x=0}^{x=\infty} - \frac{dg(x,x')}{dx}\phi(x)\Big|_{x=0}^{x=\infty}\\ &= \int_0^\infty g(x,x')f(x)dx.\end{aligned} \tag{2.2.5}$$

In the boundary terms, $\phi^{'}(0)$ and $\phi(0)$ are known. To get rid of unknown terms, we require

$$g(\infty, x') = 0 \quad \text{and} \quad \frac{dg}{dx}(\infty, x') = 0. \tag{2.2.6}$$

Also, we choose $g(x, x')$ to satisfy

$$Lg(x,x') = \delta(x-x'). \tag{2.2.7}$$

Then we find from Eq. (2.2.5)

$$\phi(x') = bg(0,x') - a\frac{dg}{dx}(0,x') + \int_0^\infty g(x,x')f(x)dx. \tag{2.2.8}$$

Solution for *g(x,x′)*. The governing equation and the boundary conditions are given by

$$\left(\frac{d^2}{dx^2} + k^2\right) g(x, x') = \delta(x - x') \quad \text{on} \quad x \in (0, \infty) \quad \text{with} \quad x' \in (0, \infty). \quad (2.2.9)$$

Boundary condition 1:

$$g(\infty, x') = 0, \quad (2.2.10)$$

Boundary condition 2:

$$\frac{dg}{dx}(\infty, x') = 0. \quad (2.2.11)$$

For $x < x'$,

$$g(x, x') = A \sin kx + B \cos kx. \quad (2.2.12)$$

For $x > x'$,

$$g(x, x') = C \sin kx + D \cos kx. \quad (2.2.13)$$

Applying the boundary conditions, (2.2.10) and (2.2.11) above, results in $C = D = 0$. Thus

$$g(x, x') = 0 \quad \text{for} \quad x > x'. \quad (2.2.14)$$

Now, integrate the differential equation (2.2.9) across x' to obtain

$$\frac{dg}{dx}(x' + \varepsilon, x') - \frac{dg}{dx}(x' - \varepsilon, x') = 1, \quad (2.2.15)$$

$$g(x' + \varepsilon, x') = g(x' - \varepsilon, x'). \quad (2.2.16)$$

Letting $\varepsilon \to 0$, we obtain the equations for A and B.

$$\begin{cases} A \sin kx' + B \cos kx' = 0, \\ -A \cos kx' + B \sin kx' = 1/k. \end{cases}$$

Thus A and B are determined to be

$$A = (-\cos kx')/k, \quad B = (\sin kx')/k, \quad (2.2.17)$$

and Green's function is found to be

$$g(x, x') = \begin{cases} (\sin k(x' - x))/k & \text{for} \quad x < x', \\ 0 & \text{for} \quad x > x'. \end{cases} \quad (2.2.18)$$

Equation (2.2.8) becomes

$$\phi(x') = b\frac{\sin kx'}{k} + a\cos kx' + \int_0^{x'} \frac{\sin k(x'-x)}{k} f(x)dx.$$

Changing x to ξ and x' to x, and recalling that $f(x) = U(x)\phi(x)$, we find

$$\phi(x) = a\cos kx + b\frac{\sin kx}{k} + \int_0^{x} \frac{\sin k(x-\xi)}{k} U(\xi)\phi(\xi)d\xi, \tag{2.2.19}$$

which is a Volterra Integral Equation of the second kind.

Next consider the very important *scattering problem* for the Schrödinger equation:

$$\left(\frac{d^2}{dx^2} + k^2\right)\phi(x) = U(x)\phi(x) \quad \text{on} \quad -\infty < x < \infty, \tag{2.2.20}$$

where the potential $U(x) \to 0$ as $|x| \to \infty$. As such we might expect that

$$\begin{cases} \phi(x) & \to & Ae^{ikx} + Be^{-ikx} & \text{as} & x \to -\infty, \\ \phi(x) & \to & Ce^{ikx} + De^{-ikx} & \text{as} & x \to +\infty. \end{cases}$$

Now (with an $e^{-i\omega t}$ implicitly multiplying $\phi(x)$), the term e^{ikx} represents a wave going to the right while e^{-ikx} is a wave going to the left. In the scattering problem, we suppose that there is an incident wave with amplitude 1 (i.e., $A = 1$), the reflected wave with amplitude R (i.e., $B = R$) and the transmitted wave with amplitude T (i.e., $C = T$). Both R and T are still unknown. Also as $x \to +\infty$, there is no left-going wave (i.e., $D = 0$). Thus the problem is to solve

$$\left(\frac{d^2}{dx^2} + k^2\right)\phi(x) = U(x)\phi(x), \tag{2.2.21}$$

with the boundary conditions

$$\begin{cases} \phi(x \to -\infty) = e^{ikx} + Re^{-ikx}, \\ \phi(x \to +\infty) = Te^{ikx}. \end{cases} \tag{2.2.22}$$

Green's function: Multiply both sides of Eq. (2.2.21) by $g(x, x')$, integrate with respect to x from $-\infty$ to $+\infty$, and integrate twice by parts. The result is

$$\begin{aligned} &\int_{-\infty}^{+\infty} \phi(x)\left(\frac{d^2}{dx^2} + k^2\right) g(x,x')dx + g(\infty,x')\frac{d\phi}{dx}(\infty) - g(-\infty,x')\frac{d\phi}{dx}(-\infty) \\ &\quad - \frac{dg}{dx}(\infty,x')\phi(\infty) + \frac{dg}{dx}(-\infty,x')\phi(-\infty) \\ &\quad = \int_{-\infty}^{+\infty} g(x,x')U(x)\phi(x)dx. \end{aligned} \tag{2.2.23}$$

We require that Green's function satisfies

$$\left(\frac{d^2}{dx^2}+k^2\right)g(x,x')=\delta(x-x'). \tag{2.2.24}$$

Then Eq. (2.2.23) becomes

$$\begin{aligned}&\phi(x')+g(\infty,x')Tike^{ikx}-g(-\infty,x')\left[ike^{ikx}-Rike^{-ikx}\right]\\&-\frac{dg}{dx}(\infty,x')Te^{ikx}+\frac{dg}{dx}(-\infty,x')\left[e^{ikx}+Re^{-ikx}\right]\\&=\int_{-\infty}^{+\infty}g(x,x')U(x)\phi(x)dx.\end{aligned} \tag{2.2.25}$$

We require that terms involving the unknowns T and R vanish in Eq. (2.2.25), i.e.,

$$\begin{cases}\frac{dg}{dx}(\infty,x')=ikg(\infty,x'),\\ \frac{dg}{dx}(-\infty,x')=-ikg(-\infty,x').\end{cases} \tag{2.2.26}$$

These conditions, (2.2.26), are the appropriate boundary conditions for $g(x,x')$. Hence we obtain

$$\phi(x')=\left[ikg(-\infty,x')-\frac{dg}{dx}(-\infty,x')\right]e^{ikx}+\int_{-\infty}^{+\infty}g(x,x')U(x)\phi(x)dx. \tag{2.2.27}$$

Solution for g(x, x′). the governing equation for $g(x,x')$ is

$$\left(\frac{d^2}{dx^2}+k^2\right)g(x,x')=\delta(x-x'),$$

and the boundary conditions are Eq. (2.2.26). The solution to this problem is found to be

$$g(x,x')=\begin{cases}A'e^{ikx} & \text{for}\quad x>x',\\ B'e^{-ikx} & \text{for}\quad x<x'.\end{cases}$$

At $x=x'$, there exists a discontinuity in the first derivative $\frac{dg}{dx}$ of g with respect to x,

$$\frac{dg}{dx}(x'_+,x')-\frac{dg}{dx}(x'_-,x')=1,\quad g(x'_+,x')=g(x'_-,x').$$

From these two conditions, A' and B' are determined to be $A'=e^{-ikx'}/2ik$ and $B'=e^{ikx'}/2ik$. Thus Green's function $g(x,x')$ for this problem is given by

$$g(x,x')=\frac{1}{2ik}e^{ik|x-x'|}.$$

Now, the first term on the right-hand side of Eq. (2.2.27) assumes the following form:

$$ikg(-\infty, x') - \frac{dg}{dx}(-\infty, x') = 2ikB'e^{-ikx} = e^{ik(x'-x)}.$$

Hence Eq. (2.2.27) becomes

$$\phi(x') = e^{ikx'} + \int_{-\infty}^{+\infty} (e^{ik|x-x'|}/2ik)\, U(x)\phi(x)dx. \tag{2.2.28}$$

Changing x to ξ and x' to x in Eq. (2.2.28), we have

$$\phi(x) = e^{ikx} + \int_{-\infty}^{+\infty} (e^{ik|\xi-x|}/2ik)\, U(\xi)\phi(\xi)d\xi. \tag{2.2.29}$$

This is the Fredholm Integral Equation of the second kind.

Reflection: As $x \to -\infty$, $|\xi - x| = \xi - x$ so that

$$\phi(x) \to e^{ikx} + e^{-ikx} \int_{-\infty}^{+\infty} (e^{ik\xi}/2ik)\, U(\xi)\phi(\xi)d\xi.$$

From this, the *reflection coefficient* R is found:

$$R = \int_{-\infty}^{+\infty} (e^{ik\xi}/2ik)\, U(\xi)\phi(\xi)d\xi.$$

Transmission: As $x \to +\infty$, $|\xi - x| = x - \xi$ so that

$$\phi(x) \to e^{ikx}\left[1 + \int_{-\infty}^{+\infty} (e^{-ik\xi}/2ik)\, U(\xi)\phi(\xi)d\xi\right].$$

From this, the *transmission coefficient* T is found:

$$T = 1 + \int_{-\infty}^{+\infty} (e^{-ik\xi}/2ik)\, U(\xi)\phi(\xi)d\xi.$$

These R and T are still unknowns since $\phi(\xi)$ is not known, but for $|U(\xi)| \ll 1$ (weak potential), we can approximate $\phi(x)$ by e^{ikx}. Then the approximate equations for R and T are given by

$$R \simeq \int_{-\infty}^{+\infty} (e^{2ik\xi}/2ik)\, U(\xi)d\xi \quad \text{and} \quad T \simeq 1 + \int_{-\infty}^{+\infty} (1/2ik)\, U(\xi)d\xi.$$

Also, by approximating $\phi(\xi)$ by $e^{ik\xi}$ in the integrand of Eq. (2.2.29) on the right-hand side, we have, as the first approximation,

$$\phi(x) \simeq e^{ikx} + \int_{-\infty}^{+\infty} (e^{ik|\xi-x|}/2ik)\, U(\xi)e^{ik\xi}\, d\xi.$$

By continuing to iterate, we can generate the *Born series* for $\phi(x)$.

2.3 Sturm–Liouville System

Consider the linear differential operator

$$L = (1/r(x)) \left[\frac{d}{dx} \left(p(x) \frac{d}{dx} \right) - q(x) \right], \tag{2.3.1}$$

where

$$r(x), p(x) > 0 \quad \text{on} \quad 0 < x < 1, \tag{2.3.2}$$

together with the inner product defined with $r(x)$ as the weight,

$$(f, g) = \int_0^1 f(x)g(x) \cdot r(x)dx. \tag{2.3.3}$$

Examine the inner product (g, Lf) by integral by parts twice to obtain

$$\begin{aligned}(g, Lf) &= \int_0^1 dx \cdot r(x) \cdot g(x)(1/r(x)) \left[\frac{d}{dx} \left(p(x) \frac{df(x)}{dx} \right) - q(x)f(x) \right] \\ &= p(1) \left[f'(1)g(1) - f(1)g'(1) \right] - p(0) \left[f'(0)g(0) - f(0)g'(0) \right] \\ &\quad + (Lg, f).\end{aligned} \tag{2.3.4}$$

Suppose that the boundary conditions on $f(x)$ are

$$f(0) = 0 \quad \text{and} \quad f(1) = 0, \tag{2.3.5}$$

and the adjoint boundary conditions on $g(x)$ are

$$g(0) = 0 \quad \text{and} \quad g(1) = 0. \tag{2.3.6}$$

(Many other boundary conditions of the type

$$\alpha f(0) + \beta f'(0) = 0 \tag{2.3.7}$$

also work.) Then the boundary terms in Eq. (2.3.4) disappear and we have

$$(g, Lf) = (Lg, f), \tag{2.3.8}$$

i.e., L is *self-adjoint* with the given weighted inner product.

Now examine the eigenvalue problem. The basic equation and boundary conditions are given by

$$L\phi(x) = \lambda\phi(x), \quad \text{with} \quad \phi(0) = 0 \quad \text{and} \quad \phi(1) = 0,$$

i.e.,

$$\frac{d}{dx}\left[p(x)\frac{d}{dx}\phi(x)\right] - q(x)\phi(x) = \lambda r(x)\phi(x), \tag{2.3.9}$$

with the boundary conditions

$$\phi(0) = 0 \quad \text{and} \quad \phi(1) = 0. \tag{2.3.10}$$

Suppose that $\lambda = 0$ is not an eigenvalue (i.e., the homogeneous problem has no nontrivial solutions) so that Green's function exists. (Otherwise we have to define the modified Green's function.) Suppose that the second-order ordinary differential equation

$$\frac{d}{dx}\left[p(x)\frac{d}{dx}y(x)\right] - q(x)y(x) = 0 \tag{2.3.11}$$

has two independent solutions $y_1(x)$ and $y_2(x)$ such that

$$y_1(0) = 0 \quad \text{and} \quad y_2(1) = 0. \tag{2.3.12}$$

In order for $\lambda = 0$ not to be an eigenvalue, we must make sure that the only C_1 and C_2 for which $C_1 y_1(0) + C_2 y_2(0) = 0$ and $C_1 y_1(1) + C_2 y_2(1) = 0$ are not nontrivial. This requires

$$y_1(1) \neq 0 \quad \text{and} \quad y_2(0) \neq 0. \tag{2.3.13}$$

Now, to find Green's function, multiply the eigenvalue equation (2.3.9) by $G(x, x')$ and integrate from 0 to 1. Using the boundary conditions that

$$G(0, x') = 0 \quad \text{and} \quad G(1, x') = 0, \tag{2.3.14}$$

we obtain, after integrating by parts twice,

$$\int_0^1 \phi(x)\left[\frac{d}{dx}\left(p(x)\frac{dG(x, x')}{dx}\right) - q(x)G(x, x')\right]dx = \lambda \int_0^1 G(x, x')r(x)\phi(x)dx. \tag{2.3.15}$$

Requiring that Green's function $G(x, x')$ satisfies

$$\frac{d}{dx}\left(p(x)\frac{dG(x, x')}{dx}\right) - q(x)G(x, x') = \delta(x - x') \tag{2.3.16}$$

with the boundary conditions (2.3.14), we arrive at the following equation:

$$\phi(x') = \lambda \int_0^1 G(x, x')r(x)\phi(x)dx. \tag{2.3.17}$$

This is a homogeneous Fredholm integral equation of the second kind once $G(x, x')$ is known.

Solution for G(x, x′). Recalling Eqs. (2.3.12), (2.3.13), and (2.3.14), we have

$$G(x, x') = \begin{cases} Ay_1(x) + By_2(x) & \text{for} \quad x < x', \\ Cy_1(x) + Dy_2(x) & \text{for} \quad x > x'. \end{cases}$$

From the boundary conditions (2.3.14) of $G(x, x')$, and (2.3.12) and (2.3.13) of $y_1(x)$ and $y_2(x)$, we immediately have

$$B = 0 \quad \text{and} \quad C = 0.$$

Thus we have

$$G(x, x') = \begin{cases} Ay_1\,(x) & \text{for} \quad x < x', \\ Dy_2\,(x) & \text{for} \quad x > x'. \end{cases}$$

In order to determine A and D, integrate Eq. (2.3.16) across x' with respect to x, and make use of the continuity of $G(x, x')$ at $x = x'$, which results in

$$\begin{aligned} &p(x')\left[\frac{dG}{dx}(x'_+, x') - \frac{dG}{dx}(x'_-, x')\right] = 1, \\ &G(x'_+, x') = G(x'_-, x'), \end{aligned}$$

or

$$\begin{aligned} &Ay_1(x') = Dy_2(x'), \\ &Dy'_2(x') = Ay'_1(x')p(x'). \end{aligned}$$

Noting that

$$W(y_1(x), y_2(x)) \equiv y_1(x)y'_2(x) - y_2(x)y'_1(x) \tag{2.3.18}$$

is the *Wronskian* of the differential equation (2.3.11), we obtain A and D as

$$\begin{cases} A = y_2(x')/[p(x')W(y_1(x'), y_2(x'))], \\ D = y_1(x')/[p(x')W(y_1(x'), y_2(x')]. \end{cases}$$

Now, it can be easily proven that

$$p(x)W(y_1(x), y_2(x)) = \text{constant}, \tag{2.3.19}$$

for the differential equation (2.3.11). Denoting this constant by

$$p(x)W(y_1(x), y_2(x)) = C',$$

we simplify A and D as

$$\begin{cases} A = y_2(x')/C', \\ D = y_1(x')/C'. \end{cases}$$

Thus Green's function $G(x, x')$ for the Sturm–Liouville system is given by

$$G(x, x') = \begin{cases} y_1(x)\, y_2(x')/C' & \text{for} \quad x < x', \\ y_1(x')\, y_2(x)/C' & \text{for} \quad x > x', \end{cases} \tag{2.3.20a}$$

$$= y_1(x_<)y_2(x_>)/C' \quad \text{for} \quad \begin{cases} x_< = ((x + x')/2) - |x - x'|/2, \\ x_> = ((x + x')/2) + |x - x'|/2. \end{cases} \tag{2.3.20b}$$

The Sturm–Liouville eigenvalue problem is equivalent to the homogeneous Fredholm integral equation of the second kind,

$$\phi(x) = \lambda \int_0^1 G(\xi, x) r(\xi)\phi(\xi) d\xi. \tag{2.3.21}$$

We remark that the Sturm–Liouville eigenvalue problem turns out to have a complete set of eigenfunctions in the space $\mathbb{L}_2(0, 1)$ as long as $p(x)$ and $r(x)$ are analytic and positive on $(0, 1)$.

The kernel of Eq. (2.3.21) is

$$K(\xi, x) = r(\xi)G(\xi, x).$$

This kernel can be symmetrized by defining

$$\psi(x) = \sqrt{r(x)}\phi(x).$$

Then the integral equation (2.3.21) becomes

$$\psi(x) = \lambda \int_0^1 \sqrt{r(\xi)}G(\xi, x)\sqrt{r(x)}\psi(\xi) d\xi. \tag{2.3.22}$$

Now, the kernel of Eq. (2.3.22),

$$\sqrt{r(\xi)}G(\xi, x)\sqrt{r(x)},$$

is symmetric since $G(\xi, x)$ is symmetric.

Symmetry of Green's function, called *reciprocity*, is true in general for any *self-adjoint operator*. The proof of this fact is as follows: consider

$$L_x G(x, x') = \delta(x - x'), \tag{2.3.23}$$

$$L_x G(x, x'') = \delta(x - x''). \tag{2.3.24}$$

Take the inner product of Eq. (2.3.23) with $G(x, x'')$ from the left and Eq. (2.3.24) with $G(x, x')$ from the right.

$$(G(x, x''), L_x G(x, x')) = (G(x, x''), \delta(x - x')),$$

$$(L_x G(x, x''), G(x, x')) = (\delta(x - x''), G(x, x')).$$

Since L_x is assumed to be self-adjoint, subtracting the above two equations results in

$$G^*(x', x'') = G(x'', x'). \tag{2.3.25}$$

Then G is Hermitian. If G is real, we have

$$G(x', x'') = G(x'', x'),$$

i.e., $G(x', x'')$ is symmetric.

2.4 Green's Function for Time-Dependent Scattering Problem

The time-dependent Schrödinger equation assumes the following form after setting $\hbar = 1$ and $2m = 1$:

$$\left(i\frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2}\right)\psi(x, t) = V(x, t)\psi(x, t). \tag{2.4.1}$$

Assume

$$\begin{cases} \lim_{|t|\to\infty} V(x, t) = 0, \\ \lim_{t\to-\infty} \exp\,[i\omega_0 t]\psi(x, t) = \exp\,[ik_0 x], \end{cases} \tag{2.4.2}$$

from which we find

$$\omega_0 = k_0^2. \tag{2.4.3}$$

Define Green's function $G(x, t; x', t')$ by requiring

$$\psi(x, t) = \exp[i(k_0 x - k_0^2 t)] + \int_{-\infty}^{+\infty} dt' \int_{-\infty}^{+\infty} dx' G(x, t; x', t') V(x', t')\psi(x', t'). \tag{2.4.4}$$

In order to satisfy partial differential equation (2.4.1), we require

$$\left(i\frac{\partial}{\partial t}+\frac{\partial^2}{\partial x^2}\right)G(x,t;x',t')=\delta(t-t')\delta(x-x'). \tag{2.4.5}$$

We also require that

$$G(x,t;x',t')=0 \quad \text{for} \quad t<t'. \tag{2.4.6}$$

Note that the initial condition at $t=-\infty$ is satisfied as well as *Causality*. Note also that the set of equations could be obtained by the methods we were employing in the previous two examples. To solve the above equations, Eqs. (2.4.5) and (2.4.6), we Fourier transform in time and space, i.e., we write

$$\begin{cases} \tilde{G}(k,\omega;x',t')=\displaystyle\int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}dte^{-ikx}e^{-i\omega t}G(x,t;x',t'), \\ G(x,t;x',t')=\displaystyle\int_{-\infty}^{+\infty}\frac{dk}{2\pi}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}e^{+ikx}e^{+i\omega t}\tilde{G}(k,\omega;x',t'). \end{cases} \tag{2.4.7}$$

Taking the Fourier transform of the original equation (2.4.5), we find

$$G(x,t;x',t')=\int_{-\infty}^{+\infty}\frac{dk}{2\pi}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\left(\frac{-1}{\omega+k^2}\right)e^{ik(x-x')}e^{i\omega(t-t')}. \tag{2.4.8}$$

Where do we use the condition that $G(x,t;x',t')=0 \quad \text{for} \quad t<t'$? Consider the ω integration in the complex ω plane as in Figure 2.1,

$$\int_{-\infty}^{+\infty}d\omega\frac{1}{\omega+k^2}e^{i\omega(t-t')}. \tag{2.4.9}$$

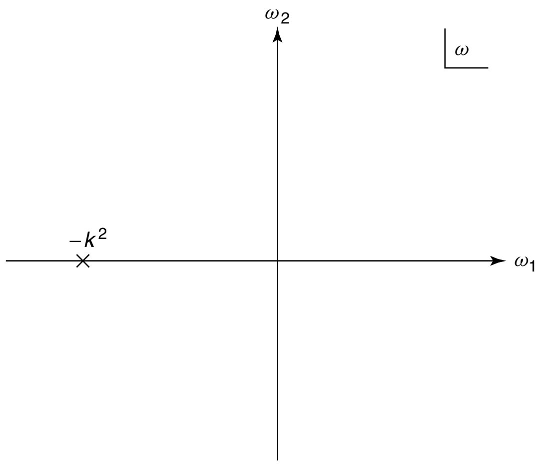

Fig. 2.1 The location of the singularity of the integrand of Eq. (2.4.9) in the complex ω plane.

We find that there is a singularity right on the path of integration at $\omega = -k^2$. We either have to go above or below it. Upon writing ω as $\omega = \omega_1 + i\omega_2$, we have the following bound:

$$\left|e^{i\omega(t-t')}\right| = \left|e^{i\omega_1(t-t')}\right| \left|e^{-\omega_2(t-t')}\right| = e^{-\omega_2(t-t')}. \tag{2.4.10}$$

For $t < t'$, we close the contour in the lower half plane to do the contour integral. Since we want G to be zero in this case, we want no singularities inside the contour in that case. This prompts us to take the contour to be as in Figure 2.2. For $t > t'$, when we close the contour in the upper half plane, we get the contribution from the pole at $\omega = -k^2$.

The result of calculation is given by

$$\int_{-\infty}^{+\infty} d\omega \frac{1}{\omega + k^2} e^{i\omega(t-t')} = \begin{cases} 2\pi i \cdot e^{ik(x-x')-ik^2(t-t')}, & t > t', \\ 0, & t < t'. \end{cases} \tag{2.4.11}$$

We remark that the idea of the deformation of the contour to satisfy causality is often expressed by taking the singularity to be at $-k^2 + i\varepsilon$ $(\varepsilon > 0)$ as in Figure 2.3, whence we replace the denominator $\omega + k^2$ with $\omega + k^2 - i\varepsilon$,

$$G(x, t; x', t') = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \left(\frac{-1}{\omega + k^2 - i\varepsilon}\right) e^{ik(x-x')+i\omega(t-t')}. \tag{2.4.12}$$

This shifts the singularity above the real axis and is equivalent, as $\varepsilon \to 0^+$, to our previous solution. After the ω integration in the complex ω plane is performed, the k integral can be done by completing the square in the exponent of Eq. (2.4.12), but the resulting Gaussian integration is a bit more complicated than the diffusion equation.

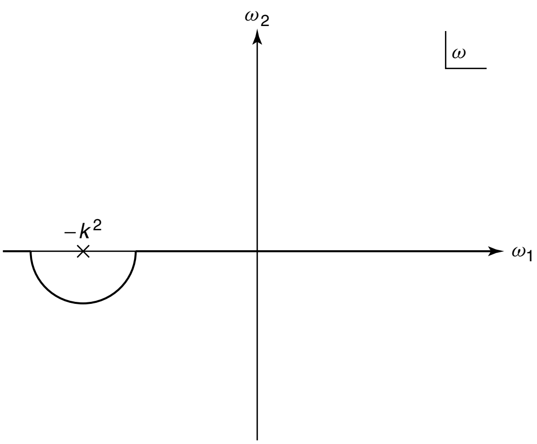

Fig. 2.2 The contour of the complex ω integration of Eq. (2.4.9).

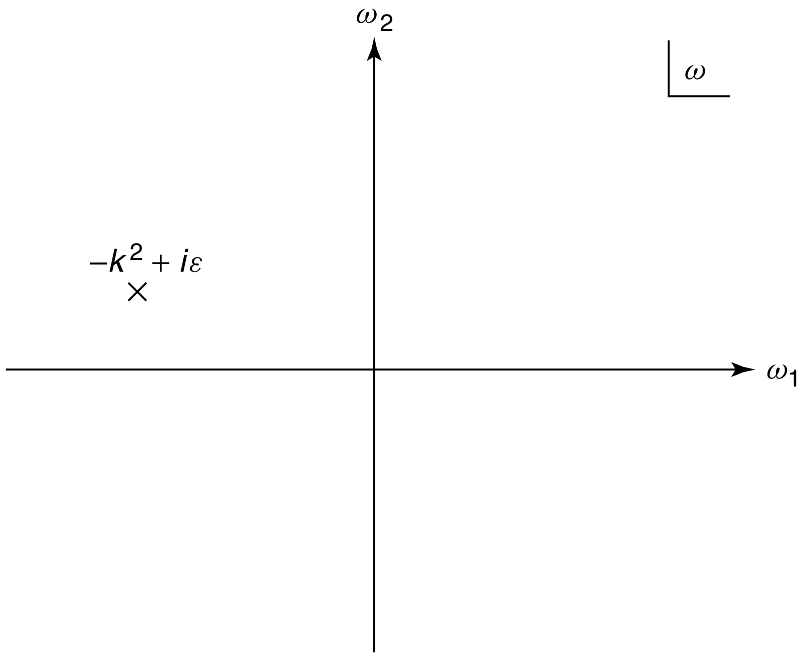

Fig. 2.3 The singularity of the integrand of Eq. (2.4.9) at $\omega = -k^2$ gets shifted to $\omega = -k^2 + i\varepsilon$ $(\varepsilon > 0)$ in the upper half plane of the complex ω plane.

The result is given by

$$G(x, t; x', t') = \begin{cases} \sqrt{\frac{i}{4\pi(t-t')}} e^{i(x-x')^2/4(t-t')} & \text{for} \quad t > t', \\ 0 & \text{for} \quad t < t', \end{cases} \tag{2.4.13}$$

where $\hbar = 1$ and $2m = 1$. In the case of the diffusion equation,

$$\left(-\frac{1}{D}\frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2} \right) \psi(x, t) = 0, \tag{2.4.14}$$

Eq. (2.4.13) reduces to Green's function for the diffusion equation,

$$G(x, t; x', t') = \begin{cases} \sqrt{\frac{1}{4\pi\kappa(t-t')}} e^{-(x-x')^2/4D(t-t')} & \text{for} \quad t > t', \\ 0 & \text{for} \quad t < t', \end{cases} \tag{2.4.15}$$

which satisfies the following equation:

$$\left(-\frac{1}{D}\frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2} \right) G(x, t; x', t') = \delta(t - t')\delta(x - x'), \tag{2.4.16}$$

$$G(x, t; x', t') = 0 \quad \text{for} \quad t < t',$$

where the diffusion constant D is given by

$$D = \frac{K}{C\rho} = \frac{\text{(thermal conductivity)}}{\text{(specific heat)} \times \text{(density)}}.$$

These two expressions, Eqs. (2.4.13) and (2.4.15), are related by the analytic continuation, $t \to -it$. The diffusion constant D plays the role of the inverse of the Planck constant $\hbar$.

We shall devote the next section for the more formal discussion of the scattering problem.

2.5
Lippmann–Schwinger Equation

In the nonrelativistic scattering problem of quantum mechanics, we have the *macroscopic causality* of Stueckelberg: when we regard the potential $V(t, \mathbf{r})$ as a function of t, we have no scattered wave, $\psi_{\text{scatt}}(t, \mathbf{r}) = 0$, for $t < T$, if $V(t, \mathbf{r}) = 0$ for $t < T$. We employ the *adiabatic switching hypothesis*: we can take the limit $T \to -\infty$ after the computation of the scattered wave, $\psi_{\text{scatt}}(t, \mathbf{r})$. We derive the Lippmann–Schwinger equation, and prove the orthonormality of the outgoing wave and the incoming wave and the unitarity of the S matrix. We then discuss optical theorem and asymptotic wavefunctions. Lastly, we discuss the rearrangement collision and the final state interaction to get in touch with Born approximation.

Lippmann–Schwinger Equation: We shall begin with the time-dependent Schrödinger equation with the time-dependent potential $V(t, \mathbf{r})$,

$$i\hbar\frac{\partial}{\partial t}\psi(t, \mathbf{r}) = [H_0 + V(t, \mathbf{r})]\psi(t, \mathbf{r}).$$

In order to use the macroscopic causality, we assume

$$V(t, \mathbf{r}) = \begin{cases} V(\mathbf{r}) & \text{for} \quad t \geq T, \\ 0 & \text{for} \quad t < T. \end{cases}$$

For $t < T$, the particle obeys the free equation,

$$i\hbar\frac{\partial}{\partial t}\psi(t, \mathbf{r}) = H_0\psi(t, \mathbf{r}). \tag{2.5.1}$$

We write the solution of Eq. (2.5.1) as $\psi_{\text{inc}}(t, \mathbf{r})$. The wavefunction for the general time t is written as

$$\psi(t, \mathbf{r}) = \psi_{\text{inc}}(t, \mathbf{r}) + \psi_{\text{scatt}}(t, \mathbf{r}),$$

where we have

$$\left(i\hbar\frac{\partial}{\partial t} - H_0\right)\psi_{\text{scatt}}(t,\mathbf{r}) = V(t,\mathbf{r})\psi(t,\mathbf{r}). \tag{2.5.2}$$

We introduce the retarded Green's function for Eq. (2.5.2) as

$$\begin{cases} (i\hbar(\partial/\partial t) - H_0)K_{\text{ret}}(t,\mathbf{r};t',\mathbf{r}') = \delta(t-t')\delta^3(\mathbf{r}-\mathbf{r}'), \\ K_{\text{ret}}(t,\mathbf{r};t',\mathbf{r}') = 0 \quad \text{for} \quad t < t'. \end{cases} \tag{2.5.3}$$

Formal solution to Eq. (2.5.2) is given by

$$\psi_{\text{scatt}}(t,\mathbf{r}) = \int_{-\infty}^{\infty}\int K_{\text{ret}}(t,\mathbf{r};t',\mathbf{r}')V(t',\mathbf{r}')\psi(t',\mathbf{r}')dt'd\mathbf{r}'. \tag{2.5.4}$$

We note that the integrand of Eq. (2.5.4) survives only for $t \geq t' \geq T$. We now take the limit $T \to -\infty$, thus losing the t-dependence of $V(t,\mathbf{r})$,

$$\psi_{\text{scatt}}(t,\mathbf{r}) = \int_{-\infty}^{\infty}\int K_{\text{ret}}(t,\mathbf{r};t',\mathbf{r}')V(\mathbf{r}')\psi(t',\mathbf{r}')dt'd\mathbf{r}'.$$

When H_0 has no explicit space–time dependence, we have from the translation invariance that

$$K_{\text{ret}}(t,\mathbf{r};t',\mathbf{r}') = K_{\text{ret}}(t-t';\mathbf{r}-\mathbf{r}').$$

Adding $\psi_{\text{inc}}(t,\mathbf{r})$ to $\psi_{\text{scatt}}(t,\mathbf{r})$, Eq. (2.5.4), we have

$$\begin{aligned}\psi(t,\mathbf{r}) &= \psi_{\text{inc}}(t,\mathbf{r}) + \psi_{\text{scatt}}(t,\mathbf{r}) \\ &= \psi_{\text{inc}}(t,\mathbf{r}) + \int_{-\infty}^{\infty}\int K_{\text{ret}}(t-t';\mathbf{r}-\mathbf{r}')V(\mathbf{r}')\psi(t',\mathbf{r}')dt'd\mathbf{r}'.\end{aligned} \tag{2.5.5}$$

This equation is the integral equation determining the total wavefunction, given the incident wave. We rewrite this equation in a time-independent form. For this purpose, we set

$$\begin{aligned}\psi_{\text{inc}}(t,\mathbf{r}) &= \exp[-iEt/\hbar]\psi_{\text{inc}}(\mathbf{r}), \\ \psi(t,\mathbf{r}) &= \exp[-iEt/\hbar]\psi(\mathbf{r}).\end{aligned}$$

Then, from Eq. (2.5.5), we obtain

$$\psi(\mathbf{r}) = \psi_{\text{inc}}(\mathbf{r}) + \int G(\mathbf{r}-\mathbf{r}';E)V(\mathbf{r}')\psi(\mathbf{r}')d\mathbf{r}'. \tag{2.5.6}$$

Here $G(\mathbf{r}-\mathbf{r}';E)$ is given by

$$G(\mathbf{r}-\mathbf{r}';E) = \int_{-\infty}^{\infty} dt' \exp\left[\frac{iE(t-t')}{\hbar}\right]K_{\text{ret}}(t-t';\mathbf{r}-\mathbf{r}'). \tag{2.5.7}$$

Setting

$$K_{\text{ret}}(t-t^{'};\mathbf{r}-\mathbf{r}^{'}) = \int \frac{dEd^3p}{(2\pi)^4}\exp[\{i\mathbf{p}(\mathbf{r}-\mathbf{r}^{'}) - iE(t-t^{'})\}/\hbar]K(E,\mathbf{p}),$$

$$\delta(t-t^{'})\delta^3(\mathbf{r}-\mathbf{r}^{'}) = \int \frac{dEd^3p}{(2\pi)^4}\exp[\{i\mathbf{p}(\mathbf{r}-\mathbf{r}^{'}) - iE(t-t^{'})\}/\hbar],$$

substituting into Eq. (2.5.3), and writing $H_0 = \mathbf{p}^2/2m$, we obtain

$$\left(E - \frac{\mathbf{p}^2}{2m}\right)K(E,\mathbf{p}) = 1.$$

The solution consistent with the *retarded boundary condition* is

$$K(E,\mathbf{p}) = \frac{1}{E - (\mathbf{p}^2/2m) + i\varepsilon}, \quad \text{with } \varepsilon \text{ positive infinitesimal.}$$

Namely,

$$K_{\text{ret}}(t-t^{'};\mathbf{r}-\mathbf{r}^{'}) = \frac{1}{(2\pi)^4}\int dEd^3p\frac{\exp[\{i\mathbf{p}(\mathbf{r}-\mathbf{r}^{'}) - iE(t-t^{'})\}/\hbar]}{E-(\mathbf{p}^2/2m)+i\varepsilon}.$$

Substituting this into Eq. (2.5.7) and setting $E = (\hbar\mathbf{k})^2/2m$, we obtain

$$G(\mathbf{r}-\mathbf{r}^{'};E) = \frac{1}{(2\pi)^3}\int d^3p\frac{\exp[i\mathbf{p}(\mathbf{r}-\mathbf{r}^{'})/\hbar]}{E-(\mathbf{p}^2/2m)+i\varepsilon} = -\frac{m}{2\pi}\frac{\exp[ik\left|\mathbf{r}-\mathbf{r}^{'}\right|]}{\left|\mathbf{r}-\mathbf{r}^{'}\right|}.$$

In Eq. (2.5.6), since the Fourier transform of $G(\mathbf{r}-\mathbf{r}^{'};E)$ is written as

$$\frac{1}{E-H_0+i\varepsilon},$$

Eq. (2.5.6) can be written formally as

$$\Psi = \Phi + \frac{1}{E-H_0+i\varepsilon}V\Psi, \quad E > 0, \tag{2.5.8}$$

where we wrote $\Psi = \psi(\mathbf{r})$, $\Phi = \psi_{\text{inc}}(\mathbf{r})$, and the incident wave Φ satisfies the free particle equation,

$$(E-H_0)\Phi = 0.$$

Operating $(E-H_0)$ on Eq. (2.5.8) from the left, we obtain the Schrödinger equation,

$$(E-H_0)\Psi = V\Psi. \tag{2.5.9}$$

We shall note that Eq. (2.5.9) is the *differential equation* whereas Eq. (2.5.8) is the *integral equation* which embodies the boundary condition.

For the bound state problem ($E < 0$), since the operator $(E - H_0)$ is negative definite and has the unique inverse, we have

$$\Psi = \frac{1}{E - H_0} V\Psi, \quad E < 0. \tag{2.5.10}$$

We call Eqs. (2.5.8) and (2.5.10) as the Lippmann–Schwinger equation (the L–S equation in short). $+i\varepsilon$ in the denominator of Eq. (2.5.8) makes the scattered wave the outgoing spherical wave. The presence of $+i\varepsilon$ in Eq. (2.5.8) enforces the *outgoing wave condition*. It is convenient mathematically to introduce also $-i\varepsilon$ into Eq. (2.5.8), which makes the scattered wave the incoming spherical wave and thus enforces the *incoming wave condition*. We construct two kinds of the wavefunctions:

$$\Psi_a^{(+)} = \Phi_a + \frac{1}{E_a - H_0 + i\varepsilon} V\Psi_a^{(+)}, \quad \text{outgoing wave condition,} \tag{2.5.11.+}$$

$$\Psi_a^{(-)} = \Phi_a + \frac{1}{E_a - H_0 - i\varepsilon} V\Psi_a^{(-)}, \quad \text{incoming wave condition.} \tag{2.5.11.−}$$

The formal solution to the L–S equation was obtained by G. Chew and M. Goldberger. By iteration of Eq. (2.5.11.+), we have

$$\begin{aligned}
\Psi_a^{(+)} &= \Phi_a + \frac{1}{E_a - H_0 + i\varepsilon}\left(1 + V\frac{1}{E_a - H_0 + i\varepsilon} + \cdots\right) V\Phi_a \\
&= \Phi_a + \frac{1}{E_a - H_0 + i\varepsilon}\left(1 - V\frac{1}{E_a - H_0 + i\varepsilon}\right)^{-1} V\Phi_a \\
&= \Phi_a + \frac{1}{E_a - H + i\varepsilon} V\Phi_a.
\end{aligned}$$

Here we used the operator identity $A^{-1}B^{-1} = (BA)^{-1}$ and $H = H_0 + V$ represents the total Hamiltonian.

We write the formal solution for $\Psi_a^{(+)}$ and $\Psi_a^{(-)}$ together:

$$\Psi_a^{(+)} = \Phi_a + \frac{1}{E_a - H + i\varepsilon} V\Phi_a, \tag{2.5.12.+}$$

$$\Psi_a^{(-)} = \Phi_a + \frac{1}{E_a - H - i\varepsilon} V\Phi_a. \tag{2.5.12.−}$$

Orthonormality of $\Psi_a^{(+)}$: We will prove the orthonormality only for $\Psi_a^{(+)}$:

$$\begin{aligned}
(\Psi_b^{(+)}, \Psi_a^{(+)}) &= (\Phi_b, \Psi_a^{(+)}) + \left(\frac{1}{E_b - H + i\varepsilon} V\Phi_b, \Psi_a^{(+)}\right) \\
&= (\Phi_b, \Psi_a^{(+)}) + \left(\Phi_b, V\frac{1}{E_b - H - i\varepsilon} \Psi_a^{(+)}\right) \\
&= (\Phi_b, \Psi_a^{(+)}) + \frac{1}{E_b - E_a - i\varepsilon}(\Phi_b, V\Psi_a^{(+)})
\end{aligned}$$

$$= (\Phi_b, \Phi_a) + \left(\Phi_b, \frac{1}{E_a - H_0 + i\varepsilon} V\Psi_a^{(+)}\right)$$
$$+ \frac{1}{E_b - E_a - i\varepsilon}(\Phi_b, V\Psi_a^{(+)})$$
$$= \delta_{ba} + \left(\frac{1}{E_a - E_b + i\varepsilon} + \frac{1}{E_b - E_a - i\varepsilon}\right)(\Phi_b, V\Psi_a^{(+)})$$
$$= \delta_{ba}. \tag{2.5.13}$$

Thus $\Psi_a^{(+)}$ forms a complete and orthonormal basis. The same proof goes through for $\Psi_a^{(-)}$ also. Frequently, the orthonormality of $\Psi_a^{(\pm)}$ is assumed on the outset. We have proven the orthonormality of $\Psi_a^{(\pm)}$ using the L–S equation and the formal solution due to G. Chew and M. Goldberger.

In passing, we state that, in relativistic quantum field theory in the L.S.Z. formalism, the outgoing wave $\Psi_a^{(+)}$ is called the *in-state* and is written as $\Psi_a^{(\text{in})}$, and the incoming wave $\Psi_a^{(-)}$ is called the *out-state* and is written as $\Psi_a^{(\text{out})}$.

Unitarity of the *S* matrix: We define the S matrix by

$$S_{ba} = (\Psi_b^{(-)}, \Psi_a^{(+)}) = (\Psi_b^{(\text{out})}, \Psi_a^{(\text{in})}). \tag{2.5.14}$$

This definition states that the S matrix transforms one complete set to the other complete set. By making use of the formal solution of G. Chew and M. Goldberger first and then using the L–S equation as before, we obtain

$$S_{ba} = \delta_{ba} + \left(\frac{1}{E_a - E_b + i\varepsilon} + \frac{1}{E_b - E_a + i\varepsilon}\right)(\Phi_b, V\Psi_a^{(+)}) \tag{2.5.1}$$
$$= \delta_{ba} - 2\pi i\delta(E_b - E_a)(\Phi_b, V\Psi_a^{(+)}). \tag{2.5.15}$$

We define the T matrix by

$$T_{ba} = (\Phi_b, V\Psi_a^{(+)}). \tag{2.5.16}$$

Then we have

$$S_{ba} = \delta_{ba} - 2\pi i\delta(E_b - E_a)T_{ba}. \tag{2.5.17}$$

If the S matrix is unitary, it satisfies

$$\hat{S}^\dagger \hat{S} = \hat{S}\hat{S}^\dagger = 1. \tag{2.5.18}$$

These unitarity conditions are equivalent to the following conditions in terms of the T matrix:

$$T_{ba}^\dagger - T_{ba} = \begin{cases} 2\pi i \sum_n T_{bn}^\dagger \delta(E_b - E_n) T_{na}, \\ 2\pi i \sum_n T_{bn}\delta(E_b - E_n) T_{na}^\dagger, \end{cases} \quad \text{with} \quad E_b = E_a. \tag{2.5.19,20}$$

In order to prove the unitarity of the S matrix, Eq. (2.5.18), it suffices to prove Eqs. (2.5.19) and (2.5.20), which are expressed in terms of the T matrix.

We first note

$$T_{ba}^{\dagger} = T_{ab}^{*} = (\Phi_a, V\Psi_b^{(+)})^{*} = (V\Psi_b^{(+)}, \Phi_a) = (\Psi_b^{(+)}, V\Phi_a).$$

Then

$$T_{ba}^{\dagger} - T_{ba} = (\Psi_b^{(+)}, V\Phi_a) - (\Phi_b, V\Psi_a^{(+)}).$$

Inserting the formal solution of G. Chew and M. Goldberger to $\Psi_a^{(+)}$ and $\Psi_b^{(+)}$ above, we have

$$\begin{aligned} T_{ba}^{\dagger} - T_{ba} &= (\Phi_b, V\Phi_a) + \left(\frac{1}{E_b - H + i\varepsilon} V\Phi_b, V\Phi_a\right) \\ &\quad - (\Phi_b, V\Phi_a) - \left(\Phi_b, V\frac{1}{E_a - H + i\varepsilon} V\Phi_a\right) \\ &= (V\Phi_b, \left(\frac{1}{E_b - H - i\varepsilon} - \frac{1}{E_b - H + i\varepsilon}\right) V\Phi_a) \\ &= (V\Phi_b, 2\pi i\delta(E_b - H)V\Phi_a), \end{aligned} \tag{2.5.21}$$

where, in the one line above the last line of Eq. (2.5.21), we used the fact that $E_b = E_a$. Inserting the complete orthonormal basis, $\Psi^{(-)}$, between the product of the operators in Eq. (2.5.21), we have

$$\begin{aligned} T_{ba}^{\dagger} - T_{ba} &= \sum_n (V\Phi_b, \Psi_n^{(-)}) 2\pi i\delta(E_b - E_n)(\Psi_n^{(-)}, V\Phi_a) \\ &= 2\pi i \sum_n T_{bn}^{\dagger}\delta(E_b - E_n)T_{na}. \end{aligned}$$

This is Eq. (2.5.19).

If we insert the complete orthonormal basis, $\Psi^{(+)}$, between the product of the operators in Eq. (2.5.21), we obtain

$$\begin{aligned} T_{ba}^{\dagger} - T_{ba} &= \sum_n (V\Phi_b, \Psi_n^{(+)}) 2\pi i\delta(E_b - E_n)(\Psi_n^{(+)}, V\Phi_a) \\ &= 2\pi i \sum_n T_{bn}\delta(E_b - E_n)T_{na}^{\dagger}. \end{aligned}$$

This is Eq. (2.5.20). Thus the S matrix defined by Eq. (2.5.14) is unitary. The unitarity of the S matrix is *equivalent* to the fact that the outgoing wave set $\{\Psi_a^{(+)}\}$ and the incoming wave set $\{\Psi_b^{(-)}\}$ form the complete orthonormal basis, respectively.

Actually, the S matrix has to be unitary since the S matrix transforms one complete orthonormal set to another complete orthonormal set.

Optical Theorem: The probability per unit time for the transition $a \to b$ is given by

$$w_{ba} = 2\pi\,\delta(E_b - E_a)\,|T_{ba}|^2\,. \tag{2.5.22}$$

If we sum over the final state b, we have

$$w_a = \sum_b w_{ba} = 2\pi \sum_b \delta(E_b - E_a)\,|T_{ba}|^2\,. \tag{2.5.23}$$

We compare the right-hand side of Eq. (2.5.23) with the special case of the unitarity condition, Eq. (2.5.19),

$$T^{\dagger}_{aa} - T_{aa} = 2\pi i \sum_n T^{\dagger}_{an}\delta(E_a - E_n)T_{na}.$$

We immediately obtain

$$w_a = -2\,\mathrm{Im}\,T_{aa}. \tag{2.5.24}$$

The cross section for $a \to$ (arbitrary state) is given by

$$\sigma_a = \frac{V}{v_{\mathrm{rel}}} w_a, \tag{2.5.25}$$

where V is the normalization volume and v_{rel} is the relative velocity. From Eqs. (2.5.24) and (2.5.25), we obtain

$$\sigma_a = -\frac{2V}{v_{\mathrm{rel}}}\,\mathrm{Im}\,T_{aa}. \tag{2.5.26}$$

This relationship is called the optical theorem and holds true under any circumstance. We note that the *total cross section σ_a includes the inelastic channels as well as the elastic channel.*

Asymptotic form: We shall consider the asymptotic form of the wavefunction $\Psi_a^{(+)}$ in the L–S equation. We construct the asymptotic form of $\Psi_a^{(+)}$ as the resulting expression for the S matrix acting upon the free wavefunction Φ_a,

$$\Psi_a^{(+)} \approx \hat{S}\Phi_a = \Phi_a - 2\pi i\delta(E_a - H_0)V\Psi_a^{(+)}. \tag{2.5.27}$$

We compare Eq. (2.5.27) with the L–S equation for $\Psi_a^{(+)}$,

$$\Psi_a^{(+)} = \Phi_a + \frac{1}{E_a - H_0 + i\varepsilon} V\Psi_a^{(+)}.$$

We find that the asymptotic form of $\Psi_a^{(+)}$ is obtained by the replacement,

$$\frac{1}{E_a - H_0 + i\varepsilon} \to -2\pi i\delta(E_a - H_0).$$

Equivalently, in closed form, we have

$$\Psi_a^{(+)} \approx \Phi_a - 2\pi i\delta(E_a - H_0)[(E_a - H_0)\Psi_a^{(+)}], \tag{2.5.28}$$

where $(E_a - H_0)$ acts on $\Psi_a^{(+)}$ and then $\delta(E_a - H_0)$ acts on $[(E_a - H_0)\Psi_a^{(+)}]$.

We note the following:

$$(E_a - H_0)S\Phi_a = 0. \tag{2.5.29}$$

Namely, the asymptotic form of $\Psi_a^{(+)}$ satisfies the free particle equation. We can say that the initial state and the final state described by the asymptotic form of $\Psi_a^{(\pm)}$ are the free particle state. In relativistic quantum field theory, the initial state and the final state are the interacting system, respectively. Lehmann, Symanzik, and Zimmermann (L.S.Z.) extended the notion of the asymptotic wavefunction to the *asymptotic condition* in relativistic quantum field theory and successfully constructed relativistic quantum field theory in the L.S.Z. formalism axiomatically, with the notion of the in-state and the out-state.

The asymptotic condition, roughly speaking, is equivalent to the adiabatic switching hypothesis of the interaction.

Rearrangement collision: So far, all the computations are formal. The L–S equation shows its utmost power for the rearrangement collision where the splitting of H into H_0 and V is *not unique*. The typical process is

$$\mathrm{n} + \mathrm{d} \to \mathrm{n} + \mathrm{n}' + \mathrm{p}. \tag{2.5.30}$$

In this case, the total Hamiltonian is

$$H = T_\mathrm{p} + T_\mathrm{n} + T_{\mathrm{n}'} + V_{\mathrm{np}} + V_{\mathrm{n'p}} + V_{\mathrm{nn'}}. \tag{2.5.31}$$

Here T's represent the kinetic energies and V's the two-body potentials.

The decomposition of the total Hamiltonian H in the initial state is

$$\begin{cases} H_0^{\text{initial}} = T_\mathrm{p} + T_\mathrm{n} + T_{\mathrm{n}'} + V_{\mathrm{n'p}}, \\ V^{\text{initial}} = V_{\mathrm{np}} + V_{\mathrm{nn'}}. \end{cases}$$

Here we included $V_{\mathrm{n'p}}$ in H_0 in order to form the deuteron in the initial state. The decomposition of the total Hamiltonian H in the final state is

$$\begin{cases} H_0^{\text{final}} = T_\mathrm{p} + T_\mathrm{n} + T_{\mathrm{n}'}, \\ V^{\text{final}} = V_{\mathrm{np}} + V_{\mathrm{nn'}} + V_{\mathrm{n'p}}. \end{cases}$$

In order to discuss the reaction wherein the decomposition of the total Hamiltonian is different in the initial and final states, we introduce the two decompositions

$$H = \underbrace{H_a + V_a}_{\text{initial state}} = \underbrace{H_b + V_b}_{\text{final state}}, \tag{2.5.32}$$

where H_a and H_b are the free Hamiltonians in the initial and final states, respectively. We introduce the free wavefunctions , Φ_a and Φ_b, such that

$$(H_a - E_a)\Phi_a = (H_b - E_b)\Phi_b = 0. \tag{2.5.33}$$

In order that the reaction $a \to b$ is possible, from the energy conservation, we must have $E_a = E_b$.

We begin with the general collision problem starting from the initial state wavefunction, Φ_a, and construct the asymptotic form of the final state wavefunction. The formal solution to the L–S equation which satisfies the outgoing wave condition, starting from Φ_a, is

$$\Psi_a^{(+)} = \Phi_a + \frac{1}{E_a - H + i\varepsilon} V_a \Phi_a. \tag{2.5.34}$$

In order to construct the asymptotic wavefunction corresponding to the final state, we use the operator identity

$$\frac{1}{A} - \frac{1}{B} = \frac{1}{B}(B - A)\frac{1}{A}.$$

Then we have

$$\frac{1}{E_a - H + i\varepsilon} - \frac{1}{E_a - H_b + i\varepsilon} = \frac{1}{E_a - H_b + i\varepsilon} V_b \frac{1}{E_a - H + i\varepsilon}. \tag{2.5.35}$$

Substituting Eq. (2.5.35) into (2.5.34), we have

$$\Psi_a^{(+)} = \Phi_a + \frac{1}{E_a - H_b + i\varepsilon}\left(1 + V_b \frac{1}{E_a - H + i\varepsilon}\right) V_a \Phi_a. \tag{2.5.36}$$

In the second term of Eq. (2.5.36), we now make the replacement for the asymptotic wavefunction,

$$\frac{1}{E_a - H_b + i\varepsilon} \to -2\pi i \delta(E_a - H_b) \quad \text{with} \quad E_a = E_b. \tag{2.5.37}$$

The transition matrix element T_{ba} is now expressed as

$$\begin{aligned} T_{ba} &= \left(\Phi_b, \left(1 + V_b \frac{1}{E_a - H + i\varepsilon}\right) V_a \Phi_a\right) \\ &= \left(\left(1 + \frac{1}{E_b - H - i\varepsilon} V_b\right) \Phi_b, V_a \Phi_a\right) \\ &= (\Psi_b^{(-)}, V_a \Phi_a). \end{aligned} \tag{2.5.38}$$

We have two ways of expressing T_{ba}. Namely,

$$T_{ba} = (\Psi_b^{(-)}, V_a \Phi_a) = (\Phi_b, V_b \Psi_a^{(+)}). \tag{2.5.39}$$

We prove the latter equality with the use of the formal solution of G. Chew and M. Goldberger. By taking the difference of $(\Psi_b^{(-)}, V_a\Phi_a)$ and $(\Phi_b, V_b\Psi_a^{(+)})$, on setting $E_a = E_b = E$, we obtain

$$\begin{aligned}
(\Psi_b^{(-)}, V_a\Phi_a) - (\Phi_b, V_b\Psi_a^{(+)}) &= (\Phi_b, V_a\Phi_a) - (\Phi_b, V_b\Phi_a) \\
&\quad + \left(\Phi_b, V_b \frac{1}{E - H + i\varepsilon} V_a\Phi_a\right) \\
&\quad - \left(\Phi_b, V_b \frac{1}{E - H + i\varepsilon} V_a\Phi_a\right) \\
&= (\Phi_b, (V_a - V_b)\Phi_a) = (\Phi_b, (H_b - H_a)\Phi_a) \\
&= (E_b - E_a)(\Phi_b, \Phi_a) = 0.
\end{aligned}$$

We established Eq. (2.5.39). In the *plane wave Born approximation*, we replace $\Psi_b^{(-)}$ and $\Psi_a^{(+)}$ with Φ_b and Φ_a, respectively, and we have

$$T_{ba}^{(\mathrm{Born})} = (\Phi_b, V_a\Phi_a) = (\Phi_b, V_b\Phi_a).$$

Final state interaction: In the presence of the final state interaction, we split the total Hamiltonian into the following form:

$$H = H_0 + U + V, \tag{2.5.40}$$

(1) U is strong and V is weak,

$$|U| \gg |V|,$$

(2) we have the exact solution for $H_0 + U$,

$$(H_0 + U)\chi_a^{(\pm)} = E_a\chi_a^{(\pm)}. \tag{2.5.41}$$

For the electron in interaction with the electromagnetic field, for example, we have

$$\begin{aligned}
H_0 &= (\text{free electron}) + (\text{free electromagnetic field}), \\
U &= (\text{Coulomb potential}), \\
V &= (\text{interaction between electron and electromagnetic field}).
\end{aligned}$$

We represent the eigenstate of the total Hamiltonian H as $\Psi^{(\pm)}$:

$$H\Psi_a^{(\pm)} = E_a\Psi_a^{(\pm)}. \tag{2.5.42}$$

We have the following relationship between $\Psi_a^{(\pm)}$ and $\chi_a^{(\pm)}$:

$$\begin{aligned}
\Psi_a^{(\pm)} &= \chi_a^{(\pm)} + \frac{1}{E_a - H \pm i\varepsilon} V\chi_a^{(\pm)} \\
&= \chi_a^{(\pm)} + \frac{1}{E_a - (H_0 + U) \pm i\varepsilon} V\Psi_a^{(\pm)}.
\end{aligned} \tag{2.5.43}$$

This is the L–S equation relating $\Psi_a^{(\pm)}$ to $\chi_a^{(\pm)}$, with $H_0 + U$ as the free Hamiltonian.

We can envision the photoelectric effect for the present treatment:

$$\chi_a^{(+)} \ : \ \text{(1s electron)} \times \text{(1 photon state)},$$
$$\chi_b^{(-)} \ : \ \text{(electron in scattering state)} \times \text{(0 photon state)},$$

with $(\chi_b^{(-)}, \chi_a^{(+)}) = 0$.

From the S matrix, $S_{ba} = (\Psi_b^{(-)}, \Psi_a^{(+)})$, we pick up the first-order term in V,

$$(\Psi_b^{(-)}, \Psi_a^{(+)}) = (\chi_b^{(-)}, \chi_a^{(+)}) + \left(\chi_b^{(-)}, \frac{1}{E_a - (H_0 + U) + i\varepsilon} V \chi_a^{(+)}\right) + \left(\frac{1}{E_b - (H_0 + U) - i\varepsilon} V \chi_b^{(-)}, \chi_a^{(+)}\right) + \cdots. \tag{2.5.44}$$

In many applications, we frequently have $(\chi_b^{(-)}, \chi_a^{(+)}) = 0$ which we shall assume. Keeping the first-order term in V in Eq. (2.5.44), we have

$$S_{ba} = -2\pi i \delta(E_a - E_b)(\chi_b^{(-)}, V \chi_a^{(+)}). \tag{2.5.45}$$

Hence we have the following T matrix:

$$T_{ba} = (\chi_b^{(-)}, V \chi_a^{(+)}). \tag{2.5.46}$$

The relationship between Φ_b and $\chi_b^{(-)}$ is given by

$$\begin{aligned} \chi_b^{(-)} &= \Phi_b + \frac{1}{E_b - H_0 - i\varepsilon} U \chi_b^{(-)} \\ &= \Phi_b + \frac{1}{E_b - (H_0 + U) - i\varepsilon} U \Phi_b. \end{aligned} \tag{2.5.47}$$

This is the L–S equation with $H_0 + U$ as the total Hamiltonian.

Thus the Born approximation to Eq. (2.5.46) is given by

$$T_{ba}^{(\text{Born})} = (\Phi_b, V \chi_a^{(+)}), \tag{2.5.48}$$

and is frequently called as the *distorted wave Born approximation*.

We shall consider the following reaction:

$$\mathrm{p} + \mathrm{p} \to \mathrm{p} + \mathrm{n} + \pi^+.$$

We represent the potential energy between the two nucleons by U and the interaction term necessary to produce π^+ by V. We assume $|U| \gg |V|$. The wavefunction of the initial state $\chi_{\mathrm{pp}}^{(+)}$ satisfies the L–S equation,

$$\chi_{\mathrm{pp}}^{(+)} = \Phi_{\mathrm{pp}} + \frac{1}{E_a - H_a + i\varepsilon} U_{\mathrm{pp}} \chi_{\mathrm{pp}}^{(+)}.$$

Here, H_a represents the kinetic energy of the two-nucleon system and U_{pp} represents the potential energy between the two protons, corresponding to V_a. E_a represents the total energy of the initial state. The wavefunction of the final state $\chi^{(-)}_{\mathrm{pn}\pi}$ satisfies the L–S equation,

$$\chi^{(-)}_{\mathrm{pn}\pi^+} = \Phi_{\mathrm{pn}\pi^+} + \frac{1}{E_a - H_b - i\varepsilon} V_b \chi^{(-)}_{\mathrm{pn}\pi^+}.$$

Here, H_b represents the sum of the kinetic energies of p, n and π^+, and the rest mass of π^+, and V_b represents the three-body interaction term. We shall assume that the interaction between π^+ and the nucleons is weak and we can approximate V_b by U_{pn}. Then the wavefunction of the final state $\chi^{(-)}_{\mathrm{pn}\pi^+}$ is separated as

$$\chi^{(-)}_{\mathrm{pn}\pi^+} \approx g^{(-)}_{\mathrm{pn}} h_{\pi^+}.$$

The T matrix T_{ba} is given by

$$T_{ba} = (\chi^{(-)}_b, V\chi^{(+)}_a) \approx (g^{(-)}_{\mathrm{pn}} h_{\pi^+}, V\chi^{(+)}_{\mathrm{pp}}).$$

The final proton–neutron wavefunction $g^{(-)}_{\mathrm{pn}}$ satisfies the L–S equation,

$$g^{(-)}_{\mathrm{pn}} = g^{(0)}_{\mathrm{pn}} + \frac{1}{E_{\mathrm{p}} + E_{\mathrm{n}} - H_{\mathrm{p}} - H_{\mathrm{n}} - i\varepsilon} U_{\mathrm{pn}} g^{(-)}_{\mathrm{pn}}.$$

Here, H_{p} and H_{n} represent the kinetic energy operators of the proton and the neutron, respectively, and E_{p} and E_{n} represent their eigenvalues. This is the approach followed by K. Watson. We shall not go into any further computational details.

Inclusion of the spin–spin force and the tensor force,

$$V(r) = V_0(r) + V_{\text{spin-spin}}(r)(\vec{\sigma}_{\mathrm{n}}\vec{\sigma}_{\mathrm{p}}) + V_{\text{tensor}}(r)\left(3\frac{(\vec{\sigma}_{\mathrm{n}}\vec{\mathbf{r}})(\vec{\sigma}_{\mathrm{p}}\vec{\mathbf{r}})}{r^2} - (\vec{\sigma}_{\mathrm{n}}\vec{\sigma}_{\mathrm{p}})\right),$$

is immediate with the use of the spin-singlet and spin-triplet basis for the spin wavefunction of the two-nucleon system.

For further details on rearrangement collision and final state interaction, we refer the reader to the advanced textbooks on quantum mechanics and nuclear physics.

2.6
Scalar Field Interacting with Static Source

Consider the quantized real scalar field $\phi(\vec{x}, t)$ interacting with the time-independent c-number source $\rho(\vec{x})$. Let the equation of motion for the field $\phi(\vec{x}, t)$ be

$$\ddot{\phi}(\vec{x}, t) - s^2\vec{\nabla}^2\phi(\vec{x}, t) + s^2\mu^2\phi(\vec{x}, t) = \rho(\vec{x}), \quad \text{with} \quad \mu \neq 0. \tag{2.6.1}$$

The Hamiltonian of the system is given by

$$H = \frac{1}{2}\int d\vec{x}\{\Pi^2(\vec{x},t) + s^2(\vec{\nabla}\phi(\vec{x},t))^2 + s^2\mu^2\phi^2(\vec{x},t) - 2\rho(\vec{x})\phi(\vec{x},t)\}. \tag{2.6.2}$$

Imposing the equal-time canonical commutation relations,

$$[\Pi(\vec{x},t),\phi(\vec{x}',t)] = -i\hbar\delta(\vec{x}-\vec{x}'),$$
$$[\Pi(\vec{x},t),\Pi(\vec{x}',t)] = [\phi(\vec{x},t),\phi(\vec{x}',t)] = 0,$$

the Heisenberg equations of motion become

$$\begin{aligned} i\hbar\dot{\phi}(\vec{x},t) &= [\phi(\vec{x},t),H] = i\hbar\Pi(\vec{x},t),\\ i\hbar\dot{\Pi}(\vec{x},t) &= [\Pi(\vec{x},t),H] = i\hbar\{s^2\vec{\nabla}^2\phi(\vec{x},t) - s^2\mu^2\phi(\vec{x},t) + \rho(\vec{x})\}, \end{aligned} \tag{2.6.3a}$$

which agree with (2.6.1). We can write the q-number operators satisfying (2.6.3a) in the large box of volume V in Schrödinger picture as

$$\begin{aligned} \phi(\vec{x},0) \equiv \phi(\vec{x}) &= \sqrt{\hbar/V}\textstyle\sum_{\vec{k}}(1/\sqrt{2\omega_{\vec{k}}})\{a_{\vec{k}}\exp[i\vec{k}\cdot\vec{x}] + a^\dagger_{\vec{k}}\exp[-i\vec{k}\cdot\vec{x}]\},\\ \Pi(\vec{x},0) \equiv \Pi(\vec{x}) &= -i\sqrt{\hbar/V}\textstyle\sum_{\vec{k}}\sqrt{\omega_{\vec{k}}/2}\{a_{\vec{k}}\exp[i\vec{k}\cdot\vec{x}] - a^\dagger_{\vec{k}}\exp[-i\vec{k}\cdot\vec{x}]\}, \end{aligned}$$

where

$$\begin{aligned} [a_{\vec{k}},a^\dagger_{\vec{k}'}] &= \delta_{\vec{k}\vec{k}'},\\ [a_{\vec{k}},a_{\vec{k}'}] &= [a^\dagger_{\vec{k}},a^\dagger_{\vec{k}'}] = 0, \end{aligned} \qquad \omega_{\vec{k}} = s\sqrt{\vec{k}^2+\mu^2}. \tag{2.6.3b}$$

Substituting the expansions of $\phi(\vec{x})$ and $\Pi(\vec{x})$ into (2.6.2), we obtain

$$H = \sum_{\vec{k}}\left\{\hbar\omega_{\vec{k}}\left(a^\dagger_{\vec{k}}a_{\vec{k}} + \frac{1}{2}\right) + \lambda_{\vec{k}}a_{\vec{k}} + \lambda^*_{\vec{k}}a^\dagger_{\vec{k}}\right\}.$$

Here, we set

$$\rho(\vec{k}) = \frac{1}{\sqrt{V}}\int d\vec{x}\rho(\vec{x})\exp[i\vec{k}\cdot\vec{x}], \quad \lambda_{\vec{k}} = -\sqrt{\frac{\hbar}{2\omega_{\vec{k}}}}\rho(\vec{k}).$$

We can express the Hamiltonian as

$$H = \sum_{\vec{k}}\left\{\hbar\omega_{\vec{k}}\left(a^\dagger_{\vec{k}} + \frac{\lambda_{\vec{k}}}{\hbar\omega_{\vec{k}}}\right)\left(a_{\vec{k}} + \frac{\lambda^*_{\vec{k}}}{\hbar\omega_{\vec{k}}}\right) - \frac{|\lambda_{\vec{k}}|^2}{\hbar\omega_{\vec{k}}} + \frac{\hbar\omega_{\vec{k}}}{2}\right\}. \tag{2.6.4}$$

The last two terms above are the c-number terms which do not cause any problem. The first term above represents the shift of $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$ by the c-numbers. Since the shifted $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$ also satisfy the commutation relation

(2.6.3b), we can express the shift as a result of the unitary transformation. If we define

$$v \equiv \exp\left[\sum_{\vec{k}} \frac{1}{\hbar\omega_{\vec{k}}}\left(\lambda_{\vec{k}} a_{\vec{k}} - \lambda^*_{\vec{k}} a^\dagger_{\vec{k}}\right)\right],$$

we have

$$v^{-1} a_{\vec{k}} v = a_{\vec{k}} - \frac{\lambda^*_{\vec{k}}}{\hbar\omega_{\vec{k}}}, \quad v^{-1} a^\dagger_{\vec{k}} v = a^\dagger_{\vec{k}} - \frac{\lambda_{\vec{k}}}{\hbar\omega_{\vec{k}}}.$$

Here we used the Baker–Campbell–Hausdorff formula,

$$\exp[-iB] A \exp[iB] = A + i[A, B] + \frac{i^2}{2!}[[A, B], B] + \cdots. \tag{2.6.5}$$

Applying the unitary transformation v to the Hamiltonian H, we obtain

$$v^{-1} H v = \sum_{\vec{k}} \left\{\hbar\omega_{\vec{k}} a^\dagger_{\vec{k}} a_{\vec{k}} - \frac{\left|\lambda_{\vec{k}}\right|^2}{\hbar\omega_{\vec{k}}} + \frac{\hbar\omega_{\vec{k}}}{2}\right\} \equiv H_0.$$

We can obtain the eigenvalue of H_0 immediately. We set the difference of H and H_0 as

$$H^{\text{int}} \equiv H - H_0 = \sum_{\vec{k}} \left\{\lambda_{\vec{k}} a_{\vec{k}} + \lambda^*_{\vec{k}} a^\dagger_{\vec{k}} + \frac{\left|\lambda_{\vec{k}}\right|^2}{\hbar\omega_{\vec{k}}}\right\}.$$

In order to understand the meaning of the Interaction Picture, using this H_0, we compute $H^{\text{int}}(t)$ defined by

$$H^{\text{int}}(t) \equiv \exp\left[i\frac{H_0 t}{\hbar}\right] H^{\text{int}} \exp\left[-i\frac{H_0 t}{\hbar}\right].$$

We obtain, with the use of (2.6.5),

$$H^{\text{int}}(t) = \sum_{\vec{k}} \left\{\lambda_{\vec{k}} a_{\vec{k}} \exp[-i\omega_{\vec{k}} t] + \lambda^*_{\vec{k}} a^\dagger_{\vec{k}} \exp[i\omega_{\vec{k}} t] + \frac{\left|\lambda_{\vec{k}}\right|^2}{\hbar\omega_{\vec{k}}}\right\}.$$

Connecting the state vector $\left|\Psi(t)\right\rangle$ in the Interaction Picture and the state vector $|\Phi\rangle$ in the Heisenberg Picture by

$$\left|\Psi(t)\right\rangle = U(t)\left|\Phi\right\rangle,$$

we have

$$i\hbar\frac{d}{dt} U(t) = H^{\text{int}}(t) U(t), \tag{2.6.6}$$

with the initial condition, $U(0) = 1$.

In what follows, we solve (2.6.6) explicitly and show that $U(t)$ and v are related through

$$U^{-1}(-\infty) = v.$$

To solve (2.6.6), we set

$$U(t) = \exp[iG(t)]U_1(t), \tag{2.6.7}$$

where $G(t)$ and $U_1(t)$ are unknown. Substituting (2.6.7) into (2.6.6), we obtain

$$\begin{aligned} &i\hbar\frac{d}{dt}U_1(t) \\ &= \Big\{\exp[-iG(t)]H^{\text{int}}(t)\exp[iG(t)] \\ &\quad - i\hbar\exp[-iG(t)]\frac{d}{dt}\exp[iG(t)]\Big\}U_1(t). \end{aligned}$$

Using the formula

$$\exp[-iG(t)]\frac{d}{dt}\exp[iG(t)] = i\dot{G}(t) + \frac{i^2}{2!}[\dot{G}(t), G(t)] + \frac{i^3}{3!}[[\dot{G}(t), G(t)], G(t)] + \cdots,$$

we obtain

$$\begin{aligned} H'^{\text{int}}(t) &\equiv \exp[-iG(t)]H^{\text{int}}(t)\exp[iG(t)] - i\hbar\exp[-iG(t)]\frac{d}{dt}\exp[iG(t)] \\ &= H^{\text{int}}(t) + i[H^{\text{int}}(t), G(t)] + \frac{i^2}{2!}[[H^{\text{int}}(t), G(t)], G(t)] + \cdots \\ &\quad - i\hbar\{i\dot{G}(t) + \frac{i^2}{2!}[\dot{G}(t), G(t)] + \frac{i^3}{3!}[[\dot{G}(t), G(t)], G(t)] + \cdots\}. \end{aligned}$$

So we set

$$G(t) \equiv -\frac{1}{\hbar}\int_0^t dt_1 H^{\text{int}}(t_1).$$

Since, in this model, we have

$$[H^{\text{int}}(t_1), H^{\text{int}}(t_2)] = 2i\sum_{\vec{k}}\left|\lambda_{\vec{k}}\right|^2 \sin\omega_{\vec{k}}(t_2 - t_1) = c\text{-number},$$

the third terms onward and many others in each infinite series vanish. So we have

$$\begin{aligned} H'^{\text{int}}(t) &= i[H^{\text{int}}(t), G(t)] + i\frac{\hbar}{2}[\dot{G}(t), G(t)] \\ &= -\frac{i}{2\hbar}\int_0^t dt_1[H^{\text{int}}(t), H^{\text{int}}(t_1)] \end{aligned}$$

$$= \frac{1}{\hbar}\int_0^t dt_1 \sum_{\vec{k}} \left|\lambda_{\vec{k}}\right|^2 \sin\omega_{\vec{k}}(t_1 - t)$$

$$= \sum_{\vec{k}} \frac{\left|\lambda_{\vec{k}}\right|^2}{\hbar\omega_{\vec{k}}}(\cos\omega_{\vec{k}}t - 1).$$

Hence, we immediately obtain $U_1(t)$ as

$$U_1(t) = \exp\left[-\frac{i}{\hbar}\int_0^t dt_1 \sum_{\vec{k}} \frac{\left|\lambda_{\vec{k}}\right|^2}{\hbar\omega_{\vec{k}}}(\cos\omega_{\vec{k}}t_1 - 1)\right]$$

$$= \exp\left[-\frac{i}{\hbar}\sum_{\vec{k}} \frac{\left|\lambda_{\vec{k}}\right|^2}{\hbar\omega_{\vec{k}}^2}\sin\omega_{\vec{k}}t\right]\exp\left[\frac{i}{\hbar}\sum_{\vec{k}} \frac{\left|\lambda_{\vec{k}}\right|^2}{\hbar\omega_{\vec{k}}}t\right].$$

Substituting this into the decomposition for $U(t)$, we obtain

$$U(t) = \exp\left[-\frac{i}{\hbar}\int_0^t dt_1 H^{\text{int}}(t_1) - \frac{i}{\hbar}\sum_{\vec{k}} \frac{\left|\lambda_{\vec{k}}\right|^2}{\hbar\omega_{\vec{k}}^2}\sin\omega_{\vec{k}}t + \frac{i}{\hbar}\sum_{\vec{k}} \frac{\left|\lambda_{\vec{k}}\right|^2}{\hbar\omega_{\vec{k}}}t\right]$$

$$= \exp\left[-\frac{i}{\hbar}\left\{\int_0^t dt_1 \sum_{\vec{k}}(\lambda_{\vec{k}}a_{\vec{k}}\exp[-i\omega_{\vec{k}}t_1] + \lambda_{\vec{k}}^* a_{\vec{k}}^\dagger \exp[i\omega_{\vec{k}}t_1]) + \sum_{\vec{k}} \frac{\left|\lambda_{\vec{k}}\right|^2}{\hbar\omega_{\vec{k}}^2}\sin\omega_{\vec{k}}t\right\}\right]$$

$$= \exp\left[\sum_{\vec{k}}\left[\frac{1}{\hbar\omega_{\vec{k}}}\{\lambda_{\vec{k}}a_{\vec{k}}(\exp[-i\omega_{\vec{k}}t] - 1) - \lambda_{\vec{k}}^* a_{\vec{k}}^\dagger(\exp[i\omega_{\vec{k}}t] - 1)\} - i\frac{\left|\lambda_{\vec{k}}\right|^2}{\hbar^2\omega_{\vec{k}}^2}\sin\omega_{\vec{k}}t\right]\right].$$

Since we have chosen μ to be nonzero constant, $\omega_{\vec{k}}$ never becomes zero. For each oscillatory term in the exponent, we invoke the following identity and its complex conjugate:

$$\frac{1}{\omega_{\vec{k}}}\exp[i\omega_{\vec{k}}t] = -i\lim_{\varepsilon\to 0^+}\int_t^\infty dt_1 \exp[i\omega_{\vec{k}}t_1]\exp[-\varepsilon\left|t_1\right|] \underset{t\to-\infty}{\longrightarrow} -2\pi i\delta(\omega_{\vec{k}}) = 0.$$

Thus we finally obtain

$$U^{-1}(-\infty) = \exp\left[\frac{1}{\hbar\omega_{\vec{k}}}\sum_{\vec{k}}\{\lambda_{\vec{k}}a_{\vec{k}} - \lambda_{\vec{k}}^* a_{\vec{k}}^\dagger\}\right] = v.$$

2.7 Problems for Chapter 2

2.1. (due to H. C.). Solve

$$\phi(x) = 1 + \lambda \int_0^1 (xy + x^2y^2)\phi(y)dy.$$

(a) Show that this is equivalent to a 2×2 matrix equation.

(b) Find the eigenvalues and the corresponding eigenvectors of the kernel.

(c) Find the solution of the inhomogeneous equation if $\lambda \neq$ eigenvalues.

(d) Solve the corresponding Fredholm integral equation of the first kind.

2.2. (due to H. C.). Solve

$$\phi(x) = 1 + \lambda \int_0^x xy\phi(y)dy.$$

Discuss the solution of the homogeneous equation.

2.3. (due to H. C.). Consider the integral equation,

$$\phi(x) = f(x) + \lambda \int_{-\infty}^{+\infty} e^{-(x^2+y^2)}\phi(y)dy, \quad -\infty < x < \infty.$$

(a) Solve this equation for

$$f(x) = 0.$$

For what values of λ, does it have nontrivial solutions?

(b) Solve this equation for

$$f(x) = x^m, \quad \text{with} \quad m = 0, 1, 2, \ldots.$$

Does this inhomogeneous equation have any solutions when λ is equal to an eigenvalue of the kernel?

Hint: You may express your results in terms of the *Gamma function*,

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad \mathrm{Re}\, z > 0.$$

2.4. (due to H. C.). Solve the following integral equation,

$$u(\theta) = 1 + \lambda \int_0^{2\pi} \sin(\phi - \theta) u(\phi) d\phi, \quad 0 \le \theta < 2\pi,$$

where $u(\theta)$ is periodic with period 2π. Does the kernel of this equation have any real eigenvalues?

Hint: Note

$$\sin(\phi - \theta) = \sin\phi \cos\theta - \cos\phi \sin\theta.$$

2.5. (due to D. M.) Consider the integral equation

$$\phi(x) = 1 + \lambda \int_0^1 \frac{x^n - y^n}{x - y} \phi(y) dy, \quad 0 \le x \le 1.$$

(a) Solve this equation for $n = 2$. For what values of λ, does the equation have no solutions?

(b) Discuss how you would solve this integral equation for arbitrary positive integer n.

2.6. (due to D. M.) Solve the integral equation,

$$\phi(x) = 1 + \int_0^1 (1 + x + y + xy)^\nu \phi(y) dy, \quad 0 \le x \le 1, \quad \nu : \text{real}.$$

Hint: Note that the kernel

$$(1 + x + y + xy)^\nu,$$

can be factorized.

2.7. In the Fredholm integral equation of the second kind, if the kernel is given by

$$K(x, y) = \sum_{n=1}^{N} g_n(x) h_n(y),$$

show that the integral equation is equivalent to an $N \times N$ matrix equation.

2.8. (due to H. C.). Consider the motion of an harmonic oscillator with a time-dependent spring constant,

$$\frac{d^2}{dt^2}x + \omega^2 x = -A(t)x,$$

where ω is a constant and $A(t)$ is a complicated function of t. Transform this differential equation together with the boundary conditions

$$x(T_i) = x_i \quad \text{and} \quad x(T_f) = x_f,$$

to an integral equation.

Hint: Construct a Green's function $G(t, t')$ satisfying

$$G(T_i, t') = G(T_f, t') = 0.$$

2.9. Generalize the discussion of Section 2.4 to the case of three spatial dimensions and transform the Schrödinger equation with the initial condition,

$$\lim_{t\to-\infty} e^{i\omega t}\psi(\vec{x}, t) = e^{ikz},$$

to an integral equation.

Hint: Construct a Green's function $G(t, t')$ satisfying

$$\left(i\frac{\partial}{\partial t} + \vec{\nabla}^2\right) G(\vec{x}, t; \vec{x}', t') = \delta(t - t')\delta^3(\vec{x} - \vec{x}'),$$
$$G(\vec{x}, t; \vec{x}', t') = 0 \quad \text{for} \quad t < t'.$$

2.10. (due to H. C.). Consider the equation

$$\left[-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} - m^2\right]\phi(x, t) = U(x, t)\phi(x, t).$$

If the initial and final conditions are

$$\phi(x, -T) = f(x) \quad \text{and} \quad \phi(x, T) = g(x),$$

transform the equation to an integral equation.

Hint: Consider Green's function

$$\left[-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} - m^2\right] G(x, t; x', t') = \delta(x - x')\delta(t - t').$$

2.11. (due to D. M.) The time-independent Schrödinger equation with the periodic potential, $V(x) = -(a^2 + k^2 \cos^2 x)$, reads

$$\frac{d^2}{dx^2}\psi(x) + (a^2 + k^2 \cos^2 x)\psi(x) = 0.$$

Show directly that even periodic solutions of this equation, which are even *Mathieu functions*, satisfy the homogeneous integral equation,

$$\psi(x) = \lambda \int_{-\pi}^{\pi} \exp[k \cos x \cos y]\psi(y)dy.$$

Hint: Show that $\phi(x)$ defined by

$$\phi(x) \equiv \int_{-\pi}^{\pi} \exp[k \cos x \cos y]\psi(y)dy$$

is even and periodic, and satisfies the above time-independent Schrödinger equation. Thus, $\psi(x)$ is the constant multiple of $\phi(x)$,

$$\psi(x) = \lambda\phi(x).$$

2.12. (due to H. C.). Consider the differential equation,

$$\frac{d^2}{dt^2}\phi(t) = \lambda e^{-t}\phi(t), \quad 0 \le t < \infty, \quad \lambda = \text{constant},$$

together with the initial conditions,

$$\phi(0) = 0 \quad \text{and} \quad \phi'(0) = 1.$$

(a) Find the partial differential equation for Green's function $G(t, t')$. Determine the form of $G(t, t')$ when $t \neq t'$.

(b) Transform the differential equation for $\phi(t)$ together with the initial conditions to an integral equation. Determine the conditions on $G(t, t')$.

(c) Determine $G(t, t')$.

(d) Substitute your answer for $G(t, t')$ into the integral equation and verify explicitly that the integral equation is equivalent to the differential equation together with the initial conditions.

(e) Does the initial value problem have a solution for all λ? If so, is the solution unique?

2.13. In the Volterra integral equation of the second kind, if the kernel is given by

$$K(x, y) = \sum_{n=1}^{N} g_n(x)h_n(y),$$

show that the integral equation can be reduced to an ordinary differential equation of the Nth order.

2.14. Consider the partial differential equation of the form

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\phi(x,t) = p(x,t) - \lambda \frac{\partial^2}{\partial x^2}\phi(x,t) \cdot \frac{\partial}{\partial x}\phi(x,t),$$

where

$$-\infty < x < \infty, \quad t \geq 0,$$

and λ is a constant, with the initial conditions specified by

$$\phi(x,0) = a(x),$$

and

$$\frac{\partial}{\partial t}\phi(x,0) = b(x).$$

This partial differential equation describes the displacement of a vibrating string under the distributed load $p(x,t)$.

(a) Find Green's function for this partial differential equation.

(b) Express this initial value problem in terms of an integral equation using Green's function found in (a). Explain how you would find an approximate solution if λ were small.

Hint: By applying the Fourier transform in x, find a function $\phi_0(x,t)$ which satisfies the wave equation,

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\phi_0(x,t) = 0,$$

and the given initial conditions.

2.15. Show that Green's function $G(\vec{r};\vec{r}^{\,\prime})$ for the Poisson equation in three spatial dimensions,

$$\vec{\nabla}^2 G(\vec{r};\vec{r}^{\,\prime}) = -4\pi\delta(\vec{r}-\vec{r}^{\,\prime}),$$

is given by

$$G(\vec{r};\vec{r}^{\,\prime}) = \frac{1}{\left|\vec{r}-\vec{r}^{\,\prime}\right|} = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta),$$

where θ is the angle between $\vec{r}$ and $\vec{r}^{\,\prime}$, $P_l(\cos\theta)$ is the lth-order Legendre polynomial of the first kind, and $r_<$ ($r_>$) is the smaller (the larger) of the lengths, $|\vec{r}|$ and $\left|\vec{r}^{\,\prime}\right|$,

$$r_< = \frac{1}{2}\left(|\vec{r}| + \left|\vec{r}^{\,\prime}\right|\right) - \frac{1}{2}\left||\vec{r}| - \left|\vec{r}^{\,\prime}\right|\right| \quad \text{and}$$

$$r_> = \frac{1}{2}\left(|\vec{r}| + \left|\vec{r}^{\,\prime}\right|\right) + \frac{1}{2}\left||\vec{r}| - \left|\vec{r}^{\,\prime}\right|\right|.$$

2.16. Show that Green's function $G(\vec{r};\vec{r}')$ for the Helmholtz equation in three spatial dimension,

$$(\vec{\nabla}^2 + k^2)G(\vec{r};\vec{r}') = -4\pi\delta(\vec{r}-\vec{r}'),$$

is given by

$$G(\vec{r};\vec{r}') = \frac{\exp[ik\left|\vec{r}-\vec{r}'\right|]}{\left|\vec{r}-\vec{r}'\right|} = k\sum_{l=0}^{\infty}(2l+1)j_l(kr_<)h_l^{(1)}(kr_>)P_l(\cos\theta),$$

where $j_l(kr)$ is the lth-order *spherical Bessel function* and $h_l^{(1)}(kr)$ is the lth-order *spherical Hankel function of the first kind*. Show that Green's function $G(\vec{r};\vec{r}')$ which is even in k for the Helmholtz equation is given by

$$G(\vec{r};\vec{r}') = \frac{\cos k\left|\vec{r}-\vec{r}'\right|}{\left|\vec{r}-\vec{r}'\right|} = k\sum_{l=0}^{\infty}(2l+1)j_l(kr_<)n_l(kr_>)P_l(\cos\theta),$$

where $n_l(kr)$ is the lth-order *spherical Neumann function*,

$$n_l(kr) = \mathrm{Re}\, h_l^{(1)}(kr).$$

2.17. Consider the differential equation for spherical Bessel functions,

$$\left\{-\frac{d^2}{dr^2} - \frac{2}{r}\frac{d}{dr} + \frac{l(l+1)}{r^2} - k^2\right\}\left\{\begin{array}{c} h_l^{(1)}(kr) \\ j_l(kr)\end{array}\right\} = 0,$$

with the boundary conditions

$$\left\{\begin{array}{c} h_l^{(1)}(kr) \\ j_l(kr)\end{array}\right\} \to \left\{\begin{array}{c} (1/i^{l+1})(\exp[ikr]/kr) \\ \sin(kr - l\pi/2)/kr\end{array}\right\} \quad \text{as} \quad kr\to\infty.$$

(a) We define

$$\left\{\begin{array}{c} w_l(kr) \\ u_l(kr)\end{array}\right\} = \left\{\begin{array}{c} krh_l^{(1)}(kr), \\ krj_l(kr).\end{array}\right\}$$

Show that $u_l(kr)$ and $w_l(kr)$ satisfy the following differential equation:

$$\left\{-\frac{d^2}{dx^2} + \frac{l(l+1)}{x^2} - 1\right\}\left\{\begin{array}{c} w_l(x) \\ u_l(x)\end{array}\right\} = 0,$$

with the boundary conditions

$$\left\{ \begin{array}{c} w_l(x \to \infty) \\ u_l(0) \end{array} \right\} = \left\{ \begin{array}{c} \exp[ix] \\ 0 \end{array} \right\}$$

and the normalization determined by

$$\left\{ \begin{array}{c} w_l(x) \\ u_l(x) \end{array} \right\} \to \left\{ \begin{array}{c} (1/i^{l+1})(\exp[ix]) \\ \sin(x - l\pi/2) \end{array} \right\} \quad \text{as} \quad x \to \infty.$$

(b) Define the operators H_l, A_l^+, and A_l^- by

$$H_l \equiv \left\{ -\frac{d^2}{dx^2} + \frac{l(l+1)}{x^2} \right\},$$

and

$$A_l^+ \equiv -\frac{d}{dx} + \frac{l}{x}, \quad A_l^- \equiv \frac{d}{dx} + \frac{l}{x}.$$

Show that

$$A_l^+ A_l^- = H_l \quad \text{and} \quad A_l^- A_l^+ = H_{l-1}.$$

(c) If

$$H_l \psi_l = \psi_l,$$

show that

$$A_l^+ A_l^- \psi_l = \psi_l \quad \text{and} \quad A_l^- A_l^+ A_l^- \psi_l = A_l^- \psi_l,$$

and hence

$$H_{l-1}(A_l^- \psi_l) = (A_l^- \psi_l).$$

A_l^- is a lowering operator which takes ψ_l into ψ_{l-1}. Similarly, since

$$A_{l+1}^- A_{l+1}^+ = H_l,$$

show that

$$A_{l+1}^- A_{l+1}^+ \psi_l = \psi_l \quad \text{and} \quad A_{l+1}^+ A_{l+1}^- A_{l+1}^+ \psi_l = A_{l+1}^+ \psi_l,$$

and hence

$$H_{l+1}(A_{l+1}^+ \psi_l) = (A_{l+1}^+ \psi_l).$$

A_{l+1}^+ is a raising operator which takes ψ_l into ψ_{l+1}.

(d) As $x \to \infty$, by observing

$$A^+ \to -\frac{d}{dx}, \quad A^- \to \frac{d}{dx},$$

show that

$$\begin{cases} A^+ w_l & \to \quad \exp[ix]/i^{l+2}, \\ A^- w_l & \to \quad \exp[ix]/i^l, \end{cases} \quad \text{and} \quad \begin{cases} A^+ u_l & \to \quad \sin(x-(l+1)\pi/2), \\ A^- u_l & \to \quad \sin(x-(l-1)\pi/2). \end{cases}$$

Namely, A^+ and A^- correctly maintain the asymptotic forms of w_l and u_l and hence the normalizations of w_l and u_l.

(e) Starting from the solutions for $l = 0$,

$$w_0 = \exp[ix]/i \quad \text{and} \quad u_0 = \sin x,$$

obtain w_l and u_l and hence $h_l^{(1)}$ and j_l.

2.18. Consider the scattering off a spherically symmetric potential $V(r)$.

(a) Prove that the free Green's function in the spherical polar coordinate is given by

$$\frac{\hbar^2}{2m}\left\langle \vec{x} \left| \frac{1}{E - \hat{H}_0 + i\varepsilon} \right| \vec{x}' \right\rangle = -ik\sum_l \sum_m Y_{l,m}(\hat{r})Y^*_{l,m}(\hat{r}')j_l(kr_<)h_l^{(1)}(kr_>),$$

where $r_<$ ($r_>$) stands for the smaller (larger) of r and r'.

(b) The Lippmann–Schwinger equation can be written for spherical waves as

$$\left|Elm(+)\right\rangle = \left|Elm\right\rangle + \frac{1}{E - \hat{H}_0 + i\varepsilon} V \left|Elm(+)\right\rangle.$$

Using a), show that this equation, written in the $\vec{x}$-representation, leads to an integral equation for the radial function, $A_l(k;\, r)$, as follows:

$$A_l(k;\, r) = j_l(kr) - (\frac{2mik}{\hbar^2})\int_0^\infty j_l(kr_<)h_l^{(1)}(kr_>)V(r')A_l(k;\, r')r'^2 dr',$$

where the radial function, $A_l(k;\, r)$, is defined by

$$\left\langle \vec{x} \,\middle|\, Elm(+) \right\rangle = c_l A_l(k;\, r) Y_{l,m}(\hat{r}),$$

with the normalization constant c_l given by

$$c_l = \frac{i^l}{\hbar}\sqrt{\frac{2mk}{\pi}}.$$

By taking r very large, obtain the scattering amplitude for the lth partial wave as

$$f_l(k) = \exp[i\delta_l]\frac{\sin\delta_l}{k} = -\frac{2m}{\hbar^2}\int_0^\infty j_l(kr)A_l(k;\, r)V(r)r^2 dr.$$

Hint for Problem 2.18: The wavefunction for the free wave in the spherical polar coordinate is given by

$$\langle \vec{x}\,|\,Elm\rangle = c_l j_l(kr)Y_{l,m}(\hat{r}).$$

When $r \to \infty$, use the asymptotic formula for $j_l(kr)$ and $h_l^{(1)}(kr)$.

2.19. Consider a scattering of nonrelativistic particle off delta-shell potential. The potential we consider is

$$U(r) = -\lambda\delta(r-a), \tag{A}$$

i.e., a force field that vanishes everywhere except on a sphere of radius a. The strength parameter λ has the dimension $(\text{length})^{-1}$. One can look upon (A) as a crude model of the interaction experienced by a neutron when it interacts with a nucleus of radius a.

(a) Show that the radial integral equation

$$A_l(k;\, r) = j_l(kr) + \int_0^\infty G_k^{(l)}(r;\, r')U(r')A_l(k;\, r')r'^2 dr',$$

with

$$G_k^{(l)}(r;\, r') = -ikj_l(kr_<)h_l(kr_>),$$

reduces to the algebraic equation,

$$A_l(k;\, r) = j_l(kr) + ik\lambda a^2 A_l(k;\, a) \times \begin{cases} j_l(kr)h_l(ka), & r < a, \\ j_l(ka)h_l(kr), & r > a. \end{cases} \tag{B}$$

Obtain, by setting $r \to a$ in (B),

$$A_l(k;\, a) = \frac{j_l(ka)}{1 - ik\lambda a^2 j_l(ka)h_l(ka)}.$$

(b) Construct the partial wave scattering amplitude from the general formula

$$\exp[i\delta_l]\sin\delta_l = -k\int_0^\infty j_l(kr)\,U(r)A_l(k;\,r)r^2dr,$$

as

$$\exp[i\delta_l]\sin\delta_l = k\lambda a^2 j_l(ka)A_l(k;\,a) = \frac{k\lambda a^2[j_l(ka)]^2}{1 - ik\lambda a^2 j_l(ka)h_l(ka)}. \qquad \text{(C)}$$

The tangent of δ_l is also a convenient quantity for some purposes. Using $h_l(z) = j_l(z) + in_l(z)$, we obtain

$$\tan\delta_l = \frac{k\lambda a^2[j_l(ka)]^2}{1 + k\lambda a^2 j_l(ka)n_l(ka)}.$$

It is natural to express all length in units of a, and all wave numbers in units of $1/a$. Hence define the dimensionless variables as $\rho = r/a, \xi = ka, g = \lambda a$, in terms of which, show that (C) is expressed as

$$\exp[i\delta_l]\sin\delta_l = \frac{g\xi[j_l(\xi)]^2}{1 - i\xi g j_l(\xi)h_l(\xi)}. \qquad \text{(D)}$$

(c) The existence of bound states requires the occurrence of poles in the radial continuum wavefunctions when the latter are treated as functions of the complex variable k. Hence we must determine the location of the pure imaginary zeros of the function

$$D_l(g;\,\zeta) = 1 - i\zeta g j_l(\zeta)h_l(\zeta),$$

where $\zeta = \xi + i\eta$. We call the iterative solution of the radial integral equation the Born series for A_l, and we call the corresponding expansion,

$$\frac{1}{k}\exp[i\delta_l]\sin\delta_l = -\int_0^\infty [j_l(kr)]^2\,U(r)r^2dr - \int_0^\infty j_l(kr)\,U(r)G_k^{(l)}(r,r')\,U(r')j_l(kr')r^2r'^2drdrr' + \cdots,$$

the Born series for the partial wave amplitude. Show that

$$\exp[i\delta_l]\sin\delta_l = g\xi[j_l(\xi)]^2\sum_{n=0}^{\infty} g^n[i\xi j_l(\xi)h_l(\xi)]^n,$$

and its radius of convergence in the complex g-plane is

$$|g| < \frac{1}{\xi\,|j_l(\xi)h_l(\xi)|} \equiv g_l^{(B)}(\xi).$$

(d) Show that the actual form of the radial wavefunction for a bound state can be obtained by evaluating the residue of (B) at the bound state pole. If this pole is at $\zeta = i\eta_b^{(l)}$, say, show that with N the normalization constant,

$$R_l(\rho) = \begin{cases} N j_l(i\eta_b^{(l)}\rho)h_l(i\eta_b^{(l)}), & \rho < 1, \\ N h_l(i\eta_b^{(l)}\rho)j_l(i\eta_b^{(l)}), & \rho > 1. \end{cases}$$

(e) Consider the behavior of the S-wave cross section as a function of energy; this is given by $\sin^2\delta_0(\xi)$. From (D), show that

$$\sin^2\delta_0(\xi) = \frac{g^2\sin^4\xi}{(\xi - (1/2)g\sin 2\xi)^2 + g^2\sin^4\xi}. \tag{E}$$

This function attains its maximum value of unity at the roots of

$$2\xi = g\sin 2\xi. \tag{F}$$

Show that the resonances fall into two very different classes, $\xi_b^{(n)}$ and $\xi_s^{(n)}$: (a) $\xi_b^{(n)}$ near $\xi = \pi/2, 3\pi/2, 5\pi/2\ldots$, which are *broad* because $\sin^4\xi \approx 1$ at these points, and (b) $\xi_s^{(n)}$ near $\xi = \pi, 2\pi, 3\pi, \ldots$, which are *very sharp* because $\sin^4\xi \approx 0$ there. Verify these statements. The location of the sharp resonances follows immediately from (F); that is,

$$\xi_s^{(n)} \simeq n\pi\left(1 + \frac{1}{g}\right) = \operatorname{Re}\zeta_n \quad \text{if} \quad g \gg n.$$

Furthermore, $\sin^2\xi_s^{(n)} \simeq (n\pi/g)^2$. Show that we can write (E) in the Breit–Wigner form

$$\sin^2\delta_0(\xi) \simeq \frac{|\operatorname{Im}\zeta_n|^2}{(\xi - \operatorname{Re}\zeta_n)^2 + |\operatorname{Im}\zeta_n|^2}, \tag{G}$$

in the immediate vicinity of the sharp resonances. Show that as n increases, the resonances become broader, and the whole discussion culminating in (G) are only valid if the width $|\operatorname{Im}\zeta_n|$ is small compared to the distance between neighboring resonances, i.e.,

$$|\operatorname{Im}\zeta_n| \ll \operatorname{Re}\zeta_{n+1} - \operatorname{Re}\zeta_n. \tag{H}$$

Once (H) is violated, the resonances soon disappear.

Hint for Problem 2.19: For the clean-cut general discussion of the delta-shell potential problem along the line of the present problem, as well as the analytic properties of physical quantities of our interest as a function of k and energy, we refer the reader to the following textbook.

Gottfried, K.: *Quantum Mechanics, Vol. I: Fundamentals*, W.A. Benjamin, New York. (1966). Chapter III, Section 15.

2.20. Consider the S-wave scattering off a spherically symmetric potential $U(r)$. The radial Schrödinger equation is given by

$$\frac{d^2}{dr^2}u(r) + k^2 u(r) = U(r)u(r),$$

with the boundary conditions

$$u(0) = 0 \quad \text{and} \quad u(r) \sim \frac{\sin(kr + \delta)}{\sin\delta} \quad \text{as} \quad r \to \infty.$$

(a) Consider the rigid sphere potential problem,

$$U(r) = \begin{cases} \infty & \text{for} \quad r < b, \\ 0 & \text{for} \quad r \geq b. \end{cases}$$

Obtain the exact solution for the rigid sphere potential problem.

(b) Consider the spherical box potential problem,

$$U(r) = \begin{cases} U(> 0) & \text{for} \quad r < b, \\ 0 & \text{for} \quad r \geq b. \end{cases}$$

For the spherical box potential problem, show that δ is given by

$$k\cot\delta = \frac{(K\cot Kb)(k\cot kb) + k^2}{k\cot kb - K\cot Kb},$$

from the continuity of the wavefunction at $r = b$, where K is given by

$$K = \sqrt{k^2 - U}.$$

(c) Consider the bound state for the spherical box potential problem,

$$U(r) = \begin{cases} U(< 0) & \text{for} \quad r < b, \\ 0 & \text{for} \quad r \geq b. \end{cases}$$

From the continuity of the wavefunction at $r = b$, show that γ ($\gamma^2 = -k^2 > 0$) is determined by

$$K \cot Kb = -\gamma,$$
$$K = \sqrt{k^2 - U} = \sqrt{|U| - \gamma^2}.$$

2.21. Consider the S-wave scattering off a spherically symmetric potential $U(r)$. The radial Schrödinger equation is given by

$$\frac{d^2}{dr^2}u(r) + k^2 u(r) = U(r)u(r),$$

with the boundary conditions

$$u(0) = 0 \quad \text{and} \quad u(r) \sim \frac{\sin(kr + \delta)}{\sin \delta} \quad \text{as} \quad r \to \infty.$$

The scattering length a and the effective range r_{eff} are defined by

$$k \cot \delta = -\frac{1}{a} + \frac{1}{2} r_{\text{eff}} k^2 + \cdots.$$

Potential of range b has a bound state of energy

$$E = -E_{\text{B}} = -\kappa_{\text{B}}^2 < 0,$$

(i) Spherical box potential:

$$U(r) = \begin{cases} U(<0) & \text{for} \quad r < b, \\ 0 & \text{for} \quad r \geq b. \end{cases}$$

(a) Set up the equations for binding energy $E = -E_{\text{B}}$ and phase shift δ ($E > 0$). Weakly bound means that

$$Kb > \frac{\pi}{2}, \quad K = \sqrt{k^2 - U},$$

just above $\pi/2$, but not much. Define an energy Δ by

$$b\sqrt{(|U| - \Delta)} \equiv \frac{\pi}{2},$$

and q by

$$\Delta = q^2.$$

Then

$$\Delta \ll |U|.$$

(b) Show that the binding energy E_B is related to Δ by

$$\kappa_B b \simeq \frac{1}{2}(qb)^2.$$

(c) For positive E but $E \ll |U|$, define χ by

$$\chi \equiv -Kb\cot(Kb).$$

Show that

$$\chi \simeq \frac{1}{2}(k^2+q^2)b^2 \simeq \frac{1}{2}(kb)^2 + \kappa_B b.$$

(d) From this, show that the scattering length a is given by

$$a \equiv -\lim_{k\to 0}\left(\frac{\tan\delta}{k}\right) \simeq b\frac{1+\kappa_B b}{\kappa_B b} \gg b.$$

(ii) δ shell potential:

$$U(r) = -\kappa_0\delta(r-b).$$

(e) Set up the equations for binding energy $E = -E_B$ and phase shift δ $(E > 0)$.

(f) Show that the bound state energy satisfies

$$1+\kappa_B b \simeq \kappa_0 b.$$

(g) Show that in this case too, the scattering length a is given by

$$a \equiv -\lim_{k\to 0}\left(\frac{\tan\delta}{k}\right) \simeq b\frac{1+\kappa_B b}{\kappa_B b}.$$

This is the same result as for the spherical box potential of the same range and the same binding energy.

(h) You are in a position to show, without any effort, that if Δ is negative, that is, no bound state exists, the scattering length has an opposite sign, that is negative.

This problem is not too terribly intellectual, but it is a good exercise in approximation techniques.

2.22. Consider a time-dependent wave packet. A solution of the time-dependent Schrödinger equation,

$$\left(i\hbar\frac{\partial}{\partial t} - H\right)|\psi\rangle = 0,$$

can be constructed by superposition of energy eigenfunctions as

$$\langle\vec{r}\,|\psi^{(+)}(t)\rangle = \int d^3k \left\langle\vec{r}\,\middle|\vec{k}^{(+)}\right\rangle \exp[-\frac{i}{\hbar}E_k t]a(\vec{k}-\vec{k}_0),$$

with

$$E_k = \frac{(\hbar\vec{k})^2}{2m},$$

where $a(\vec{k}-\vec{k}_0)$ is an amplitude containing a narrow band of $\vec{k}$-values only. It is suggested that you make a packet in the z-direction (along $\vec{k}_0$) only,

$$a(\vec{k}-\vec{k}_0) = C\exp\left[-\frac{1}{2}D^2(k_z-k_0)^2\right]\delta(k_x)\delta(k_y).$$

D is the spatial width of the packet in the z-direction.

A narrow packet means that we assume

$$\frac{1}{D} = \Delta k \ll k_0,$$

and

$$\Delta k \cdot r_0 = \frac{r_0}{D} \ll 1,$$

where r_0 is the range the potential $V(r)$ (that is $V(r) \approx 0$ for $r > r_0$). In addition, we make a dynamic assumption that k_0 is not the position of a narrow resonance. This allows us to assume that, for $r \lesssim r_0$,

$$\langle r\,|k^{(+)}\rangle \cong \left\langle r\,\middle|k_0^{(+)}\right\rangle \quad \text{or} \quad V(r)\langle r\,|k^{(+)}\rangle \cong V(r)\left\langle r\,\middle|k_0^{(+)}\right\rangle.$$

Analogously, with all the assumptions made above, we can construct an incident wave packet as

$$\langle\vec{r}\,|\phi(t)\rangle = \int d^3k \left\langle\vec{r}\,\middle|\vec{k}\right\rangle \exp[-\frac{i}{\hbar}E_k t]a(\vec{k}-\vec{k}_0).$$

(a) Show that

$$\langle\vec{r}\,|\phi(t)\rangle \cong \exp\left[ik_0 z - \frac{i}{\hbar}E_0 t\right]F(z-v_0 t), \quad E_0 = \frac{(\hbar\vec{k}_0)^2}{2m}.$$

Establish the form of $F(z-v_0t)$. Show that it is a packet of width D.

(b) Show that

$$\langle \vec{r} \,|\psi(t)\rangle \cong \langle \vec{r} \,|\phi(t)\rangle + f(\theta)\frac{\exp[ik_0 r - \frac{i}{\hbar}E_0 t]}{r}F(|\vec{r}| - v_0 t),$$

where $f(\theta)$ is the scattering amplitude.

(c) Show that the scattering occurs only in the time interval

$$|t| < \frac{D}{v_0}.$$

(d) Show that prior to the scattering,

$$t < -D/v_0,$$

we have

$$\langle \vec{r} \,|\psi(t)\rangle = \langle \vec{r} \,|\phi(t)\rangle \,,$$

and that the scattered wave is present only after the scattering,

$$t > D/v_0.$$

Hint for Problem 2.22: We cite the following article for the description of the time-dependent scattering problem.

Low, F.E.: Brandeis Summer School Lectures. (1959).

2.23. Correct the error in the following proof:

$$T_{ba} = (\Phi_b, V\Psi_a^{(+)}) = (\Phi_b, (H - H_0)\Psi_a^{(+)}) = (E_a - E_b)(\Phi_b, \Psi_a^{(+)}).$$

Hence

$$T_{ba} = 0 \quad \text{when} \quad E_a = E_b.$$

Hint: Use the L–S equation for $\Psi_a^{(+)}$ in the last term of the above expression.

2.24. Consider the general properties of the S matrix. The S matrix

$$\left\langle \vec{k}' \middle| S \middle| \vec{k} \right\rangle = \left\langle \vec{k}'^{(-)} \middle| \vec{k}^{(+)} \right\rangle = \int d^3 r \left\langle \vec{k}'^{(-)} \middle| \vec{r} \right\rangle \left\langle \vec{r} \middle| \vec{k}^{(+)} \right\rangle$$

for the scattering of a spinless particle by a central force potential has the following properties:

(i) it is unitary,

$$S^\dagger S = SS^\dagger = 1,$$

(ii) it is diagonal in the energy,

$$S \sim \delta(E_k - E_{k'}),$$

(iii) it is a scalar, that is, a function of $\vec{k}^2$, $\vec{k}'^2$, and $\vec{k}\cdot\vec{k}'$ only.

(a) Show that the general form of a matrix satisfying (i), (ii), and (iii) is

$$\left\langle \vec{k}' \middle| S \middle| \vec{k} \right\rangle = \frac{\delta(k' - k)}{4\pi k' k} \sum_l (2l+1) P_l(\hat{\vec{k}}' \cdot \hat{\vec{k}}) \exp[2i\delta_l(k)].$$

For the proof, use the fact that the simplest unitary matrix satisfying (i), (ii), and (iii) is the unit matrix,

$$\left\langle \vec{k}' \middle| \vec{k} \right\rangle = \delta(\vec{k}' - \vec{k}) = \frac{\delta(k' - k)}{4\pi k' k} \sum_l (2l+1) P_l(\hat{\vec{k}}' \cdot \hat{\vec{k}})$$
$$= \frac{\delta(k' - k)}{k' k} \sum_{l,m} Y_{lm}(\hat{\vec{k}}') Y^*_{lm}(\hat{\vec{k}}).$$

(b) Show that S is a symmetric matrix,

$$\left\langle \vec{k}' \middle| S \middle| \vec{k} \right\rangle = \left\langle \vec{k} \middle| S \middle| \vec{k}' \right\rangle.$$

For the proof of this, you need to consider the rotational invariance and the time reversal relation,

$$\left\langle \vec{r} \middle| \vec{k}^{(+)} \right\rangle = \left\langle \vec{r} \middle| -\vec{k}^{(-)} \right\rangle^* \equiv \left\langle -\vec{k}^{(-)} \middle| \vec{r} \right\rangle.$$

2.25. Consider the scattering cross section for spin-1/2 particle. In this problem, we obtain the differential cross section $\sigma(\theta)$ and total cross section σ_{tot} of spin-1/2 particle off the spin-dependent potential of the form

$$V(r) = V_0(r) + V_{LS}(r)(\vec{L}\cdot\vec{\sigma}).$$

(a) We first construct the scattering amplitude $\langle m_{s'} | f(\theta) | m_s \rangle$, where $|m_s\rangle$ is the eigenstate of $\vec{\sigma}_{\text{op}}$ and $P_l(\cos\theta)$ is the eigenfunction of $(\vec{L}_{\text{op}})^2$,

$$(\vec{L}_{\text{op}})^2 P_l(\cos\theta) = l(l+1) P_l(\cos\theta).$$

Out of $P_l(\cos\theta)\,|m_s\rangle$ which is the eigenstate of $(\vec{L}_{\text{op}})^2$ and $\vec{\sigma}_{\text{op}}$, we select the eigenkets of $(\vec{L}\cdot\vec{\sigma})$ for which the potential is diagonal,

$$\vec{J} \equiv \vec{L} + \frac{1}{2}\vec{\sigma} \qquad \vec{J}^2 = \vec{L}^2 + \frac{3}{4} + (\vec{L}\cdot\vec{\sigma}).$$

When $\vec{J}^2$ and $\vec{L}^2$ are diagonal, so is $(\vec{L}\cdot\vec{\sigma})$,

$$\begin{aligned} j = l + \tfrac{1}{2}: \quad (\vec{L}\cdot\vec{\sigma})\,\left|l, j = l + \tfrac{1}{2}\right\rangle &= l\left|l, j = l + \tfrac{1}{2}\right\rangle, \\ j = l - \tfrac{1}{2}: \quad (\vec{L}\cdot\vec{\sigma})\,\left|l, j = l - \tfrac{1}{2}\right\rangle &= -(l+1)\left|l, j = l - \tfrac{1}{2}\right\rangle. \end{aligned}$$

Show that the projection operators $P_{l,j}$ that project the state of the definite j and definite l out of $P_l(\cos\theta)\,|m_s\rangle$ are given by

$$P_{l,j=l+1/2} = \frac{l + 1 + (\vec{L}\cdot\vec{\sigma})}{2l+1}, \qquad P_{l,j=l-1/2} = \frac{l - (\vec{L}\cdot\vec{\sigma})}{2l+1},$$

which satisfy the property of the projection operator,

$$P_{l,j}P_{l,j'} = \delta_{jj'}P_{l,j}, \qquad \sum_j P_{l,j} = 1.$$

(b) For the complete set of the basic kets $P_{l,j}P_l(\cos\theta)\,|m_s\rangle$, $(\vec{L}\cdot\vec{\sigma})$ is diagonal now. The matrix element of the S matrix is given by

$$\begin{aligned} &\left\langle \vec{k}', m_s' \right| S \left| \vec{k}, m_s \right\rangle \\ &= \frac{\delta(k' - k)}{4\pi k' k} \sum_l (2l+1) \sum_j \exp[2i\delta_{l,j}] \left\langle m_s' \right| P_{l,j}P_l(\cos\theta) \left| m_s \right\rangle. \end{aligned}$$

By definition of the T matrix and the scattering amplitude $f(\theta)$, we have

$$\begin{aligned} &\left\langle \vec{k}', m_s' \right| (S - 1) \left| \vec{k}, m_s \right\rangle \\ &= -2\pi i\delta(E_{k'} - E_k) \left\langle \vec{k}', m_s' \right| T \left| \vec{k}, m_s \right\rangle \\ &= i\frac{\delta(E_{k'} - E_k)}{2\pi k'} \left\langle \vec{k}', m_s' \right| f(\theta) \left| \vec{k}, m_s \right\rangle. \end{aligned}$$

Show that the scattering amplitude $f(\theta)$ is given by

$$\begin{aligned} &\left\langle \vec{k}', m_s' \right| f(\theta) \left| \vec{k}, m_s \right\rangle \\ &= \frac{1}{k} \sum_l (2l+1) \sum_j \frac{\exp[2i\delta_{l,j}] - 1}{2i} \left\langle m_s' \right| P_{l,j}P_l(\cos\theta) \left| m_s \right\rangle \\ &= \frac{1}{k} \sum_l (2l+1) \sum_j \exp[i\delta_{l,j}] \sin\delta_{l,j} \left\langle m_s' \right| P_{l,j}P_l(\cos\theta) \left| m_s \right\rangle. \end{aligned}$$

(c) Upon substitution of the explicit form of $P_{l,j}$ obtained above, show that

$$\left\langle \vec{k}', m_s' \middle| f(\theta) \middle| \vec{k}, m_s \right\rangle$$
$$= \frac{1}{k}\sum_l \{(l+1)\exp[i\delta_{l+}]\sin\delta_{l+} + l\exp[i\delta_{l-}]\sin\delta_{l-}\}P_l(\cos\theta)\left\langle m_s' \middle| m_s \right\rangle$$
$$+ \frac{1}{k}\sum_l \{\exp[i\delta_{l+}]\sin\delta_{l+} - \exp[i\delta_{l-}]\sin\delta_{l-}\}(\vec{L}P_l(\cos\theta))\left\langle m_s' \middle| \vec{\sigma} \middle| m_s \right\rangle,$$

which is the spin-matrix element of the spin-space operator $F(\theta)$,

$$F(\theta) = 1 \cdot f_{\mathrm{N}}(\theta) + \vec{\sigma} \cdot (\vec{L}f_{\mathrm{S}}(\theta)).$$

(d) Show that the non-spin-flip amplitude $f_{\mathrm{N}}(\theta)$ and the spin-flip amplitude $f_{\mathrm{S}}(\theta)$ are, respectively, given by

$$f_{\mathrm{N}}(\theta) = \frac{1}{k}\sum_l \{(l+1)\exp[i\delta_{l+}]\sin\delta_{l+} + l\exp[i\delta_{l-}]\sin\delta_{l-}\}P_l(\cos\theta),$$
$$f_{\mathrm{S}}(\theta) = \frac{1}{k}\sum_l \{\exp[i\delta_{l+}]\sin\delta_{l+} - \exp[i\delta_{l-}]\sin\delta_{l-}\}P_l(\cos\theta).$$

(e) Show that the final spin-averaged differential cross section $\sigma(\theta)$ is given by, for the unpolarized beam of spin-1/2 particle,

$$\sigma(\theta) = \frac{1}{2}\sum_{m_s'}\sum_{m_s} \left\langle m_s' \middle| f(\theta) \middle| m_s \right\rangle^2$$
$$= \frac{1}{2}\sum_{m_s'}\sum_{m_s} \left\langle m_s' \middle| f(\theta) \middle| m_s \right\rangle^* \left\langle m_s' \middle| f(\theta) \middle| m_s \right\rangle$$
$$= \frac{1}{2}\sum_{m_s'}\sum_{m_s} \left\langle m_s \middle| f^\dagger(\theta) \middle| m_s' \right\rangle \left\langle m_s' \middle| f(\theta) \middle| m_s \right\rangle$$
$$= \frac{1}{2}\sum_{m_s} \left\langle m_s \middle| f^\dagger(\theta) f(\theta) \middle| m_s \right\rangle = \frac{1}{2}\mathrm{tr}F^\dagger(\theta)F(\theta),$$

where $F(\theta)$ and $F^\dagger(\theta)$ with the dagger $\dagger$ in spin space are given by

$$F(\theta) = 1 \cdot f_{\mathrm{N}}(\theta) + \vec{\sigma} \cdot (\vec{L}f_{\mathrm{S}}(\theta)),$$
$$F^\dagger(\theta) = 1 \cdot f_{\mathrm{N}}^*(\theta) + \vec{\sigma} \cdot (\vec{L}f_{\mathrm{S}}(\theta))^*.$$

(f) Making use of the identity $(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \vec{a}\cdot\vec{b} + i\vec{\sigma}\cdot(\vec{a}\times\vec{b})$, $\mathrm{tr}\vec{\sigma} = 0$, and $\mathrm{tr}1 = 2$, show that

$$\sigma(\theta) = \frac{1}{2}\mathrm{tr}F^\dagger(\theta)F(\theta) = f_{\mathrm{N}}^*(\theta)f_{\mathrm{N}}(\theta) + (\vec{L}f_{\mathrm{S}}(\theta))^*(\vec{L}f_{\mathrm{S}}(\theta)).$$

(g) Plugging in the explicit form of $f_N(\theta)$ and $f_S(\theta)$ obtained as above into $\sigma(\theta)$, show that

$$\sigma(\theta) = \frac{1}{k^2}\sum_{ll'}\{(l+1)\exp[i\delta_{l+}]\sin\delta_{l+} + l\exp[i\delta_{l-}]\sin\delta_{l-}\}^* \times \{(l'+1)\exp[i\delta_{l'+}]\sin\delta_{l'+} + l'\exp[i\delta_{l'-}]\sin\delta_{l'-}\}P_l^*(\cos\theta)P_{l'}(\cos\theta) + \frac{1}{k^2}\sum_{ll'}\{\exp[i\delta_{l+}]\sin\delta_{l+} - \exp[i\delta_{l-}]\sin\delta_{l-}\}^* \times \{\exp[i\delta_{l'+}]\sin\delta_{l'+} - \exp[i\delta_{l'-}]\sin\delta_{l'-}\}(\vec{L}P_l(\cos\theta))^* (\vec{L}P_{l'}(\cos\theta)).$$

(h) Identities due to orthonormality of spherical harmonics are given by

$$\int d\Omega P_l^*(\cos\theta)P_{l'}(\cos\theta) = \frac{4\pi}{2l+1}\delta_{ll'},$$
$$\int d\Omega(\vec{L}P_l(\cos\theta))^*(\vec{L}P_{l'}(\cos\theta)) = \frac{4\pi}{2l+1}\delta_{ll'}l(l+1).$$

With the use of identities stated above, show that the total cross section σ_{tot} is given by

$$\sigma_{tot} = \int d\Omega\sigma(\theta) = \frac{4\pi}{k^2}\sum_l\{(l+1)\sin^2\delta_{l+} + l\sin^2\delta_{l-}\}.$$

(i) Show that the imaginary part of the forward scattering amplitude is given by

$$\mathrm{Im} f_N(\theta = 0) = \frac{1}{k}\sum_l\{(l+1)\sin^2\delta_{l+} + l\sin^2\delta_{l-}\}.$$

Show finally the optical theorem for spin-1/2 particle. Namely, show that

$$\sigma_{tot} = \frac{4\pi}{k}\,\mathrm{Im} f_N(\theta = 0).$$

2.26. Consider the polarization and the density matrix of spin-1/2 particle in the scattering process discussed in Problem 2.25. Two-state spin systems are represented by $|\alpha\rangle$ and $|\beta\rangle$. They satisfy

$$\langle\alpha|\beta\rangle = 0, \quad \langle\alpha|\alpha\rangle = \langle\beta|\beta\rangle = 1.$$

The density matrix ρ in the $\alpha-\beta$ representation is given by

$$\rho = |\alpha\rangle p_\alpha\langle\alpha| + |\beta\rangle p_\beta\langle\beta| \quad \text{with} \quad p_\alpha + p_\beta = 1,$$

where p_α (p_β) is the probability of finding the particle in the state $|\alpha\rangle$ ($|\beta\rangle$).

(a) Show Hermiticity of the density matrix ρ. Show also that

$$\mathrm{tr}\rho = 1, \quad \mathrm{tr}\rho^2 \leq 1.$$

(b) Unique parameterization of the density matrix ρ is given by

$$\rho = \frac{1}{2}(1 + \varepsilon(\vec{\sigma} \cdot \hat{n})), \quad \varepsilon \geq 0, \quad \hat{n} \cdot \hat{n} = 1.$$

Show that the size of ε is given by $0 \leq \varepsilon \leq 1$.

(c) We define the average of $\vec{\sigma}$ by

$$\langle \vec{\sigma} \rangle = \mathrm{tr}(\vec{\sigma}\rho).$$

Show that

$$\langle \vec{\sigma} \rangle = \varepsilon \hat{n}.$$

(d) Consider the scattering where the density matrix $\rho^{(\mathrm{i})}$ of the incident beam is defined by

$$\rho^{(\mathrm{i})} = \sum_{m_S^{(\mathrm{i})}} \left| m_S^{(\mathrm{i})} \right\rangle p_{i,m_S^{(\mathrm{i})}} \left\langle m_S^{(\mathrm{i})} \right|,$$

where $p_{i,m_S^{(\mathrm{i})}}$ is the probability of the incident beam in the state $\left| m_S^{(\mathrm{i})} \right\rangle$. Show that the density matrix $\rho^{(\mathrm{S})}(\theta)$ for the scattered beam is given by

$$\rho^{(\mathrm{S})}(\theta) = \frac{F(\theta)\rho^{(\mathrm{i})}F^\dagger(\theta)}{\mathrm{tr}(F(\theta)\rho^{(\mathrm{i})}F^\dagger(\theta))}.$$

(e) For an unpolarized incident beam,

$$\rho^{(\mathrm{i})} = \frac{1}{2}\mathbf{1},$$

show that the average of $\vec{\sigma}$ of the scattered beam is given by

$$\langle \vec{\sigma} \rangle_{\mathrm{Scattered}} = \mathrm{tr}(\vec{\sigma}\rho^{(\mathrm{S})}(\theta)) = \frac{2\,\mathrm{Re}\{f_\mathrm{N}^*(\theta)(\vec{L}f_\mathrm{S}(\theta))\}}{f_\mathrm{N}^*(\theta)f_\mathrm{N}(\theta) + (\vec{L}f_\mathrm{S}(\theta))^*(\vec{L}f_\mathrm{S}(\theta))}.$$

(f) For an unpolarized incident beam, in the first-order Born approximation, show that

$$f_{\mathrm{N}}^{\mathrm{Born}}(\theta) = -\frac{4\pi^2 m}{\hbar^2}\left[\int \frac{d^3\vec{r}}{(2\pi)^3}\exp[-i\vec{q}\cdot\vec{r}]V_0(r)\right]_{\vec{q}=\vec{k}'-\vec{k}},$$

$$\vec{L}f_{\mathrm{S}}^{\mathrm{Born}}(\theta) = -\frac{4\pi^2 m}{\hbar^2} i(\vec{k}' \times \vec{k}) \times \left[\frac{1}{q}\frac{d}{dq}\left[\int \frac{d^3\vec{r}}{(2\pi)^3}\exp[-i\vec{q}\cdot\vec{r}]V_{LS}(r)\right]\right]_{\vec{q}=\vec{k}'-\vec{k}}.$$

Namely, $f_{\mathrm{N}}^{\mathrm{Born}}(\theta)$ is real and $\vec{L}f_{\mathrm{S}}^{\mathrm{Born}}(\theta)$ is pure imaginary. Show also that

$$\langle\vec{\sigma}\rangle_{\mathrm{Scattered}}^{\mathrm{Born}} = 0, \quad \sigma_{\mathrm{tot}}^{\mathrm{Born}} = \frac{4\pi}{k}\,\mathrm{Im} f_{\mathrm{N}}^{\mathrm{Born}}(\theta = 0) = 0.$$

These results clearly show that the first-order Born approximation is not good enough in the presence of the spin–orbit coupling.

2.27. Consider the one pion-exchange potential. The one pion-exchange potential in the static limit has the form

$$V(\vec{r}) = g^2(\vec{\tau}^{(1)}\cdot\vec{\tau}^{(2)})\frac{1}{\mu^2}(\vec{\sigma}^{(1)}\cdot\vec{\nabla})(\vec{\sigma}^{(2)}\cdot\vec{\nabla})\frac{\exp[-\mu r]}{r}, \tag{A}$$

where g^2 is the coupling constant,

$$\frac{g^2}{\hbar c} = 0.08,$$

and μ is the inverse pion Compton wavelength,

$$\mu = \frac{m_\pi c}{\hbar}, \quad \text{or} \quad \hbar c\mu = m_\pi c^2.$$

(a) In carrying out the derivatives on $\exp[-\mu r]/r$, rewrite $V(\vec{r})$ as

$$V(\vec{r}) = \frac{g^2}{\hbar c}(m_\pi c^2)(\vec{\tau}^{(1)}\cdot\vec{\tau}^{(2)})\left(\vec{\sigma}^{(1)}\cdot\frac{\vec{\nabla}}{\mu}\right)\left(\vec{\sigma}^{(2)}\cdot\frac{\vec{\nabla}}{\mu}\right)\frac{\exp[-\mu r]}{\mu r},$$

and introduce the dimensionless variable $\vec{\rho}$ as

$$\vec{\rho} = \mu\vec{r}, \quad \frac{\vec{\nabla}}{\mu} = \frac{\partial}{\partial(\mu\vec{r})} = \vec{\nabla}_\rho.$$

Recalling the identity

$$\nabla_{\rho_k} = \frac{df}{d\rho}\frac{d\rho}{d\rho_k} = \frac{\rho_k}{\rho}(\frac{df}{d\rho}),$$

and that $\exp[-\rho]/\rho$ satisfies the Klein–Gordon equation with the source term, $4\pi\delta^3(\vec{\rho})$,

$$(-\vec{\nabla}_\rho^2 + 1)\frac{\exp[-\rho]}{\rho} = 4\pi\delta^3(\vec{\rho}),$$

show that, for all ρ, we have

$$\begin{aligned}&(\vec{\sigma}^{(1)} \cdot \vec{\nabla}_\rho)(\vec{\sigma}^{(2)} \cdot \vec{\nabla}_\rho)\frac{\exp[-\rho]}{\rho}\\ &= \frac{1}{3}\left\{(\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)}) + (3(\vec{\sigma}^{(1)} \cdot \hat{\rho})(\vec{\sigma}^{(2)} \cdot \hat{\rho}) - (\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)}))\left(1 + \frac{3}{\rho} + \frac{3}{\rho^2}\right)\right\}\\ &\times \frac{\exp[-\rho]}{\rho} - \frac{1}{3}(\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)}) \cdot 4\pi\delta^3(\vec{\rho}).\end{aligned}$$

(b) Writing the tensor operator S_{12} as $S_{12} \equiv 3(\vec{\sigma}^{(1)} \cdot \hat{\rho})(\vec{\sigma}^{(2)} \cdot \hat{\rho}) - (\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)})$, show that

$$\begin{aligned}V(\vec{r}) &= \frac{g^2}{\hbar c}(m_\pi c^2)\frac{1}{3}(\vec{\tau}^{(1)} \cdot \vec{\tau}^{(2)})\left\{(\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)}) + S_{12}\left(1 + \frac{3}{\mu r} + \frac{3}{(\mu r)^2}\right)\right\}\\ &\times \frac{\exp[-\mu r]}{\mu r} \qquad \text{(B)}\\ &- \frac{g^2}{\hbar c}(m_\pi c^2)\frac{1}{3}(\vec{\tau}^{(1)} \cdot \vec{\tau}^{(2)})(\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)}) \cdot 4\pi\delta^3(\mu\vec{r}). \qquad \text{(C)}\end{aligned}$$

(c) The scattering amplitude, in the first-order Born approximation, is given by the Fourier transform

$$f(\vec{q}) = -\frac{4\pi^2 m_{\text{red}}}{\hbar^2}\int \frac{d^3\vec{r}}{(2\pi)^3}\exp[-i\vec{q} \cdot \vec{r}]\{V(\vec{r}) - \text{(C)}\},$$

where $\vec{q}$ is the momentum transfer given by $\vec{q} = \vec{k}' - \vec{k}$ and m_{red} is the reduced mass of the two-nucleon system, $m_{\text{red}} = m/2$. Show that the term (C) is to be subtracted from $V(\vec{r})$. Obtain the scattering amplitude as

$$\begin{aligned}f(\vec{q}) &= +\frac{g^2}{\hbar c}\frac{1}{\mu^2}\left(\frac{m_\pi c}{\hbar}\right)\left(\frac{m}{m_\pi}\right)(\vec{\tau}^{(1)} \cdot \vec{\tau}^{(2)})\\ &\times \left\{\frac{(\vec{\sigma}^{(1)} \cdot \vec{q})(\vec{\sigma}^{(2)} \cdot \vec{q})}{\mu^2 + \vec{q}^2} - \frac{1}{3}(\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)})\right\}.\end{aligned}$$

Setting

$$\begin{aligned}\vec{S} &= \frac{1}{2}(\vec{\sigma}^{(1)} + \vec{\sigma}^{(2)}),\\ \Sigma_{12}(\vec{q}) &\equiv 2\left[(\vec{S} \cdot \vec{q})^2 - \frac{1}{3}\vec{S}^2 \cdot \vec{q}^2\right] = (\vec{\sigma}^{(1)} \cdot \vec{q})(\vec{\sigma}^{(2)} \cdot \vec{q}) - \frac{1}{3}\vec{q}^2(\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)})\\ &= \frac{1}{3}S_{12}(\hat{q})q^2,\end{aligned}$$

obtain the final form of the scattering amplitude as

$$f(\vec{q}) = +\frac{g^2}{\hbar c}\left(\frac{\hbar}{m_\pi c}\right)\left(\frac{m}{m_\pi}\right)\frac{1}{\mu^2+\vec{q}^2}(\vec{\tau}^{(1)}\cdot\vec{\tau}^{(2)})$$
$$\times\,(\Sigma_{12}(\vec{q}) - \frac{1}{3}\mu^2(\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)})).$$

(d) Consider the isospin factor, $(\vec{\tau}^{(1)}\cdot\vec{\tau}^{(2)})$. By the Pauli principle, we have Spin-triplet:

$$S=1 \quad \text{or} \quad (\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)}) = 1: \quad (\vec{\tau}^{(1)}\cdot\vec{\tau}^{(2)}) = \begin{cases} -3 & \text{for } l \text{ even,} \\ +1 & \text{for } l \text{ odd,} \end{cases}$$

Spin-singlet:

$$S=0 \quad \text{or} \quad (\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)}) = -3: \quad (\vec{\tau}^{(1)}\cdot\vec{\tau}^{(2)}) = \begin{cases} +1 & \text{for } l \text{ even,} \\ -3 & \text{for } l \text{ odd.} \end{cases}$$

Show that we can split any Born amplitude into even-parity and odd-parity parts,

$$f_{\text{even}} = \frac{1}{2}(f(\vec{q}) + f(\vec{q}^{\,'})), \quad f_{\text{odd}} = \frac{1}{2}(f(\vec{q}) - f(\vec{q}^{\,'}));$$
$$\vec{q}^{\,'} = (-\vec{k}^{'}) - \vec{k} = -(\vec{k}^{'} + \vec{k}).$$

Obtain the spin-triplet state ($S=1$) N–N scattering amplitude as

$$f_{S=1}(\vec{q},\vec{q}^{\,'}) = f_{\text{even}} + f_{\text{odd}} = \underbrace{\frac{1}{2}(f(\vec{q}) + f(\vec{q}^{\,'}))}_{\text{iso-singlet}} + \underbrace{\frac{1}{2}(f(\vec{q}) - f(\vec{q}^{\,'}))}_{\text{iso-triplet}}$$
$$= -\frac{g^2}{\hbar c}\left(\frac{\hbar}{m_\pi c}\right)\left(\frac{m}{m_\pi}\right)$$
$$\left\{\frac{1}{\mu^2+\vec{q}^2}\left(\Sigma_{12}(\vec{q}) - \frac{1}{3}\mu^2\right) + \frac{2}{\mu^2+\vec{q}^{\,'2}}\left(\Sigma_{12}(\vec{q}^{\,'}) - \frac{1}{3}\mu^2\right),\right\}$$

and the spin-singlet state ($S=0$) N–N scattering amplitude as

$$f_{S=0}(\vec{q},\vec{q}^{\,'}) = f_{\text{even}} + f_{\text{odd}} = \underbrace{\frac{1}{2}(f(\vec{q}) + f(\vec{q}^{\,'}))}_{\text{iso-triplet}} + \underbrace{\frac{1}{2}(f(\vec{q}) - f(\vec{q}^{\,'}))}_{\text{iso-singlet}}$$
$$= -\frac{g^2}{\hbar c}\left(\frac{\hbar}{m_\pi c}\right)\left(\frac{m}{m_\pi}\right)\left\{\frac{\mu^2}{\mu^2+\vec{q}^2} - \frac{2\mu^2}{\mu^2+\vec{q}^{\,'2}}\right\}.$$

(e) In order to obtain an expression for the n–p scattering differential cross section, show first that

$$(\Sigma_{12}(\vec{q}))^2 = \frac{8}{9}q^4 - \frac{2}{3}q^2\Sigma_{12}(\vec{q}), \quad \langle \Sigma_{12}(\vec{q}) \rangle = 0,$$
$$\langle (\Sigma_{12}(\vec{q}))^2 \rangle = \frac{8}{9}q^4, \quad \left\langle \Sigma_{12}(\vec{q})\Sigma_{12}(\vec{q}^{\,\prime}) \right\rangle = -\frac{4}{9}q^2 q'^2.$$

Obtain an explicit expression for the n–p scattering differential cross section,

$$\sigma_{\text{n-p}} = \frac{3}{4}\langle f^2 \rangle_{S=1} + \frac{1}{4}\langle f^2 \rangle_{S=0}.$$

(f) In the case of the p–p scattering, show that

$$f_{S=1} = 2f_{\text{odd}}, \quad f_{S=0} = 2f_{\text{even}}.$$

On this basis, obtain an explicit expression for the p–p scattering differential cross section, $\sigma_{\text{p-p}}$.

Hint: To Fourier transform the potential, the form (A) is preferable to the form (B). The derivation of one pion-exchange potential from the pseudoscalar meson theory is given as a problem in Chapter 10.

2.28. Consider a scattering of two identical particles.

(a) Consider a scattering between two identical spin 0 bosons in the center-of-mass frame. Show that a scattering can be described by the symmetrized scattering amplitude,

$$f(\vec{k}^{\prime}, \vec{k}) = f(\vec{k}^{\prime}, \vec{k}) + f(-\vec{k}^{\prime}, \vec{k}),$$

and hence the differential cross section is given by

$$\frac{d\sigma}{d\Omega} = \left| f(\vec{k}^{\prime}, \vec{k}) + f(-\vec{k}^{\prime}, \vec{k}) \right|^2.$$

In terms of the wave number k and the scattering angle θ, show that

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \left| f(k,\theta) + f(k,\pi-\theta) \right|^2 \\ &= \left| f(k,\theta) \right|^2 + \left| f(k,\pi-\theta) \right|^2 + 2\,\mathrm{Re} f(k,\theta) f^*(k,\pi-\theta). \qquad \text{(B-B)}\end{aligned}$$

Consider the partial wave expansion of $f(k,\theta)$. Show that

$$\frac{d\sigma}{d\Omega} = \frac{4}{k^2}\left| \sum_{l=0,2,4,\ldots} \sqrt{4\pi(2l+1)}\exp[i\delta_l(k)]\sin\delta_l(k)Y_{l0}(\theta) \right|^2.$$

Namely, the symmetry requirement has eliminated all partial waves of odd angular momentum.

(b) Consider now the scattering of two identical fermions with spin 1/2. The scattering amplitude is a matrix in the four-dimensional spin space of the two identical particles,

$$M(\vec{k}', \vec{k}) = M_s(\vec{k}', \vec{k}) + M_t(\vec{k}', \vec{k}),$$

where s and t refer to singlet and triplet spin states, respectively. Show that the correctly antisymmetrized amplitude is given by

$$\mathbf{M}(\vec{k}', \vec{k}) = [M_s(\vec{k}', \vec{k}) + M_s(-\vec{k}', \vec{k})] + [M_t(\vec{k}', \vec{k}) - M_t(-\vec{k}', \vec{k})].$$

When the initial state has the spin-space density matrix ρ_i, show that

$$\frac{d\sigma}{d\Omega} = \mathrm{tr}\rho_i \mathbf{M}^\dagger \mathbf{M}.$$

If neither the beam nor the target is polarized, $\rho_i = 1/4$. Show that the two pieces M_s and M_t can be obtained by use of the projection operators P_s and P_t that project onto the singlet and triplet states, respectively,

$$P_s = (1/4)(1 - \vec{\sigma}_1 \cdot \vec{\sigma}_2), \quad P_t = (1/4)(3 + \vec{\sigma}_1 \cdot \vec{\sigma}_2),$$

with the properties, $P_s P_t = 0$, $P_s + P_t = 1$; $\mathrm{tr}P_s = 1$, and $\mathrm{tr}P_t = 3$. Show that the triplet and singlet scattering amplitudes are given by

$$M_t = P_t \mathbf{M} P_t, \quad M_s = P_s \mathbf{M} P_s.$$

Consider the simple situation when the potential has the form

$$V = V_1(r) + \vec{\sigma}_1 \cdot \vec{\sigma}_2 V_2(r).$$

Show that $\mathbf{M}$ can be expressed as

$$\mathbf{M} = f_s(k, \theta) P_s + f_t(k, \theta) P_t,$$

where $f_{s,t}$ are scalars in the spin space. Show that the differential cross section becomes

$$\frac{d\sigma}{d\Omega} = |f_s(k, \theta) + f_s(k, \pi - \theta)|^2 \, \mathrm{tr}\rho_i P_s + |f_t(k, \theta) - f_t(k, \pi - \theta)|^2 \, \mathrm{tr}\rho_i P_t.$$

For a completely unpolarized initial state ($\rho_i = 1/4$), show that

$$\frac{d\sigma}{d\Omega} = \frac{1}{4} |f_s(k, \theta) + f_s(k, \pi - \theta)|^2 + \frac{3}{4} |f_t(k, \theta) - f_t(k, \pi - \theta)|^2 .$$

If the force potentials are completely spin independent, set $V_2 = 0$, and $f_s = f_t = f$. Show that the differential cross section is given by

$$\frac{d\sigma}{d\Omega} = |f(k,\theta)|^2 + |f(k,\pi-\theta)|^2 - \mathrm{Re} f^*(k,\theta) f(k,\pi-\theta). \qquad \text{(F-F)}$$

(c) Consider the Coulomb scattering of two identical particles. Applying the differential cross section formulas, (B-B) and (F-F), for the Coulomb scattering, obtain the differential cross section for each case.

Comment on Problem 2.28: In order to avoid the double counting of the scattered particles at the detector, restrict the scattering angle θ to $0 \le \theta \le \pi/2$.

2.29. Consider the Breit–Wigner resonance scattering cross section.

(a) The eigenfunctions for the noninteracting particles will be written as $\chi_{s,\varepsilon}$; with K as the free Hamiltonian, these are assumed to satisfy the Schrödinger equation,

$$K\chi_{s,\varepsilon} = \varepsilon\chi_{s,\varepsilon}.$$

The energy ε is chosen as one of the quantum variables labeling the state, the remaining state labels being written as s. We suppose that the operators whose eigenvalues are the quantities s commute with the full Hamiltonian $H = K + V$. Then the scattering eigenfunctions are written as

$$H\psi^+_{s,\varepsilon} = \varepsilon\psi^+_{s,\varepsilon}.$$

The initial state wave packet is written as

$$\phi_s(t) = \sum_\varepsilon A_\varepsilon \chi_{s,\varepsilon} \exp[-i\varepsilon t], \qquad \text{(A)}$$

where A_ε is a suitable wave packet amplitude. The wavefunction φ_s is normalized to unity,

$$(\varphi_s, \varphi_s) = 1, \quad \sum_\varepsilon |A_\varepsilon|^2 = 1.$$

The time-dependent wavefunction for the system $\Psi_s(t)$ is chosen to coincide with $\varphi_s(t)$ at the time $t = -T$ at which the experiment is begun. This is

$$\Psi_s(t) = \sum_\varepsilon A_\varepsilon \psi_{s,\varepsilon} \exp[-i\varepsilon t], \quad (\Psi_s(t), \Psi_s(t)) = 1. \qquad \text{(B)}$$

Finally, we make the special assumption that the representation s is such that in it the S-matrix is diagonal. Then our experiment is one in which the incident wave is an incoming spherical packet collapsing at a point where the interaction is to occur. The scattered wave will be an outgoing spherical packet expanding away from the local region of interaction.

Consider now a large volume S_R in the configuration space, which is characterized by a radius $\mathcal{R}$ and is sufficiently large that the particles do not interact when lying outside S_R. T is so large that the wave packets lie outside S_R with arbitrary precision when $t < -T$ or $t > T$.

Then, the "lifetime" Q of the scattering state is defined by the equation

$$Q = \lim_{\mathcal{R}\to\infty} \lim_{T\to\infty} \int_{-T}^{T} dt \int_{S_R} d\tau [\Psi_s^*(t)\Psi_s(t) - \varphi_s^*(t)\varphi_s(t)]. \tag{C}$$

Here, $d\tau$ is a volume element in the configuration space and the integral is confined to the domain S_R. The limit $\mathcal{R} \to \infty$ is understood to imply that in the limit the integral extends over the entire configuration space. Q represents the additional time, over and above the "free flight time," that the particles spend in the range of their interaction. If $Q > 0$, the particles tend to "stick together," whereas if $Q < 0$, the interaction tends to "force them apart."

(b) Using the wave packet expansions, (A) and (B), show that

$$Q = \lim_{\mathcal{R}\to\infty} 2\pi \times \left\{ \sum_{\varepsilon} |A_\varepsilon|^2 \int_{S_R} d\tau \sum_{\varepsilon'} [\psi_{s,\varepsilon'}^{+*} \delta(\varepsilon - H)\psi_{s,\varepsilon'}^{+} - \chi_{s,\varepsilon'}^{*} \delta(\varepsilon - K)\chi_{s,\varepsilon'}] \right\}. \tag{D}$$

We have the relations

$$\begin{aligned} \delta(\varepsilon - K) &= \frac{i}{2\pi}[G_0^+(\varepsilon) - G_0^-(\varepsilon)], \quad \delta(\varepsilon - H) \\ &= \frac{i}{2\pi}\left[G^+(\varepsilon) - G^-(\varepsilon)\right], \\ G_0^{\pm}(\varepsilon) &= \frac{1}{\varepsilon \pm i\eta - K}, \quad G^{\pm}(\varepsilon) = \frac{1}{\varepsilon \pm i\eta - H}. \end{aligned} \tag{E}$$

When the relations (E) are substituted into (D), show that

$$Q = i\sum_{\varepsilon} |A_\varepsilon|^2 \operatorname{tr}[G^+(\varepsilon) - G^-(\varepsilon) - G_0^+(\varepsilon) + G_0^-(\varepsilon)]. \tag{F}$$

By the symbol $\operatorname{tr}\{\cdots\}$ here, we mean only a partial trace since the variables s are held constant and the sum runs over only the energy ε'. Simplify (F) to the form

$$Q = i \sum_{\varepsilon} |A_\varepsilon|^2 \frac{d}{d\varepsilon} \left\{ \ln \det \left[\frac{G^-(\varepsilon) G_0^+(\varepsilon)}{G^+(\varepsilon) G_0^-(\varepsilon)} \right] \right\}.$$

We write

$$G_0^+ [G^+]^{-1} = 1 - G_0^+ V, \quad G_0^- [G^-]^{-1} = 1 - G_0^- V; \quad V \equiv H - K.$$

Then

$$D_G \equiv \det \left[\frac{G_0^- [G^-]^{-1}}{G_0^+ [G^+]^{-1}} \right] = \det \left[(1 - G_0^- V) \frac{1}{1 - G_0^+ V} \right].$$

Now,

$$\frac{1}{1 - G_0^+ V} = 1 + G_0^+ V \frac{1}{1 - G_0^+ V} = 1 + G_0^+ T,$$

where

$$T(\varepsilon) = V \frac{1}{1 - G_0^+(\varepsilon) V}$$

is the scattering matrix. Show that

$$\begin{aligned} D_G &= \det \left[1 + (G_0^+ - G_0^-) V \frac{1}{1 - G_0^+ V} \right] \\ &= \det \left[1 - 2\pi i \delta(\varepsilon - K) T(\varepsilon) \right] \\ &= \det \left[\delta_{\varepsilon'', \varepsilon'} - 2\pi i \delta_{\varepsilon'', \varepsilon} T_{\varepsilon'', \varepsilon'}(\varepsilon) \right]. \end{aligned}$$

Show further that

$$D_G = 1 - 2\pi i T_{\varepsilon, \varepsilon}(\varepsilon) = S(\varepsilon),$$

where $S(\varepsilon)$ is the eigenvalue of the S-matrix for the state (s, ε). Show that

$$Q = -i \sum_{\varepsilon} |A_\varepsilon|^2 \frac{d}{d\varepsilon} \ln S(\varepsilon).$$

For a wave packet of sufficiently small energy spread, show that

$$Q(\varepsilon) = -i \frac{d}{d\varepsilon} \ln S(\varepsilon). \tag{G}$$

Show also that, by writing

$$S = \exp\left[2i\delta(\varepsilon)\right],$$

where $\delta(\varepsilon)$ is a phase shift,

$$Q = 2\frac{d\delta(\varepsilon)}{d\varepsilon} = -iS^{-1}\frac{d}{d\varepsilon}S.$$

(c) Let us consider the most general conditions under which the scattering wavefunction $\psi^{+}_{s,\varepsilon}$ will have the asymptotic form

$$\psi^{+}_{s,\varepsilon} = [\mathcal{I}_{s,\varepsilon} - S_s(\varepsilon)\mathcal{O}_{s,\varepsilon}], \tag{H}$$

when the interacting systems are separated at great distances. Here $\mathcal{I}_{s,\varepsilon}$ represents an incoming wave and $\mathcal{O}_{s,\varepsilon}$ an outgoing wave. We assume that the S-matrix is diagonal in the (s, ε) representation. We do not necessarily suppose that $\psi^{+}_{s,\varepsilon}$ satisfies a Schrödinger equation, however. The wave packet amplitude $A(\varepsilon - \overline{\varepsilon})$ will be considered to be centered about a mean energy $\overline{\varepsilon}$ and to extend over only a narrow range of energy. Then the time-dependent wavefunction is

$$\Psi_s(t) = \sum_{\varepsilon} A(\varepsilon - \overline{\varepsilon})\psi^{+}_{s,\varepsilon} \exp[-i\varepsilon t].$$

The asymptotic outgoing wave packet of this is obtained from (H) as

$$\Psi_{\text{out}}(t) = -\sum_{\varepsilon} A(\varepsilon - \overline{\varepsilon})S(\varepsilon)\mathcal{O}_{s,\varepsilon} \exp[-i\varepsilon t].$$

Show that

$$\begin{aligned} S(\varepsilon) &= \exp[\ln S(\varepsilon)] \\ &= \exp\left[\ln S(\overline{\varepsilon}) + \Delta\varepsilon\frac{d}{d\overline{\varepsilon}}\ln S(\overline{\varepsilon})\right] = S(\overline{\varepsilon})\exp\left[\Delta\varepsilon\frac{d}{d\overline{\varepsilon}}\ln S(\overline{\varepsilon})\right], \end{aligned}$$

with $\Delta\varepsilon \equiv \varepsilon - \overline{\varepsilon}$. We thus have

$$\begin{aligned} \Psi_{\text{out}}(t) = -S(\overline{\varepsilon})\exp[-i\overline{\varepsilon}t]\sum_{\Delta\varepsilon} A(\Delta\varepsilon)\mathcal{O}_{s,\overline{\varepsilon}+\Delta\varepsilon} \\ \exp\left[-i\left(t + i\frac{d}{d\overline{\varepsilon}}\ln S(\overline{\varepsilon})\right)\Delta\varepsilon\right]. \end{aligned}$$

Show that the interaction between the particles leads to a time delay,

$$Q = -i\frac{d}{d\overline{\varepsilon}}\ln S(\overline{\varepsilon}), \tag{I}$$

in the appearance of the asymptotic wave packet.

(d) We label S by the channel c and energy ε only. Then, the S-matrix has the usual form

$$\langle c';\varepsilon' | S | c;\varepsilon\rangle = \delta_{(c';\varepsilon'),(c;\varepsilon)} - 2\pi i\delta(\varepsilon'-\varepsilon)T_{c'c}(\varepsilon)$$
$$= \delta_{\varepsilon',\varepsilon}[\delta_{c',c} - 2\pi i T_{c'c}(\varepsilon)] \equiv \delta_{\varepsilon',\varepsilon} S_{c'c}(\varepsilon).$$

Here $T_{c'c}$ is the element of the T matrix connecting channels c and c'. Since S is unitary, it may be diagonalized. Let it then be diagonal in a representation for which the states are labeled as (s, ε). That is,

$$\langle s',\varepsilon' | S | s,\varepsilon\rangle = \delta_{\varepsilon',\varepsilon}\delta_{s',s}[1-2\pi i T_s(\varepsilon)] \equiv \delta_{\varepsilon',\varepsilon}\delta_{s',s}S_s(\varepsilon).$$

If we write U_{cs} for the unitary matrix which transforms from the s to the c representation, then

$$S_{c'c}(\varepsilon) = \sum_s U_{c's}S_s(\varepsilon)U_{sc}^{-1}, \quad \delta_{s',s}S_s(\varepsilon) = \sum_{c,c'} U_{s'c'}^{-1}S_{c'c}(\varepsilon)U_{cs}.$$

If the scattering in the state under consideration is sufficiently larger than that in the other states, the cross section for scattering from channel c to channel c' is

$$\sigma_{c'c} = \frac{\pi}{\kappa_c^2}\frac{2J+1}{(2S_1+1)(2S_2+1)}\left|S_{c'c}(\varepsilon) - \delta_{c',c}\right|^2.$$

Here κ_c is the momentum of either of the two particles in the incident channel c, whereas S_1 and S_2 are their respective spins.

Let us ask what will be the consequence if the lifetime Q in a particular eigenstate $(s, \varepsilon, j, \ldots)$ has a maximum value at a given energy $\varepsilon = \varepsilon_0$, but that strong scattering does not occur except near $\varepsilon = \varepsilon_0$ in this state. We shall assume that $Q(\varepsilon)$ is large and positive near $\varepsilon = \varepsilon_0$, but decreases to a value of the order of the "free flight time" when ε is not close to ε_0.

A large value of Q for $\varepsilon \approx \varepsilon_0$ means, of course, that the "particles stick together" for an extended time. Show that Q^{-1} can be expanded in a Taylor series of the form

$$\frac{1}{Q} = \alpha + \beta(\varepsilon-\varepsilon_0)^2 + \gamma(\varepsilon-\varepsilon_0)^3 + \cdots, \tag{J}$$

where α, β, γ are constants and $\alpha, \beta > 0$. Normally, we might expect that

$$\beta = O(\alpha/\varepsilon_0^2), \quad \gamma = O(\beta/\varepsilon_0), \ldots.$$

Our assumption of a long lifetime near the energy ε_0 will be interpreted more precisely to mean that

$$\alpha \ll \varepsilon_0^2 \beta, \quad \gamma = O(\beta/\varepsilon_0). \tag{K}$$

The conditions (K) imply that $Q^{-1} \approx \alpha + \beta(\varepsilon - \varepsilon_0)^2$ for a range of energies such that $|\varepsilon - \varepsilon_0| \ll O(\varepsilon_0)$. Let us return to (J) and set $Z \equiv \varepsilon - \varepsilon_0$. We then write

$$\frac{1}{Q} = i\frac{dZ}{d\ln S} = \alpha + \beta Z^2 + F(Z), \quad F(Z) \equiv \gamma Z^3 + \cdots. \tag{L}$$

For convenience, we shall drop the state label s on the S-matrix for the moment. The term $F(Z)$ will be considered as small over the range of Z used.

Integrate (L) and show that

$$\ln S = \frac{i}{\sqrt{\alpha\beta}} \tan^{-1}\left(\frac{\sqrt{\alpha/\beta}}{\varepsilon_0 - \varepsilon}\right) + 2i\nu(\varepsilon), \tag{M}$$

where

$$2\nu(\varepsilon) = -\int^{\varepsilon-\varepsilon_0} \frac{F(Z)dZ}{(\alpha + \beta Z^2)[\alpha + \beta Z^2 + F(Z)]},$$

the lower limit on the integral being chosen in a manner consistent with (M).

It is convenient to replace α and β with two new parameters r and Γ, defined by the equations

$$r \equiv \frac{1}{2}(\alpha\beta)^{-1/2}, \quad \Gamma \equiv 2\sqrt{\alpha/\beta}.$$

Then (M) takes the form

$$S = \exp\left[2ri\tan^{-1}\left(\frac{\Gamma/2}{\varepsilon_0 - \varepsilon}\right) + 2i\nu(\varepsilon)\right]. \tag{N}$$

The condition (K) implies that $\Gamma/\varepsilon_0 \ll 1$, so the term involving $\tan^{-1}\left(\frac{\Gamma/2}{\varepsilon_0 - \varepsilon}\right)$ in (N) will be negligible except for $\varepsilon \simeq \varepsilon_0$. We have specified that the scattering in the state being considered is small except (possibly) near the energy ε_0. This implies that $\nu(\varepsilon)$ will not give a very important contribution to S when $\varepsilon \simeq \varepsilon_0$. To estimate the contribution from $\nu(\varepsilon)$, we take $r = 1$, so $\alpha = \Gamma/4$, $\beta = 1/\Gamma$. Then for $\varepsilon - \varepsilon_0 = \Gamma/2$, the series (J) is

$$\frac{1}{Q} = \frac{\Gamma}{4} + \frac{\Gamma}{4} + \frac{\Gamma}{4}\left(\frac{\Gamma}{2\varepsilon_0}\right) + \cdots. \tag{O}$$

Condition (K) implies that the higher terms in the series (O) are negligible. The scattering matrix $T(\varepsilon)$ is defined and analytic when ε has a sufficiently small positive imaginary part, since T was obtained as a function of $\varepsilon + i\eta$ in the limit $\eta \to 0(+)$. From this, we see that $S(\varepsilon)$ is analytic in a domain above and bounded by the real axis. If Γ is sufficiently small, the point $\varepsilon = \varepsilon_0 + i\Gamma/2$ will lie in this domain of analyticity. In order to understand the implication of this, let us write

$$\tan^{-1}\left(\frac{\Gamma/2}{\varepsilon_0 - \varepsilon}\right) = \frac{i}{2}\left[\ln\left(\varepsilon - \varepsilon_0 + i\frac{\Gamma}{2}\right) - \ln\left(\varepsilon - \varepsilon_0 - i\frac{\Gamma}{2}\right)\right]. \tag{P}$$

Now, restricting ε to the domain of analyticity of $S(\varepsilon)$, let the point ε move up from the real axis, describe a circle about the point $\varepsilon_0 + i(\Gamma/2)$, and then return to its initial point on the real axis. Show that the expression (P) does not return to its initial value, however, but acquires an additional term,

$$\tan^{-1}\left(\frac{\Gamma/2}{\varepsilon_0 - \varepsilon}\right) \to \tan^{-1}\left(\frac{\Gamma/2}{\varepsilon_0 - \varepsilon}\right) + \pi.$$

Then, moving ε around the closed contour transforms S according to $S \to S\exp[2\pi i r]$. This violates the condition that $S(\varepsilon)$ be analytic in the domain in which the contour lies, unless r is an integer. We therefore must have $r = 0, 1, 2, \ldots$. Of principal physical interest is the case that $r = 1$.

Let us suppose that $r = 1$. From (N), writing

$$\tan\delta(\varepsilon) = \frac{\Gamma/2}{\varepsilon_0 - \varepsilon},$$

show that

$$\begin{aligned} S(\varepsilon) &\equiv \exp[2i\delta(\varepsilon)]\exp[2i\nu(\varepsilon)] = \frac{1 + i\tan\delta}{1 - i\tan\delta}\exp[2i\nu(\varepsilon)] \\ &= 1 + \frac{i\Gamma}{\varepsilon_0 - \varepsilon - i(\Gamma/2)} + \frac{\varepsilon_0 - \varepsilon + i(\Gamma/2)}{\varepsilon_0 - \varepsilon - i(\Gamma/2)}(\exp[2i\nu] - 1). \end{aligned} \tag{Q}$$

The second term here represents "resonance scattering," whereas the third term is called "potential scattering." By hypothesis, it gives only a small contribution to S for $\varepsilon \simeq \varepsilon_0$.

To discuss the scattering cross section, it is convenient to reinsert the state label s on $S(\varepsilon)$, writing $S_s(\varepsilon)$. Let the state for which S has the form (Q) be s_0. For simplicity, let us neglect the scattering in other states and the potential scattering in the state s_0. Then, we may take

$$S_s = 1 \quad \text{for} \quad s \neq s_0; \qquad S_{s_0} = 1 + \frac{i\Gamma}{\varepsilon_0 - \varepsilon - i(\Gamma/2)}.$$

Show that $S_{c'c}$ in the channel representation is

$$S_{c'c} = \delta_{c'c} + i\frac{U_{c's_0} U^*_{cs_0} \Gamma}{\varepsilon_0 - \varepsilon - i(\Gamma/2)}.$$

Obtain the Breit–Wigner resonance scattering cross section as

$$\sigma_{c'c} = \frac{\pi}{\kappa_c^2} \frac{2J+1}{(2S_1+1)(2S_2+1)} \frac{\Gamma_{c'}\Gamma_c}{(\varepsilon_0 - \varepsilon)^2 + \Gamma^2/4},$$

with the channel width defined by

$$\Gamma_c \equiv \left|U_{cs_0}\right|^2 \Gamma, \quad \Gamma_{c'} \equiv \left|U_{c's_0}\right|^2 \Gamma,$$

and the full width at half the maximum defined by

$$\Gamma \equiv \sum_c \Gamma_c.$$

2.30. Consider the self-interacting Schrödinger field through the nonlocal self-interaction:

$$\left(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\vec{\nabla}^2\right)\varphi(\vec{x},t) = \int d\vec{x}'\, V(\vec{x},\vec{x}')\varphi(\vec{x}',t),$$

$$V(\vec{x},\vec{x}') = -\lambda_0\rho(\vec{x})\rho(\vec{x}'),$$

$$\lambda_0 > 0, \quad \rho(\vec{x}) = \text{a given } c\text{-number real function.}$$

(a) In order to obtain the c-number normal mode, set

$$\varphi(\vec{x},t) \sim u(\vec{x},J)\exp[-i\omega_J t],$$

and convert the differential equation into an integral equation:

$$u(\vec{x},J) = u_0(\vec{x},J) - \lambda_0 \int d\vec{x}'\, G_+(\vec{x}-\vec{x}';\hbar\omega_J)\rho(\vec{x}')(\rho u),$$

with

$$G_+(\vec{x}-\vec{x}';\hbar\omega_J) \equiv \left\langle \vec{x} \left| \frac{1}{\hbar\omega_J + (\hbar^2/2m)\vec{\nabla}^2 + i\varepsilon\hbar} \right| \vec{x}' \right\rangle,$$

$$\left(\hbar\omega_J + \frac{\hbar^2}{2m}\vec{\nabla}^2\right)u_0(\vec{x},J) = 0, \quad (\rho u) \equiv \int d\vec{x}\rho(\vec{x})u(\vec{x},J).$$

Multiplying by $\rho(\vec{x})$ in the integral equation from the left, and integrating over $\vec{x}$, obtain

$$(\rho u) = \frac{1}{D_+(\omega_J)}(\rho u_0),$$

where

$$D_+(\omega_J) = 1 + \lambda_0 \int d\vec{x} d\vec{x}^{\prime} \rho(\vec{x}) G_+(\vec{x} - \vec{x}^{\prime}; \hbar\omega_J)\rho(\vec{x}^{\prime}).$$

Since (ρu) is proportional to (ρu_0), the angular frequency of the system is given by

$$\omega_J = \frac{1}{\hbar}\frac{\hbar^2}{2m}\vec{k}^2 \equiv \omega_{\vec{k}}.$$

(b) By setting

$$u_0(\vec{x}, J) \equiv \frac{1}{(2\pi)^{3/2}}\frac{1}{\sqrt{\hbar}} \exp\,[i\vec{k}\cdot\vec{x}],$$

show that

$$u(\vec{x}, \vec{k}) = \frac{1}{(2\pi)^{3/2}}\frac{1}{\sqrt{\hbar}} \int d\vec{k}^{\prime} \left\{ \delta(\vec{k} - \vec{k}^{\prime}) - \frac{\lambda_0}{(2\pi)^3} G_+(\vec{k}^{\prime}; \omega_{\vec{k}}) \frac{\rho^*(\vec{k})\rho(\vec{k}^{\prime})}{D_+(\omega_{\vec{k}})} \right\} \exp[i\vec{k}^{\prime}\cdot\vec{x}],$$

where

$$\rho(\vec{x}) = \frac{1}{(2\pi)^3}\int d\vec{k} \exp\,[i\vec{k}\cdot\vec{x}]\rho(\vec{k}),$$

$$G_+(\vec{k}^{\prime}; \omega_{\vec{k}}) = \frac{1}{\hbar\omega_{\vec{k}} - (\hbar^2/2m)\vec{k}^{\prime 2} + i\varepsilon\hbar^{\prime}}$$

$$D_+(\omega_{\vec{k}}) = 1 + \frac{\lambda_0}{(2\pi)^3}\int d\vec{q} G_+(\vec{q}; \omega_{\vec{k}}) \left|\rho(\vec{q})\right|^2.$$

(c) Demonstrate the orthonormality of $u(\vec{x}, \vec{k})$:

$$\hbar \int d\vec{x} u^*(\vec{x}, \vec{k}) u(\vec{x}, \vec{k}^{\prime}) = \delta(\vec{k} - \vec{k}^{\prime}).$$

(d) In this model, we can have a bound state,

$$D(\omega) = 1 + \frac{\lambda_0}{(2\pi)^3}\int d\vec{q} \frac{\left|\rho(\vec{q})\right|^2}{\hbar\omega - (\hbar^2/2m)\vec{q}^2} = 0.$$

Show that $D(\omega)$ is a monotonically decreasing function such that

$$D(-\infty) = 1, \quad D(0) = 1 - \frac{\lambda_0}{(2\pi)^3} \int d\vec{q} \frac{\left|\rho(\vec{q})\right|^2}{(\hbar^2/2m)\vec{q}^2},$$

$$\frac{d}{d\omega} D(\omega) = -\frac{\lambda_0 \hbar}{(2\pi)^3} \int d\vec{q} \frac{\left|\rho(\vec{q})\right|^2}{[\hbar\omega - (\hbar^2/2m)\vec{q}^2]^2} < 0.$$

If we do not have the bound state pole, $D(\omega) \neq 0$, by direct computation, we obtain

$$\hbar \int d\vec{k} u(\vec{x}, \vec{k}) u^*(\vec{x}', \vec{k}) = \delta(\vec{x} - \vec{x}').$$

(e) If we have one point (on the negative real axis) $\omega = \omega_b$, where $D(\omega)$ vanishes, we have an extra contribution to above, which comes from

$$D(z/\hbar) \sim \left(\frac{z}{\hbar} - \omega_b\right) D'(\omega_b).$$

Show that

$$\hbar \int d\vec{k} u(\vec{x}, \vec{k}) u^*(\vec{x}', \vec{k})$$
$$= \delta(\vec{x} - \vec{x}') + \frac{\lambda_0 \hbar}{(2\pi)^3} \int d\vec{q} \int d\vec{q}' \frac{\rho(\vec{q})\rho^*(\vec{q}')}{\hbar^2(\omega_b - \omega_{\vec{q}})(\omega_b - \omega_{\vec{q}'})} \frac{1}{D'(\omega_b)}$$
$$\times \exp[i\vec{q} \cdot \vec{x}] \exp[-i\vec{q}' \cdot \vec{x}'].$$

Define the bound state wavefunction by

$$u(\vec{x}, b) \equiv \frac{1}{(2\pi)^{3/2}} \left(-\frac{\lambda_0}{D'(\omega_b)}\right)^{1/2} \int d\vec{q} \frac{\rho(\vec{q})}{\hbar(\omega_b - \omega_{\vec{q}})} \exp\,[i\vec{q} \cdot \vec{x}].$$

Show that the completeness relation becomes

$$\hbar \int d\vec{k} u(\vec{x}, \vec{k}) u^*(\vec{x}', \vec{k}) + \hbar u(\vec{x}, b) u^*(\vec{x}', b) = \delta(\vec{x} - \vec{x}').$$

(f) Show that the bound state wavefunction is normalized:

$$\hbar \int d\vec{x} u^*(\vec{x}, b) u(\vec{x}, b) = -\frac{\hbar\lambda_0}{D'(\omega_b)} \int d\vec{q} \frac{\left|\rho(\vec{q})\right|^2}{\hbar^2(\omega_b - \omega_{\vec{q}})^2} = 1.$$

Show that the configuration space wavefunction is given by

$$u(\vec{x}, b) = -\frac{m}{\hbar^2}\left(-\frac{\lambda_0}{D'(\omega_b)}\right)^{1/2} \int d\vec{x}' \rho(\vec{x}') \frac{1}{\left|\vec{x} - \vec{x}'\right|} \exp\left[-\sqrt{-\frac{2m}{\hbar^2}\omega_b}\left|\vec{x} - \vec{x}'\right|\right].$$

Show that the bound state wavefunction satisfies the homogeneous equation:

$$u(\vec{x}, b) = -\lambda_0 \int d\vec{x}' G_+(\vec{x} - \vec{x}'; \hbar\omega_b)\rho(\vec{x}') \int d\vec{x}'' \rho(\vec{x}'')u(\vec{x}'', b).$$

(g) Expand the quantum field $\varphi(\vec{x}, t)$ in terms of $u(\vec{x}, \vec{k})$ and $u(\vec{x}, b)$:

$$\varphi(\vec{x}, t) = \int d\vec{k} A^{\text{in}}(\vec{k}) u(\vec{x}, \vec{k}) \exp[-i\omega_{\vec{k}} t]\sqrt{\hbar} + Bu(\vec{x}, b) \exp[-i\omega_b t]\sqrt{\hbar},$$

where $A^{\text{in}}(\vec{k})$, $A^{\text{in}\dagger}(\vec{k}')$, B and $B^\dagger$ are the q-number operators satisfying the commutation relations:

$$\begin{cases} [A^{\text{in}}(\vec{k}), A^{\text{in}\dagger}(\vec{k}')]_\pm & = & \delta(\vec{k} - \vec{k}'), & & \\ [A^{\text{in}}(\vec{k}), A^{\text{in}}(\vec{k}')]_\pm & = & [A^{\text{in}\dagger}(\vec{k}), A^{\text{in}\dagger}(\vec{k}')]_\pm & = & 0, \\ [B, B^\dagger]_\pm & = & 1, & & \\ [B, B]_\pm & = & [B^\dagger, B^\dagger] & = & 0, \\ [B, A^{\text{in}}(\vec{k})]_\pm & = & [B, A^{\text{in}\dagger}(\vec{k})]_\pm & = & 0. \end{cases}$$

Show that $\varphi(\vec{x}, t)$ satisfies the commutation relation with the use of the completeness relation:

$$\begin{aligned} [\varphi(\vec{x}, t), \varphi^\dagger(\vec{x}', t)]_\pm &= \hbar \int d\vec{k} \int d\vec{k}' u(\vec{x}, \vec{k}) u^*(\vec{x}', \vec{k}')\delta(\vec{k} - \vec{k}') + \hbar u(\vec{x}, b)u^*(\vec{x}', b) \\ &= \delta(\vec{x} - \vec{x}'). \end{aligned}$$

(h) Show that the total Hamiltonian is given by

$$H = \int d\vec{x} \frac{\hbar^2}{2m} \vec{\nabla}\varphi^\dagger(\vec{x}, t) \cdot \vec{\nabla}\varphi(\vec{x}, t) - \lambda_0 \int d\vec{x} \int d\vec{x}' \varphi^\dagger(\vec{x}, t)\rho(\vec{x})\rho(\vec{x}')\varphi(\vec{x}', t).$$

Using the equation of motion, obtain

$$\begin{aligned} H &= \int d\vec{x} \left\{ \frac{\hbar^2}{2m} \vec{\nabla}\varphi^\dagger(\vec{x}, t) \cdot \vec{\nabla}\varphi(\vec{x}, t) + i\hbar\varphi^\dagger(\vec{x}, t)\dot{\varphi}(\vec{x}, t) \right. \\ &\quad \left. + \frac{\hbar^2}{2m} \varphi^\dagger(\vec{x}, t)\vec{\nabla}^2\varphi(\vec{x}, t) \right\} \\ &= i\hbar \int d\vec{x} \varphi^\dagger(\vec{x}, t)\dot{\varphi}(\vec{x}, t). \end{aligned}$$

From the normal mode expansion, we obtain

$$H = \int d\vec{k} \hbar\omega_{\vec{k}} A^{\text{in}\dagger}(\vec{k}) A^{\text{in}}(\vec{k}) + \hbar\omega_b B^\dagger B.$$

This is the addition of the bound state contribution to

$$H_0^{\text{in}} = \int d\vec{x} \frac{\hbar^2}{2m} \vec{\nabla}\varphi^{\text{in}\dagger}(\vec{x}, t) \cdot \vec{\nabla}\varphi^{\text{in}}(\vec{x}, t) = \int d\vec{k}\hbar\omega_{\vec{k}} A^{\text{in}\dagger}(\vec{k}) A^{\text{in}}(\vec{k}).$$

(i) Choose the state where one asymptotic field carries the momentum $\hbar\vec{k}$ as the initial state:

$$A^{\text{in}\dagger}(\vec{k})\left|0\right\rangle \equiv \left|\vec{k}\right\rangle.$$

Then, the Lippmann–Schwinger equation is

$$\left|\vec{k}^{(+)}\right\rangle = \left|k\right\rangle + \frac{1}{\hbar\omega_{\vec{k}} - H_0^{\text{in}} + i\varepsilon\hbar} H^{\text{int}} \left|\vec{k}^{(+)}\right\rangle,$$

where

$$\begin{aligned} H^{\text{int}} &\equiv -\lambda_0 \int d\vec{x}\varphi^{\text{in}\dagger}(\vec{x}, 0)\rho(\vec{x}) \int d\vec{x}^{\prime} \rho^*(\vec{x}^{\prime})\varphi^{\text{in}}(\vec{x}^{\prime}, 0) \\ &= -\frac{\lambda_0}{(2\pi)^3} \int d\vec{k} A^{\text{in}\dagger}(\vec{k})\rho(\vec{k}) \int d\vec{k}^{\prime} \rho^*(\vec{k}^{\prime}) A^{\text{in}}(\vec{k}^{\prime}). \end{aligned}$$

Multiplying by $\langle 0 \left| A^{\text{in}}(\vec{q}) \right.$ in the Lippmann–Schwinger equation from the left, we obtain

$$\int d\vec{q}\rho^*(\vec{q}) \left\langle \vec{q} \left| \vec{k}^{(+)} \right.\right\rangle = \frac{\rho^*(\vec{k})}{D_+(\omega_{\vec{k}})},$$

$$\left\langle \vec{q} \left| \vec{k}^{(+)} \right.\right\rangle = \delta(\vec{q} - \vec{k}) - \frac{\lambda_0}{(2\pi)^3} G_+(\vec{q}; \omega_{\vec{k}}) \frac{\rho(\vec{q})\rho^*(\vec{k})}{D_+(\omega_{\vec{k}})}.$$

When $\vec{k} \neq \vec{k}^{\prime}$, we obtain the transition rate as

$$w_{\vec{k},\vec{k}'} = \frac{2\pi}{\hbar} \delta(\hbar\omega_{\vec{k}} - \hbar\omega_{\vec{k}'}) \left| \frac{\lambda_0}{(2\pi)^3} \frac{\rho(\vec{k}^{\prime})\rho^*(\vec{k})}{D_+(\omega_{\vec{k}})} \right|^2 \neq 0.$$

Nonvanishing transition rate implies that the scattering takes place in this model.

3
Integral Equations of the Volterra Type

3.1
Iterative Solution to Volterra Integral Equation of the Second Kind

Consider the *inhomogeneous Volterra integral equation of the second kind,*

$$\phi(x) = f(x) + \lambda \int_0^x K(x, y)\phi(y)dy, \quad 0 \le x, y \le h, \tag{3.1.1}$$

with $f(x)$ and $K(x, y)$ *square-integrable,*

$$\|f\|^2 = \int_0^h |f(x)|^2 \, dx < \infty, \tag{3.1.2}$$

$$\|K\|^2 = \int_0^h dx \int_0^x dy \, |K(x, y)|^2 < \infty. \tag{3.1.3}$$

Also, define

$$A(x) = \int_0^x |K(x, y)|^2 \, dy. \tag{3.1.4}$$

Note that the upper limit of y integration is x. Note also that the Volterra integral equation is a special case of the Fredholm integral equation with

$$K(x, y) = 0 \quad \text{for} \quad x < y < h. \tag{3.1.5}$$

We will prove in the following facts for Eq. (3.1.1):

(1) A solution *exists* for all values of λ.
(2) The solution is *unique* for all values of λ.
(3) The iterative solution is *convergent* for all values of λ.

We start our discussion with the construction of an *iterative solution*. Consider a series solution of the usual form

$$\phi(x) = \phi_0(x) + \lambda\phi_1(x) + \lambda^2\phi_2(x) + \cdots + \lambda^n\phi_n(x) + \cdots. \tag{3.1.6}$$

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Substituting the series solution (3.1.6) into Eq. (3.1.1), we have

$$\sum_{k=0}^{\infty}\lambda^k\phi_k(x) = f(x) + \lambda\int_0^x K(x,y)\sum_{j=0}^{\infty}\lambda^j\phi_j(y)dy.$$

Collecting like powers of λ, we have

$$\phi_0(x) = f(x), \quad \phi_n(x) = \int_0^x K(x,y)\phi_{n-1}(y)dy, \quad n = 1,2,3,\ldots. \tag{3.1.7}$$

We now examine the convergence of the series solution (3.1.6). Applying the Schwarz inequality to Eq. (3.1.7), we have

$$\begin{aligned}\phi_1(x) &= \int_0^x K(x,y)f(y)dy \\ &\Rightarrow |\phi_1(x)|^2 \le A(x)\int_0^x |f(y)|^2\,dy \le A(x)\,\|f\|^2, \\ \phi_2(x) &= \int_0^x K(x,y)\phi_1(y)dy \\ &\Rightarrow |\phi_2(x)|^2 \le A(x)\int_0^x |\phi_1(y)|^2\,dy \le A(x)\int_0^x dx_1 A(x_1)\,\|f\|^2.\end{aligned}$$

In general, we have

$$|\phi_n(x)|^2 \le \frac{1}{(n-1)!}A(x)\left[\int_0^x dyA(y)\right]^{n-1}\|f\|^2, \quad n = 1,2,3,\ldots. \tag{3.1.8}$$

Define

$$B(x) \equiv \int_0^x dyA(y). \tag{3.1.9}$$

Thus, from Eqs. (3.1.8) and (3.1.9), we obtain the following bound on $\phi_n(x)$:

$$|\phi_n(x)| \le \frac{1}{\sqrt{(n-1)!}}\sqrt{A(x)}\left[B(x)\right]^{(n-1)/2}\|f\|, \quad n = 1,2,3,\ldots. \tag{3.1.10}$$

We now examine the convergence of the series solution (3.1.6):

$$\begin{aligned}\left|\phi(x) - f(x)\right| &= \left|\sum_{n=1}^{\infty}\lambda^n\phi_n(x)\right| \le \sum_{n=1}^{\infty}|\lambda|^n\frac{1}{\sqrt{(n-1)!}}\sqrt{A(x)}\left[B(x)\right]^{(n-1)/2}\|f\| \\ &= \sqrt{A(x)}\,\|f\|\sum_{n=1}^{\infty}|\lambda|^n\left[B(x)\right]^{(n-1)/2}\frac{1}{\sqrt{(n-1)!}}.\end{aligned} \tag{3.1.11}$$

Letting

$$a_n = |\lambda|^n \left[B(x)\right]^{(n-1)/2} \frac{1}{\sqrt{(n-1)!}},$$

and applying the ratio test on the right-hand side of Eq. (3.1.11), we have

$$\begin{aligned} a_{n+1}/a_n &= \left\{|\lambda|^{n+1} \left[B(x)\right]^{n/2} \sqrt{(n-1)!}\right\} / \left\{|\lambda|^n \left[B(x)\right]^{(n-1)/2} \sqrt{n!}\right\} \\ &= |\lambda| \frac{\sqrt{B(x)}}{\sqrt{n}}, \end{aligned}$$

whence we have

$$\lim_{n\to\infty} (a_{n+1}/a_n) = 0 \quad \text{for all} \quad \lambda. \tag{3.1.12}$$

Thus the series solution (3.1.6) converges for all λ, provided that $\|f\|$, $A(x)$, and $B(x)$ exist and are finite.

We have proven statements (1) and (3) with the condition that the kernel $K(x, y)$ and the inhomogeneous term $f(x)$ are square-integrable , Eqs. (3.1.2) and (3.1.3).

To show that Eq. (3.1.1) has a *unique solution* which is square-integrable ($\|\phi\| < \infty$), we shall prove that

$$R_n(x) \to 0 \quad \text{as} \quad n \to \infty, \tag{3.1.13}$$

where

$$\lambda^{n+1} R_{n+1}(x) \equiv \phi(x) - \sum_{k=0}^{k=n} \lambda^k \phi_k(x), \tag{3.1.14}$$

and

$$R_n(x) = \int_0^x K(x, y) R_{n-1}(y) dy, \quad R_0(x) = \phi(x).$$

Repeating the same procedure as above, we can establish

$$[R_n(x)]^2 \le \|\phi\|^2 A(x) [B(x)]^{n-1}/(n-1)!. \tag{3.1.15}$$

$R_{n+1}(x)/R_n(x)$ vanishes as $n \to \infty$. Returning to Eq. (3.1.14), we see that the iterative solution $\phi(x)$, Eq. (3.1.6), is *unique*. The *uniqueness* of the solution of the *inhomogeneous Volterra integral equation* implies that the *square-integrable solution of the homogeneous Volterra integral equation*

$$\psi_{\mathrm{H}}(x) = \lambda \int_0^x K(x, y) \psi_{\mathrm{H}}(y) dy \tag{3.1.16}$$

is trivial, $\psi_{\mathrm{H}}(x) \equiv 0$. Otherwise, $\phi(x) + c\psi_{\mathrm{H}}(x)$ would also be a solution of the inhomogeneous Volterra integral equation (3.1.1), in contradiction to our conclusion that the solution of Eq. (3.1.1) is unique.

3.2
Solvable Cases of the Volterra Integral Equation

We list a few solvable cases of the Volterra integral equation.

Case (1): Kernel is equal to a sum of *n factorized terms*.

We demonstrated the reduction of such integral equations into an nth-order ordinary differential equation in Problem 6 in Chapter 2.

Case (2): Kernel is *translational*.

$$K(x, y) = K(x - y). \tag{3.2.1}$$

Consider

$$\phi(x) = f(x) + \lambda \int_0^x K(x - y)\phi(y)dy \quad \text{on} \quad 0 \le x < \infty. \tag{3.2.2}$$

We shall use the *Laplace transform*,

$$L\{F(x)\} \equiv \overline{F}(s) \equiv \int_0^\infty dx F(x)e^{-sx}. \tag{3.2.3}$$

Taking the Laplace transform of the integral equation (3.2.2), we obtain

$$\overline{\phi}(s) = \overline{f}(s) + \lambda\overline{K}(s)\overline{\phi}(s). \tag{3.2.4}$$

Solving Eq. (3.2.4) for $\overline{\phi}(s)$, we obtain

$$\overline{\phi}(s) = \overline{f}(s) \big/ \left(1 - \lambda\overline{K}(s)\right). \tag{3.2.5}$$

Applying the inverse Laplace transform to Eq. (3.2.5), we obtain

$$\phi(x) = \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{ds}{2\pi i} e^{sx}\overline{f}(s) \big/ \left(1 - \lambda\overline{K}(s)\right), \tag{3.2.6}$$

where the inversion path ($\gamma \pm i\infty$) in the complex s plane lies to the right of all singularities of the integrand as indicated in Figure 3.1.

Definition: Suppose $F(t)$ is defined on $[0, \infty)$ with $F(t) = 0$ for $t < 0$. Then the Laplace transform of $F(t)$ is defined by

$$\overline{F}(s) \equiv L\{F(t)\} \equiv \int_0^\infty F(t)e^{-st}dt. \tag{3.2.7}$$

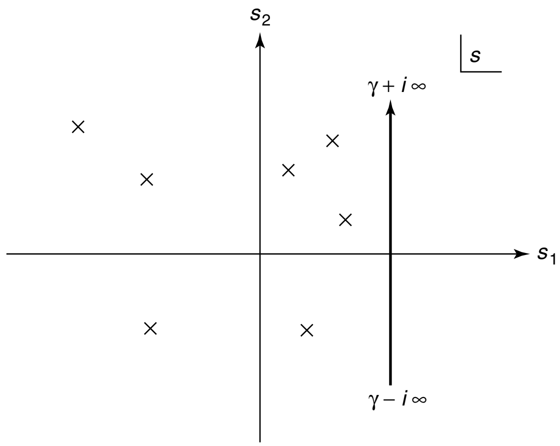

Fig. 3.1 The inversion path of the Laplace transform $\overline{\phi}(s)$ in the complex s plane from $s = \gamma - i\infty$ to $s = \gamma + i\infty$, which lies to the right of all singularities of the integrand.

The inversion is given by

$$F(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \overline{F}(s)\, e^{st} ds = L^{-1}\left\{\overline{F}(s)\right\}. \tag{3.2.8}$$

Properties:

$$L\left\{\frac{d}{dt}F(t)\right\} = s\overline{F}(s) - \overline{F}(0). \tag{3.2.9}$$

The Laplace transform of *convolution*

$$H(t) = \int_0^t G\left(t - t'\right) F\left(t'\right) dt' = \int_0^t G(t')F(t - t')dt' \tag{3.2.10}$$

is given by

$$\overline{H}(s) = \overline{G}(s)\, \overline{F}(s), \tag{3.2.11}$$

which is already used in deriving Eq. (3.2.4).
The Laplace transforms of 1 and t^n are, respectively, given by

$$L\{1\} = 1/s \tag{3.2.12}$$

and

$$L\left\{t^n\right\} = \Gamma(n+1)/s^{n+1}. \tag{3.2.13}$$

Gamma Function: $\Gamma(z)$ is defined by

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt. \tag{3.2.14}$$

Properties of Gamma Function:

$$\Gamma(1) = 1, \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \quad \Gamma(n+1) = n!, \tag{3.2.15a}$$

$$\Gamma(z+1) = z\Gamma(z), \quad \Gamma(z)\Gamma(1-z) = \pi / \sin(\pi z), \tag{3.2.15b}$$

$$\Gamma(z) \text{ is singular at } z = 0, -1, -2, \ldots. \tag{3.2.15c}$$

Derivation of the Abel integral equation: The descent time of a frictionless ball on the side of a hill is known as a function of its initial height x. Let us find the shape of the hill. Starting with initial velocity zero, the speed of the ball at height y is obtained by solving $mv^2/2 = mg(x-y)$, from which $v = \sqrt{2g(x-y)}$. Let the shape of the hill be given by $\xi = f(y)$. Then the arclength is given by

$$ds = \sqrt{(dy)^2 + (d\xi)^2} = \sqrt{1 + (f'(y))^2}\, |dy|.$$

The descent time to height $y = 0$ is given by

$$T(x) = \int dt = \int \frac{dt}{ds} ds = \int \frac{ds}{ds/dt} = \int \frac{ds}{v} = \int_{y=x}^{y=0} \frac{\sqrt{1+(f'(y))^2}}{\sqrt{2g(x-y)}} |dy|.$$

Since y is decreasing, dy is negative so that $|dy| = -dy$. Thus the descent time is given by

$$T(x) = \int_0^x \frac{\phi(y)}{\sqrt{x-y}} dy \tag{3.2.16}$$

with

$$\phi(y) = \frac{1}{\sqrt{2g}} \sqrt{1 + (f'(y))^2}. \tag{3.2.17}$$

So, given the descent time $T(x)$ as a function of the initial height x, we solve the Abel integral equation (3.2.16) for $\phi(x)$, and then solve (3.2.17) for $f'(y)$ which gives the shape of the curve $\xi = f(y)$.

We solve an example of the Volterra integral equation with a translational kernel derived above, Eq. (3.2.16).

❑ **Example 3.1.** Abel Integral Equation:

$$\int_0^x \frac{\phi(x')}{\sqrt{x-x'}}dx' = f(x), \quad \text{with} \quad f(0) = 0. \tag{3.2.18}$$

Solution. The Laplace transform of Eq. (3.2.18) is

$$\sqrt{\frac{\pi}{s}}\bar{\phi}(s) = \bar{f}(s).$$

Solving for $\bar{\phi}(s)$, we obtain

$$\bar{\phi}(s) = \sqrt{\frac{s}{\pi}}\bar{f}(s).$$

$\sqrt{s}$ is not the Laplace transform of anything. Thus we rewrite

$$\begin{aligned}\phi(x) &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{sx}(\bar{f}(s)\sqrt{s}/\sqrt{\pi})ds \\ &= \frac{1}{2\pi i}\frac{d}{dx}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{1}{\pi}\sqrt{\frac{\pi}{s}}\bar{f}(s)e^{sx}ds = \frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{f(x')}{\sqrt{x-x'}}dx',\end{aligned} \tag{3.2.19}$$

where the convolution theorem, Eqs. (3.2.10) and (3.2.11), has been applied.

As an extension of Case (2), we can solve a system of Volterra integral equations of the second kind with the translational kernels,

$$\phi_i(x) = f_i(x) + \sum_{j=1}^{n}\int_0^x K_{ij}(x-y)\phi_j(y)dy, \quad i = 1, \ldots, n, \tag{3.2.20}$$

where $K_{ij}(x)$ and $f_i(x)$ are known functions with Laplace transforms $\bar{K}_{ij}(s)$ and $\bar{f}(s)$. Taking the Laplace transform of (3.2.20), we obtain

$$\bar{\phi}_i(s) = \bar{f}_i(s) + \sum_{j=1}^{n}\bar{K}_{ij}(s)\bar{\phi}_j(s), \quad i = 1, \ldots, n. \tag{3.2.21}$$

Equation (3.2.21) is a system of linear algebraic equations for $\bar{\phi}_i(s)$'s. We can solve (3.2.21) for $\bar{\phi}_i(s)$'s easily and apply the inverse Laplace transform to $\bar{\phi}_i(s)$'s to obtain $\phi_i(x)$'s.

3.3
Problems for Chapter 3

3.1. Consider the Volterra integral equation of the first kind,

$$f(x) = \int_0^x K(x,y)\phi(y)dy, \quad 0 \le x \le h.$$

Show that by differentiating the above equation with respect to x, we can transform this integral equation to a Volterra integral equation of the second kind as long as

$$K(x,x) \neq 0.$$

3.2. Transform the radial Schrödinger equation

$$\left[\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + k^2 - V(r)\right]\psi(r) = 0, \quad \text{with} \quad \psi(r) \sim r^{l+1}$$
as $r \to 0$,

to a Volterra integral equation of the second kind.

Hint: There are two ways to define the homogeneous equation.

(i)

$$\left[\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + k^2\right]\psi_{\mathrm{H}}(r) = 0 \quad \Rightarrow \quad \psi_{\mathrm{H}}(r) = \begin{cases} krj_l(kr) \\ krh_l^{(1)}(kr) \end{cases},$$

where $j_l(kr)$ is the lth order *spherical Bessel function* and $h_l^{(1)}(kr)$ is the lth order *spherical Hankel function of the first kind.*

(ii)

$$\left[\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2}\right]\psi_{\mathrm{H}}(r) = 0 \quad \Rightarrow \quad \psi_{\mathrm{H}}(r) = r^{l+1} \text{ and } r^{-l}.$$

There exist two equivalent Volterra integral equations of the second kind for this problem.

3.3. Solve the generalized Abel equation,

$$\int_0^x \frac{\phi(y)}{(x-y)^\alpha}dy = f(x), \quad 0 < \alpha < 1.$$

3.4. Solve

$$\int_0^x \phi(y)\ln(x-y)dy = f(x), \quad \text{with} \quad f(0) = 0.$$

3.5. Solve

$$\phi(x) = 1 + \int_x^\infty e^{\alpha(x-y)}\phi(y)dy, \quad \alpha > 0.$$

Hint: Reduce the integral equation to the ordinary differential equation.

3.6. Solve

$$\phi(x) = 1 + \lambda\int_0^x e^{-(x-y)}\phi(y)dy.$$

3.7. (due to H. C.). Solve

$$\phi(x) = 1 + \int_1^x \frac{1}{x+y}\phi(y)dy, \quad x \geq 1.$$

Find the asymptotic behavior of $\phi(x)$ as $x \to \infty$.

3.8. (due to H. C.). Solve the integro-differential equation,

$$\frac{\partial}{\partial t}\phi(x,t) = -ix\phi(x,t) + \lambda\int_{-\infty}^{+\infty} g(y)\phi(y,t)dy, \quad \text{with} \quad \phi(x,0) = f(x),$$

where $f(x)$ and $g(x)$ are given. Find the asymptotic form of $\phi(x,t)$ as $t \to \infty$.

3.9. Solve

$$\int_0^x \frac{1}{\sqrt{x-y}}\phi(y)dy + \int_0^1 2xy\phi(y)dy = 1.$$

3.10. Solve

$$\lambda\int_0^x \frac{1}{(x-y)^{1/3}}\phi(y)dy + \lambda\int_0^1 \phi(y)dy = 1.$$

3.11. Solve

$$\lambda\int_0^1 K(x,y)\phi(y)dy = 1,$$

where

$$K(x,y) = \begin{cases} (x-y)^{-1/4} + xy & \text{for} \quad 0 \leq y \leq x \leq 1, \\ xy & \text{for} \quad 0 \leq x < y \leq 1. \end{cases}$$

3.12. (due to H. C.). Solve

$$\phi(x) = \lambda \int_0^x J_0(xy)\phi(y)dy,$$

where $J_0(x)$ is the 0th-order *Bessel function of the first kind.*

3.13. Solve

$$x^2\phi_\mu(x) - \int_0^x K_{(\mu)}(x,y)\phi_\mu(y)dy = \begin{cases} 0 & \text{for} \quad \mu \geq 1, \\ -x^2 & \text{for} \quad \mu = 0, \end{cases}$$

where

$$K_{(\mu)}(x,y) = -x - (x^2 - \mu^2)(x - y).$$

Hint: Setting

$$\phi_\mu(x) = \sum_{n=0}^{\infty} a_n^{(\mu)} x^n,$$

find the recursive relation of $a_n^{(\mu)}$ and solve for $a_n^{(\mu)}$.

3.14. Solve a system of the integral equations,

$$\phi_1(x) = 1 - 2\int_0^x \exp[2(x-y)]\phi_1(y)dy + \int_0^x \phi_2(y)dy,$$
$$\phi_2(x) = 4x - \int_0^x \phi_1(y)dy + 4\int_0^x (x-y)\phi_2(y)dy.$$

3.15. Solve a system of the integral equations,

$$\phi_1(x) + \phi_2(x) - \int_0^x (x-y)\phi_1(y)dy = ax,$$
$$\phi_1(x) - \phi_2(x) - \int_0^x (x-y)^2\phi_2(y)dy = bx^2.$$

3.16. (due to D. M.) Consider the Volterra integral equation of the second kind,

$$\phi(x) = f(x) + \lambda \int_0^x \exp[x^2 - y^2]\phi(y)dy, \quad x > 0.$$

(a) Sum up the iteration series exactly and find the general solution to this equation. Verify that the solution is analytic in λ.

(b) Solve this integral equation by converting it into a differential equation.

Hint: Multiply both sides by $\exp[-x^2]$ and differentiate.

3.17. (due to H. C.). The convolution of $f_1(x), f_2(x), \ldots, f_n(x)$ is defined as

$$C(x) \equiv \int_0^\infty dx_n \cdots \int_0^\infty dx_1 \prod_{i=1}^{n} f_i(x_i)\delta\left(x - \sum_{i=1}^{n} x_i\right).$$

(a) Verify that for $n = 2$, this is the convolution defined in the text, and that

$$\tilde{C}(s) = \prod_{i=1}^{n} \tilde{f}_i(s).$$

(b) If $f_1(x) = f(x)$ and $f_2(x) = f_3(x) = \cdots = f_n(x) = 1$, show that $C(x)$ is the nth integral of $f(x)$. Show also that

$$\tilde{C}(s) = \tilde{f}(s)s^{1-n},$$

and hence

$$C(x) = \int_0^x \frac{(x-y)^{n-1}}{(n-1)!} f(y)dy.$$

(c) With the result in (b), can you define the "one-third integral" of $f(x)$?

4
Integral Equations of the Fredholm Type

4.1
Iterative Solution to the Fredholm Integral Equation of the Second Kind

We consider the *inhomogeneous Fredholm Integral Equation of the second kind,*

$$\phi(x) = f(x) + \lambda \int_0^h dx' K(x, x')\phi(x'), \quad 0 \le x \le h, \tag{4.1.1}$$

and assume that $f(x)$ and $K(x, x')$ are both *square-integrable,*

$$\|f\|^2 < \infty, \tag{4.1.2}$$

$$\|K\|^2 \equiv \int_0^h dx \int_0^h dx' \left|K(x, x')\right|^2 < \infty. \tag{4.1.3}$$

Suppose that we look for an *iterative solution* in λ,

$$\phi(x) = \phi_0(x) + \lambda\phi_1(x) + \lambda^2\phi_2(x) + \cdots + \lambda^n\phi_n(x) + \cdots. \tag{4.1.4}$$

We substitute Eq. (4.1.4) into Eq. (4.1.1).

We obtain

$$\begin{aligned}
\phi_0(x) &= f(x), \\
\phi_1(x) &= \int_0^h dy_1 K(x, y_1)\phi_0(y_1) = \int_0^h dy_1 K(x, y_1) f(y_1), \\
\phi_2(x) &= \int_0^h dy_2 K(x, y_2)\phi_1(y_2) = \int_0^h dy_2 \int_0^h dy_1 K(x, y_2)K(y_2, y_1) f(y_1).
\end{aligned}$$

In general, we have

$$\begin{aligned}
\phi_n(x) &= \int_0^h dy_n K(x, y_n)\phi_{n-1}(y_n) = \int_0^h dy_n \int_0^h dy_{n-1} \cdots \int_0^h dy_1 \\
&\quad \times K(x, y_n)K(y_n, y_{n-1}) \cdots K(y_2, y_1) f(y_1).
\end{aligned} \tag{4.1.5}$$

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Bounds: First, in order to establish the bound on $\left|\phi_n(x)\right|$, define

$$A(x) \equiv \int_0^h dy \left|K(x,y)\right|^2 . \tag{4.1.6}$$

Then the square of the norm of the kernel $K(x, y)$ is given by

$$\|K\|^2 = \int_0^h dx A(x). \tag{4.1.7}$$

Applying the Schwarz inequality, each term in the iteration series (4.1.4) is bounded as follows:

$$\begin{aligned}
\phi_1(x) &= \int_0^h dy_1 K(x,y_1) f(y_1) \quad \Rightarrow \quad \left|\phi_1(x)\right|^2 \le A(x) \left\|f\right\|^2 , \\
\phi_2(x) &= \int_0^h dy_2 K(x,y_2) \phi_1(y_2) \quad \Rightarrow \quad \left|\phi_2(x)\right|^2 \le A(x) \left\|\phi_1\right\|^2 \le A(x) \left\|f\right\|^2 \left\|K\right\|^2 , \\
&\left|\phi_n(x)\right|^2 \le A(x) \left\|f\right\|^2 \left\|K\right\|^{2(n-1)} .
\end{aligned}$$

Thus the bound on $\left|\phi_n(x)\right|$ is established as

$$\left|\phi_n(x)\right| \le \sqrt{A(x)} \left\|f\right\| \left\|K\right\|^{n-1} , \quad n = 1, 2, 3, \ldots . \tag{4.1.8}$$

Now examine the whole series (4.1.4):

$$\phi(x) - f(x) = \lambda\phi_1(x) + \lambda^2\phi_2(x) + \lambda^3\phi_3(x) + \cdots .$$

Taking the absolute value of both sides, applying the triangular inequality on the right-hand side, and using the bound on $\left|\phi_n(x)\right|$ established as in Eq. (4.1.8), we have

$$\begin{aligned}
&\left|\phi(x) - f(x)\right| \le |\lambda| \left|\phi_1(x)\right| + |\lambda|^2 \left|\phi_2(x)\right| + \cdots = \sum_{n=1}^{\infty} |\lambda|^n \left|\phi_n(x)\right| \\
&\le \sum_{n=1}^{\infty} |\lambda|^n \sqrt{A(x)} \cdot \left\|f\right\| \cdot \left\|K\right\|^{n-1} = |\lambda| \sqrt{A(x)} \cdot \left\|f\right\| \cdot \sum_{n=0}^{\infty} |\lambda|^n \cdot \left\|K\right\|^n .
\end{aligned} \tag{4.1.9}$$

Now, the series on the right-hand side of inequality (4.1.9), $\sum_{n=0}^{\infty} |\lambda|^n \cdot \|K\|^n$, converges as long as

$$|\lambda| < 1 / \left\|K\right\| , \tag{4.1.10}$$

converging to

$$1/(1 - |\lambda| \cdot \|K\|).$$

Therefore, in that case (i.e., for Eq. (4.1.10)), the assumed series (4.1.4) is a convergent series, giving us a solution $\phi(x)$ to the integral equation (4.1.1), which is analytic inside the disk (4.1.10). Symbolically the integral equation (4.1.1) can be written as if it is an algebraic equation,

$$\begin{aligned} \phi &= f + \lambda K\phi \Rightarrow (1 - \lambda K)\phi = f \\ &\Rightarrow \phi = \frac{f}{1 - \lambda K} = (1 + \lambda K + \lambda^2 K^2 + \cdots)f, \end{aligned}$$

which converges only for $|\lambda K| < 1$.

Uniqueness: Inside the disk, Eq. (4.1.10), we can establish the uniqueness of the solution by showing that the corresponding homogeneous problem has no nontrivial solutions. Consider the homogeneous problem,

$$\phi_H(x) = \lambda \int_0^h K(x, y)\phi_H(y). \tag{4.1.11}$$

Applying the Schwarz inequality to Eq. (4.1.11),

$$\left|\phi_H(x)\right|^2 \leq |\lambda|^2 A(x) \|\phi_H\|^2 .$$

Integrating both sides with respect to x from 0 to h,

$$\begin{aligned} \|\phi_H\|^2 &\leq |\lambda|^2 \cdot \|\phi_H\|^2 \int_0^h A(x)dx \\ &= |\lambda|^2 \cdot \|K\|^2 \cdot \|\phi_H\|^2 , \end{aligned}$$

i.e.,

$$\|\phi_H\|^2 \cdot (1 - |\lambda|^2 \cdot \|K\|^2) \leq 0. \tag{4.1.12}$$

Since $|\lambda| \cdot \|K\| < 1$, inequality (4.1.12) can only be satisfied if and only if

$$\|\phi_H\| = 0$$

or

$$\phi_H \equiv 0. \tag{4.1.13}$$

Thus the homogeneous problem has only the trivial solution. Hence the inhomogeneous problem has a unique solution.

4.2
Resolvent Kernel

Returning to the series solution (4.1.4), we find that upon making the following definitions of the *iterated kernels*:

$$K_1(x, y) = K(x, y), \tag{4.2.1}$$

$$K_2(x, y) = \int_0^h dy_2 K(x, y_2) K(y_2, y), \tag{4.2.2}$$

$$K_3(x, y) = \int_0^h dy_3 \int_0^h dy_2 K(x, y_3) K(y_3, y_2) K(y_2, y), \tag{4.2.3}$$

$$K_n(x, y) = \int_0^h dy_n \int_0^h dy_{n-1} \cdots \int_0^h dy_2 K(x, y_n) K(y_n, y_{n-1}) \cdots K(y_2, y), \tag{4.2.4}$$

we may write each term in the series (4.1.4) as

$$\phi_1(x) = \int_0^h dy K_1(x, y) f(y), \tag{4.2.5}$$

$$\phi_2(x) = \int_0^h dy K_2(x, y) f(y), \tag{4.2.6}$$

$$\phi_3(x) = \int_0^h dy K_3(x, y) f(y), \tag{4.2.7}$$

$$\phi_n(x) = \int_0^h dy K_n(x, y) f(y). \tag{4.2.8}$$

As such, we have

$$\phi(x) = f(x) + \sum_{n=1}^{\infty} \lambda^n \int_0^h dy K_n(x, y) f(y). \tag{4.2.9}$$

Now, define the *resolvent kernel* $H(x, y; \lambda)$ to be

$$-H(x, y; \lambda) \equiv K_1(x, y) + \lambda K_2(x, y) + \lambda^2 K_3(x, y) + \cdots = \sum_{n=1}^{\infty} \lambda^{n-1} K_n(x, y). \tag{4.2.10}$$

Then solution (4.2.9) can be expressed compactly as

$$\phi(x) = f(x) - \lambda \int_0^h dy H(x, y; \lambda) f(y), \tag{4.2.11}$$

with the resolvent $H(x, y; \lambda)$ defined by Eq. (4.2.10). We have in effect shown that $H(x, y; \lambda)$ exists and is analytic for

$$|\lambda| < 1 / \|K\| . \tag{4.2.12}$$

❑ **Example 4.1.** Solve the Fredholm Integral Equation of the second kind,

$$\phi(x) = f(x) + \lambda \int_0^1 e^{x-y}\phi(y)dy. \tag{4.2.13}$$

Solution. We have, for the iterated kernels,

$$\begin{aligned} K_1(x,y) &= K(x,y) = e^{x-y}, \\ K_2(x,y) &= \int_0^1 d\xi\, K(x,\xi)K(\xi,y) = e^{x-y}, \\ K_n(x,y) &= e^{x-y} \text{ for all } n. \end{aligned} \tag{4.2.14}$$

Then we have as the resolvent kernel of this problem

$$-H(x,y;\lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} e^{x-y} = e^{x-y}(1 + \lambda + \lambda^2 + \lambda^3 + \cdots). \tag{4.2.15}$$

For

$$|\lambda| < 1, \tag{4.2.16}$$

we have

$$H(x,y;\lambda) = -e^{x-y}/(1-\lambda). \tag{4.2.17}$$

Thus the solution to this problem is given by

$$\phi(x) = f(x) + \frac{\lambda}{1-\lambda}\int_0^1 dy e^{x-y} f(y). \tag{4.2.18}$$

We remark that in this case not only do we know the radius of convergence for the series solution (or for the resolvent kernel) as in Eq. (4.2.16), we also know the nature of the singularity (a simple pole at $\lambda = 1$). In fact, our solution (4.2.18) is valid for all values of λ, even those which have $|\lambda| > 1$, with the exception of $\lambda = 1$.

Properties of the resolvent: We now derive some properties of the resolvent $H(x,y;\lambda)$. Consider the original integral operator written as

$$\tilde{K} = \int_0^h dy K(x,y), \tag{4.2.19}$$

and the operator corresponding to the resolvent as

$$\tilde{H} = \int_0^h dy H(x,y;\lambda). \tag{4.2.20}$$

We note that the integrals in Eqs. (4.2.19) and (4.2.20) are with respect to the second argument. The operators, $\tilde{K}^2$, $\tilde{H}^2$, $\tilde{K}\tilde{H}$, or $\tilde{H}\tilde{K}$ are defined in the usual way:

$$\tilde{K}^2 = \int_0^h dy_2 \int_0^h dy_1 K(x, y_2) K(y_2, y_1). \tag{4.2.21}$$

We wish to show that the two operators $\tilde{K}$ and $\tilde{H}$ commute, i.e.,

$$\tilde{K}\tilde{H} = \tilde{H}\tilde{K}. \tag{4.2.22}$$

The original integral equation can be written as

$$\phi = f + \lambda \tilde{K}\phi \tag{4.2.23}$$

while the solution by the resolvent takes the form

$$\phi = f - \lambda \tilde{H} f. \tag{4.2.24}$$

With the identity operator $\tilde{I}$,

$$\tilde{I} = \int_0^h dy \delta(x - y), \tag{4.2.25}$$

Eqs. (4.2.23) and (4.2.24) can be written as

$$f = (\tilde{I} - \lambda\tilde{K})\phi, \quad \phi = (\tilde{I} - \lambda\tilde{H})f. \tag{4.2.26,27}$$

Then, combining Eqs. (4.2.26) and (4.2.27), we obtain

$$f = (\tilde{I} - \lambda\tilde{K})(\tilde{I} - \lambda\tilde{H})f \quad \text{and} \quad \phi = (\tilde{I} - \lambda\tilde{H})(\tilde{I} - \lambda\tilde{K})\phi.$$

In other words, we have

$$(\tilde{I} - \lambda\tilde{K})(\tilde{I} - \lambda\tilde{H}) = \tilde{I} \quad \text{and} \quad (\tilde{I} - \lambda\tilde{H})(\tilde{I} - \lambda\tilde{K}) = \tilde{I}.$$

Thus we obtain

$$\tilde{K} + \tilde{H} = \lambda\tilde{K}\tilde{H} \quad \text{and} \quad \tilde{K} + \tilde{H} = \lambda\tilde{H}\tilde{K}.$$

Hence we have established the identity

$$\tilde{K}\tilde{H} = \tilde{H}\tilde{K}, \tag{4.2.28a}$$

i.e., $\tilde{K}$ and $\tilde{H}$ commute. This can be written explicitly as

$$\int_0^h dy_2 \int_0^h dy_1 K(x, y_2) H(y_2, y_1) = \int_0^h dy_2 \int_0^h dy_1 H(x, y_2) K(y_2, y_1).$$

Let both of these operators act on the function $\delta(y_1 - y)$. Then we find

$$\int_0^h d\xi\, K(x,\xi)H(\xi,y) = \int_0^h d\xi\, H(x,\xi)K(\xi,y). \tag{4.2.28b}$$

Similarly, the operator equation

$$\tilde{K} + \tilde{H} = \lambda \tilde{K}\tilde{H} \tag{4.2.29a}$$

may be written as

$$H(x,y;\lambda) = -K(x,y) + \lambda \int_0^h K(x,\xi)H(\xi,y;\lambda)d\xi. \tag{4.2.29b}$$

We will find this to be a useful relation later.

4.3 Pincherle–Goursat Kernel

Let us now examine the problem of determining a more explicit formula for the resolvent which points out more clearly the nature of the singularities of $H(x,y;\lambda)$ in the complex λ plane. We examine two types of kernels in sequence. First, we look at the case of a kernel which is given by a finite sum of separable terms (the so-called Pincherle–Goursat kernel). Secondly, we examine the case of a general kernel which we decompose into a sum of a Pincherle–Goursat kernel and a remainder which can be made as small as possible.

Pincherle–Goursat kernel: Suppose that we are given the kernel which is a finite sum of separable terms,

$$K(x,y) = \sum_{n=1}^{N} g_n(x)h_n(y), \tag{4.3.1}$$

i.e., we are given the following integral equation:

$$\phi(x) = f(x) + \lambda \int_0^h \sum_{n=1}^{N} g_n(x)h_n(y)\phi(y)dy. \tag{4.3.2}$$

Define β_n to be

$$\beta_n \equiv \int_0^h h_n(y)\phi(y)dy. \tag{4.3.3}$$

Then the integral equation (4.3.2) takes the form

$$\phi(x) = f(x) + \lambda \sum_{k=1}^{N} g_k(x)\beta_k. \tag{4.3.4}$$

Substituting Eq. (4.3.4) into expression (4.3.3) for β_n, we have

$$\beta_n = \int_0^h dy h_n(y) f(y) + \int_0^h dy h_n(y) \cdot \lambda \sum_{k=1}^{N} g_k(y) \beta_k. \tag{4.3.5}$$

Hence, we let

$$A_{nk} = \int_0^h dy h_n(y) g_k(y), \quad \alpha_n = \int_0^h dy h_n(y) f(y). \tag{4.3.6,7}$$

Equation (4.3.5) takes the form

$$\beta_n = \alpha_n + \lambda \sum_{k=1}^{N} A_{nk} \beta_k,$$

or

$$\sum_{k=1}^{N} (\delta_{nk} - \lambda A_{nk}) \beta_k = \alpha_n, \tag{4.3.8a}$$

which is equivalent to the $N \times N$ matrix equation

$$(I - \lambda A) \vec{\beta} = \vec{\alpha}, \tag{4.3.8b}$$

where the α's are known and the $\beta^{'}$s are unknown. If the determinant of the matrix $(I - \lambda A)$ is denoted by $\tilde{D}(\lambda)$,

$$\tilde{D}(\lambda) = \det(I - \lambda A),$$

the inverse of $(I - \lambda A)$ can be written as

$$(I - \lambda A)^{-1} = (1/\tilde{D}(\lambda)) \cdot D,$$

with D a matrix whose ijth element is the cofactor of the jith element of $I - \lambda A$. (We recall that the cofactor of a_{ij} is given by $(-1)^{i+j} \det M_{ij}$, with $\det M_{ij}$ the minor determinant obtained by deleting the row and column to which a_{ij} belongs.) Therefore,

$$\beta_n = (1/\tilde{D}(\lambda)) \cdot \sum_{k=1}^{N} D_{nk} \alpha_k. \tag{4.3.9}$$

From Eqs. (4.3.4) and (4.3.9), we obtain the solution $\phi(x)$ as

$$\phi(x) = f(x) + (\lambda/\tilde{D}(\lambda)) \sum_{n=1}^{N} \sum_{k=1}^{N} g_n(x) D_{nk} \alpha_k. \tag{4.3.10}$$

Writing out α_k explicitly, we have

$$\phi(x) = f(x) + (\lambda/\tilde{D}(\lambda))\sum_{n=1}^{N}\sum_{k=1}^{N} g_n(x) D_{nk} \int_0^h dy h_k(y) f(y). \tag{4.3.11}$$

Comparing Eq. (4.3.11) with the definition of the resolvent $H(x, y; \lambda)$,

$$\phi(x) = f(y) - \lambda \int_0^h dy H(x, y; \lambda) f(y), \tag{4.3.12}$$

we obtain the resolvent for the case of the Pincherle–Goursat kernel as

$$-H(x, y; \lambda) = (1/\tilde{D}(\lambda))\sum_{n=1}^{N}\sum_{k=1}^{N} g_n(x) D_{nk} h_k(y). \tag{4.3.13}$$

Note that this is a ratio of two polynomials in λ.

We remark that the cofactors of the matrix $(I - \lambda A)$ are polynomials in λ and hence have no singularities in λ. Thus the numerator of $H(x, y; \lambda)$ has no singularities. Then the only singularities of $H(x, y; \lambda)$ occur at the zeros of the denominator, $\tilde{D}(\lambda) = \det(I - \lambda A)$, which is a polynomial of degree N in λ. Therefore, $H(x, y; \lambda)$ in this case has at most N singularities which are poles in the complex λ plane.

General kernel: By approximating a general kernel as a sum of a Pincherle–Goursat kernel, we can now prove that in any finite region of the complex λ plane, there can be at most finitely many singularities. Consider the integral equation

$$\phi(x) = f(x) + \lambda \int_0^h K(x, y)\phi(y) dy, \tag{4.3.14}$$

with a general square-integrable kernel $K(x, y)$. Suppose we are interested in examining the singularities of $H(x, y; \lambda)$ in the region $|\lambda| < 1/\varepsilon$ in the complex λ plane (with ε quite small). We can always find an approximation to the kernel $K(x, y)$ in the form (with N quite large)

$$K(x, y) = \sum_{n=1}^{N} g_n(x) h_n(y) + R(x, y), \tag{4.3.15}$$

with

$$\|R\| < \varepsilon. \tag{4.3.16}$$

The integral equation (4.3.14) then becomes

$$\phi(x) = f(x) + \lambda \int_0^h \sum_{n=1}^{N} g_n(x) h_n(y)\phi(y) dy + \lambda \int_0^h R(x, y)\phi(y) dy.$$

Define

$$F(x) = f(x) + \lambda \int_0^h \sum_{n=1}^{N} g_n(x)h_n(y)\phi(y)dy. \tag{4.3.17}$$

Then

$$\phi(x) = F(x) + \lambda \int_0^h R(x,y)\phi(y)dy.$$

Let $H_R(x, y; \lambda)$ be the resolvent kernel corresponding to $R(x, y)$,

$$-H_R(x,y;\lambda) = R(x,y) + \lambda R_2(x,y) + \lambda^2 R_3(x,y) + \cdots,$$

whence we have

$$\phi(x) = F(x) - \lambda \int_0^h H_R(x,y;\lambda)F(y)dy. \tag{4.3.18}$$

Substituting the given expression (4.3.17) for $F(x)$ into Eq. (4.3.18), we have

$$\begin{aligned}\phi(x) = {} & f(x) + \lambda \int_0^h \sum_{n=1}^{N} g_n(x)h_n(y)\phi(y)dy \\ & -\lambda \int_0^h H_R(x,y;\lambda)\left[f(y) + \lambda \int_0^h \sum_{n=1}^{N} g_n(y)h_n(z)\phi(z)dz\right]dy.\end{aligned}$$

Define

$$\tilde{F}(x) \equiv f(x) - \lambda \int_0^h H_R(x,y;\lambda)f(y)dy.$$

Then

$$\begin{aligned}\phi(x) = {} & \tilde{F}(x) + \lambda \int_0^h dy \sum_{n=1}^{N} g_n(x)h_n(y)\phi(y) \\ & -\lambda^2 \int_0^h dy \int_0^h dz H_R(x,y;\lambda)\left(\sum_{n=1}^{N} g_n(y)h_n(z)\right)\phi(z).\end{aligned} \tag{4.3.19}$$

In the above expression (4.3.19), interchange y and z in the last term on the right-hand side,

$$\phi(x) = \tilde{F}(x) + \lambda \int_0^h dy \left[\sum_{n=1}^{N} g_n(x)h_n(y) - \lambda \int_0^h dz H_R(x,z;\lambda) \sum_{n=1}^{N} g_n(z)h_n(y)\right]\phi(y).$$

Then we have

$$\phi(x) = \tilde{F}(x) + \lambda \int_0^h dy \left(\sum_{n=1}^{N} G_n(x; \lambda) h_n(y) \right) \phi(y)$$

where

$$G_n(x; \lambda) = g_n(x) - \lambda \int_0^h dz H_R(x, z; \lambda) g_n(z).$$

We have thus reduced the integral equation with the general kernel to one with a *Pincherle–Goursat-type kernel*. The only difference is that the entries in the new kernel

$$\sum_{n=1}^{N} G_n(x; \lambda) h_n(y)$$

also depend on λ through the dependence of $G_n(x; \lambda)$ on λ. However, we know that for $|\lambda| \cdot \|R\| < 1$, the resolvent $H_R(x, y; \lambda)$ is analytic in λ. Hence $G_n(x; \lambda)$ is also analytic in λ. Therefore, the singularities in the complex λ plane are still found by setting $\det(I - \lambda A) = 0$, where the knth element of A is given by

$$A_{kn} = \int_0^h dy h_k(y) G_n(y; \lambda) = A_{kn}(\lambda).$$

Since the entries A_{kn} depend on λ analytically, the function $\det(I - \lambda A)$ is an analytic function of λ (but not necessarily a polynomial of degree N), and hence it has finitely many zeros in the region $|\lambda| \cdot \|R\| < 1$, or

$$|\lambda| < 1/\varepsilon < 1/\|R\|.$$

This concludes the proof that in any disk $|\lambda| < 1/\varepsilon$, there are finitely many singularities of λ for the integral equation (4.3.14).

4.4 Fredholm Theory for a Bounded Kernel

We now consider the case of a general kernel as approached by Fredholm. We shall show that the resolvent kernel can be written as a ratio of the entire functions of λ, whence the singularities in λ occur when the function in the denominator is zero.

Consider

$$\phi(x) = f(x) + \lambda \int_0^h K(x, y) \phi(y) dy, \quad 0 \le x \le h. \tag{4.4.1}$$

Discretize the above equation by letting

$$\varepsilon = h/N, \quad x_i = i\varepsilon, \quad y_j = j\varepsilon, \quad i,j = 0,1,2,\dots,N.$$

Also let

$$\phi_i = \phi(x_i), \quad f_i = f(x_i), \quad K_{ij} = K(x_i, y_j).$$

The discrete version of the integral equation (4.4.1) takes the form

$$\phi_i = f_i + \lambda \sum_{j=1}^{N} K_{ij}\phi_j \varepsilon,$$

i.e.,

$$\sum_{j=1}^{N} (\delta_{ij} - \lambda\varepsilon K_{ij})\phi_j = f_i. \tag{4.4.2}$$

Define $\tilde{D}(\lambda)$ to be

$$\tilde{D}(\lambda) = \det(I - \lambda\varepsilon K).$$

Writing out $\tilde{D}(\lambda)$ explicitly, we have

$$\tilde{D}(\lambda) = \begin{vmatrix} 1-\lambda\varepsilon K_{11}, & -\lambda\varepsilon K_{12}, & \cdot\ \cdot\ \cdot & -\lambda\varepsilon K_{1N} \\ -\lambda\varepsilon K_{21}, & 1-\lambda\varepsilon K_{22}, & \cdot\ \cdot\ \cdot & -\lambda\varepsilon K_{2N} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ -\lambda\varepsilon K_{N1}, & -\lambda\varepsilon K_{N2}, & \cdot\ \cdot\ \cdot & 1-\lambda\varepsilon K_{NN} \end{vmatrix}.$$

This determinant can be expanded as

$$\tilde{D}(\lambda) = \tilde{D}(0) + \lambda\tilde{D}'(0) + \frac{\lambda^2}{2!}\tilde{D}''(0) + \cdots + \frac{\lambda^N}{N!}\tilde{D}^{(N)}(0).$$

Using the fact that

$$\frac{d}{d\lambda}\left|\vec{a}_1, \vec{a}_2, \dots, \vec{a}_N\right| = \left|\frac{d}{d\lambda}\vec{a}_1, \vec{a}_2, \dots, \vec{a}_N\right| + \left|\vec{a}_1, \frac{d}{d\lambda}\vec{a}_2, \dots, \vec{a}_N\right| + \cdots + \left|\vec{a}_1, \vec{a}_2, \dots, \frac{d}{d\lambda}\vec{a}_N\right|,$$

we finally obtain (after considerable algebra)

$$\tilde{D}(\lambda) = 1 - \lambda\varepsilon\sum_{i=1}^{N} K_{ii} + \frac{\lambda^2\varepsilon^2}{2!}\sum_{i=1}^{N}\sum_{j=1}^{N}\begin{vmatrix} K_{ii}, & K_{ij} \\ K_{ji}, & K_{jj} \end{vmatrix}$$
$$- \frac{\lambda^3\varepsilon^3}{3!}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\begin{vmatrix} K_{ii}, & K_{ij}, & K_{ik} \\ K_{ji}, & K_{jj}, & K_{jk} \\ K_{ki}, & K_{kj}, & K_{kk} \end{vmatrix} + \cdots.$$

In the limit as $n \to \infty$, each sum when multiplied by ε is approximates a corresponding integral, i.e.,

$$\sum_{i=1}^{N}\varepsilon K_{ii} \to \int_0^h K(x,x)dx, \sum_{i=1}^{N}\sum_{j=1}^{N}\varepsilon^2\begin{vmatrix} K_{ii}, & K_{ij} \\ K_{ji}, & K_{jj} \end{vmatrix}$$
$$\to \int_0^h dx \int_0^h dy \begin{vmatrix} K(x,x), & K(x,y) \\ K(y,x), & K(y,y) \end{vmatrix}.$$

Define

$$K\begin{pmatrix} x_1, & x_2, & \cdot & \cdot & \cdot & x_n \\ y_1, & y_2, & \cdot & \cdot & \cdot & y_n \end{pmatrix}$$
$$\equiv \begin{vmatrix} K(x_1,y_1), & K(x_1,y_2), & \cdot & \cdot & \cdot & K(x_1,y_n) \\ K(x_2,y_1), & \cdot & \cdot & \cdot & \cdot & K(x_2,y_n) \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ K(x_n,y_1), & \cdot & & & & K(x_n,y_n) \end{vmatrix}.$$

Then, in the limit as $N \to \infty$, we find (on calling $\tilde{D}$ as D)

$$D(\lambda) = 1 + \sum_{n=1}^{\infty}((-1)^n\lambda^n/n!)D_n,$$

with

$$D_n = \int_0^h dx_1 \int_0^h dx_2 \cdots \int_0^h dx_n K\begin{pmatrix} x_1, & x_2, & \cdot & \cdot & \cdot & x_n \\ x_1, & x_2, & \cdot & \cdot & \cdot & x_n \end{pmatrix}.$$

We expect singularities in the resolvent $H(x, y; \lambda)$ to occur only when the determinant $D(\lambda)$ vanishes. Thus we hope to show that $H(x, y; \lambda)$ can be expressed as the ratio

$$H(x, y; \lambda) \equiv D(x, y; \lambda) / D(\lambda). \tag{4.4.3}$$

So we need to obtain the numerator $D(x, y; \lambda)$ and show that it is entire. We also show that the power series given above for $D(\lambda)$ has an infinite radius of convergence and thus represents an analytic function.

To this end, we make use of the fact that $K(x, y)$ is *bounded* and also invoke the *Hadamard inequality* which says

$$\left|\det[\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n]\right| \leq \|\vec{v}_1\| \, \|\vec{v}_2\| \cdots \|\vec{v}_n\| .$$

This has the interpretation that the volume of the parallelepiped whose edges are $\vec{v}_1$ through $\vec{v}_n$ is less than the product of the lengths of those edges. Suppose that $\left|K(x, y)\right|$ is bounded by A on $x, y \in [0, h]$. Then

$$K\begin{pmatrix} x_{1,} & \cdots & x_n \\ x_1, & \cdots & x_n \end{pmatrix} = \begin{vmatrix} K(x_1, x_1), & \cdots & K(x_1, x_n) \\ \cdot & & \cdot \\ K(x_n, x_1), & \cdots & K(x_n, x_n) \end{vmatrix}$$

is bounded by

$$\left|K\begin{pmatrix} x_1, & \cdots & x_n \\ x_1, & \cdots & x_n \end{pmatrix}\right| \leq (\sqrt{n}A)^n,$$

since the norm of each column is less than $\sqrt{n}A$. This implies

$$|D_n| = \left|\int_0^h dx_1 \cdots \int_0^h dx_n K\begin{pmatrix} x_{1,} & \cdots & x_n \\ x_1, & \cdots & x_n \end{pmatrix}\right| \leq h^n n^{n/2} A^n.$$

Thus

$$\left|\sum_{n=1}^{\infty}((-1)^n \lambda^n / n!) D_n\right| \leq \sum_{n=1}^{\infty}(|\lambda|^n \, h^n n^{n/2} A^n) / n!. \tag{4.4.4}$$

Letting

$$a_n = (|\lambda|^n \, h^n n^{n/2} A^n) / n!,$$

and applying the ratio test to the right-hand side of inequality (4.4.4), we have

$$\begin{aligned} \lim_{n\to\infty} a_{n+1} / a_n &= \lim_{n\to\infty} (|\lambda|^{n+1} \, h^{n+1} (n+1)^{(n+1)/2} A^{n+1} \cdot n!) / \\ &\qquad (|\lambda|^n \, h^n n^{n/2} A^n \cdot (n+1)!) \\ &= \lim_{n\to\infty} \left[|\lambda| \, hA(1 + \frac{1}{n})^{n/2} \frac{1}{\sqrt{n+1}} \right] = 0. \end{aligned}$$

Hence the series converges for all λ. We conclude that $D(\lambda)$ is an entire function of λ.

The last step we need to take is to find the numerator $D(x, y; \lambda)$ of the resolvent and show that it too is an entire function of λ. For this purpose, we recall that the resolvent itself $H(x, y; \lambda)$ satisfies the integral equation

$$H(x, y; \lambda) = -K(x, y) + \lambda \int_0^h K(x, z) H(z, y; \lambda) dz. \tag{4.4.5}$$

Therefore, on multiplying the integral equation (4.4.5) by $D(\lambda)$ and using definition (4.4.3) of $D(x, y; \lambda)$, we have

$$D(x, y; \lambda) = -K(x, y) D(\lambda) + \lambda \int_0^h K(x, z) D(z, y; \lambda) dz. \tag{4.4.6}$$

Recall that $D(\lambda)$ has the expansion

$$D(\lambda) = \sum_{n=0}^{\infty} ((-\lambda)^n / n!) D_n \quad \text{with} \quad D_0 = 1. \tag{4.4.7}$$

We seek an expansion for $D(x, y; \lambda)$ of the form

$$D(x, y; \lambda) = \sum_{n=0}^{\infty} ((-\lambda)^n / n!) C_n(x, y). \tag{4.4.8}$$

Substituting Eqs. (4.4.7) and (4.4.8) into the integral equation (4.4.6) for $D(x, y; \lambda)$, we find

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} C_n(x, y) \\ &\quad = -\sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} D_n K(x, y) - \sum_{n=0}^{\infty} \int_0^h \frac{(-\lambda)^{n+1}}{n!} K(x, z) C_n(z, y) dz. \end{aligned}$$

Collecting like powers of λ, we get

$$\begin{aligned} C_0(x, y) &= -K(x, y) \quad \text{for} \quad n = 0, \\ C_n(x, y) &= -D_n K(x, y) - n \int_0^h K(x, z) C_{n-1}(z, y) dz \quad \text{for} \quad n = 1, 2, \ldots . \end{aligned}$$

Let us calculate the first few of these:

$$\begin{aligned} C_0(x, y) &= -K(x, y). \\ C_1(x, y) &= -K(x, y) D_1 + \int_0^h K(x, z) K(z, y) dz = -\int_0^h dx_1 K \begin{pmatrix} x, & x_1 \\ y, & x_1 \end{pmatrix}, \end{aligned}$$

$$C_2(x,y) = -\int_0^h dx_1 \int_0^h dx_2 \left[K(x,y)K\begin{pmatrix} x_1, & x_2 \\ x_1, & x_2 \end{pmatrix} \right.$$
$$\left. -K(x,x_1)K\begin{pmatrix} x_1, & x_2 \\ y, & x_2 \end{pmatrix} + K(x,x_2)K\begin{pmatrix} x_1, & x_2 \\ y, & x_1 \end{pmatrix} \right]$$
$$= -\int_0^h dx_1 \int_0^h dx_2 K\begin{pmatrix} x, & x_1, & x_2 \\ y, & x_1, & x_2 \end{pmatrix}.$$

In general, we have

$$C_n(x,y) = -\int_0^h dx_1 \int_0^h dx_2 \cdots \int_0^h dx_n K\begin{pmatrix} x, & x_1, & x_2, & \cdots & x_n \\ y, & x_1, & x_2, & \cdots & x_n \end{pmatrix}.$$

Therefore, we have the numerator $D(x,y;\lambda)$ of $H(x,y;\lambda)$,

$$D(x,y;\lambda) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} C_n(x,y), \tag{4.4.9}$$

with

$$C_n(x,y) = -\int_0^h dx_1 \cdots \int_0^h dx_n K\begin{pmatrix} x, & x_1, & x_2, & \cdots & x_n \\ y, & x_1, & x_2, & \cdots & x_n \end{pmatrix}, \quad n = 1,2,\ldots, \tag{4.4.10}$$

$$C_0(x,y) = -K(x,y). \tag{4.4.11}$$

We prove that the power series for $D(x,y;\lambda)$ converges for all λ. First, by the Hadamard inequality, we have the following bounds:

$$\left| K\begin{pmatrix} x, & x_1, & \cdots & x_n \\ y, & x_1, & \cdots & x_n \end{pmatrix} \right| \le (\sqrt{n+1}A)^{n+1},$$

since K above is the $(n+1)\times(n+1)$ determinant with each entry less than A, i.e.,

$$\left|K(x,y)\right| < A.$$

Then the bound on $C_n(x,y)$ is given by

$$\left|C_n(x,y)\right| \le h^n(\sqrt{n+1}A)^{n+1}.$$

Thus we have the bound on $D(x,y;\lambda)$ as

$$\left|D(x,y;\lambda)\right| \le \sum_{n=0}^{\infty} \frac{|\lambda|^n}{n!} h^n (\sqrt{n+1}A)^{n+1}. \tag{4.4.12}$$

Letting

$$a_n = \frac{|\lambda|^n}{n!} h^n (\sqrt{n+1}A)^{n+1},$$

we apply the ratio test on the right-hand side of inequality (4.4.12):

$$\lim_{n\to\infty} \frac{a_n}{a_{n-1}} = \lim_{n\to\infty} |\lambda|\, hA \frac{\sqrt{n+1}}{n} \left(1+\frac{1}{n}\right)^{n/2} = 0.$$

Hence the power series expansion for $D(x, y; \lambda)$ converges for all λ, and $D(x, y; \lambda)$ is an entire function of λ.

Finally, we can prove that whenever $H(x, y; \lambda)$ exists (i.e., for all λ such that $D(\lambda) \neq 0$), the solution to the integral equation (4.4.1) is *unique*. This is best done using the operator notation introduced in Section 4.2. Consider the homogeneous problem

$$\phi_{\mathrm{H}} = \lambda \tilde{K} \phi_{\mathrm{H}} \tag{4.4.13}$$

and multiply both sides by $\tilde{H}$ to find

$$\tilde{H}\phi_{\mathrm{H}} = \lambda \tilde{H}\tilde{K}\phi_{\mathrm{H}}.$$

Use the identity

$$\lambda \tilde{H}\tilde{K} = \tilde{K} + \tilde{H}$$

to get

$$\tilde{H}\phi_{\mathrm{H}} = \tilde{K}\phi_{\mathrm{H}} + \tilde{H}\phi_{\mathrm{H}}.$$

Hence we get

$$\tilde{K}\phi_{\mathrm{H}} = 0,$$

which implies

$$\phi_{\mathrm{H}} = \lambda \tilde{K}\phi_{\mathrm{H}} = 0. \tag{4.4.14}$$

Thus the homogeneous problem has no nontrivial solutions ($\phi_{\mathrm{H}} = 0$), and the inhomogeneous problem has a unique solution.

Summary of the Fredholm Theory for a Bounded Kernel

The integral equation

$$\phi(x) = f(x) + \lambda \int_0^h K(x, y)\phi(y)dy$$

has the solution

$$\phi(x) = f(x) - \lambda \int_0^h H(x, y; \lambda) f(y) dy,$$

with the resolvent kernel given by

$$H(x, y; \lambda) = D(x, y; \lambda) / D(\lambda),$$

where

$$D(\lambda) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} D_n$$

$$D_n = \int_0^h dx_1 \cdots \int_0^h dx_n K\begin{pmatrix} x_1, & \cdots & x_n \\ x_1, & \cdots & x_n \end{pmatrix}; \quad D_0 = 1$$

and

$$D(x, y; \lambda) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} C_n(x, y),$$

$$C_n(x, y) = -\int_0^h dx_1 \cdots \int_0^h dx_n K\begin{pmatrix} x, & x_1, & \cdots & x_n \\ y, & x_1, & \cdots & x_n \end{pmatrix};$$

$$C_0(x, y) = -K(x, y),$$

where

$$K\begin{pmatrix} z_1, & z_2, & \cdots & z_n \\ w_1, & w_2, & \cdots & w_n \end{pmatrix} \equiv \begin{vmatrix} K(z_1, w_1), & K(z_1, w_2), & \cdots & K(z_1, w_n) \\ K(z_2, w_1), & K(z_2, w_2), & \cdots & K(z_2, w_n) \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ K(z_n, w_1), & K(z_n, w_2), & \cdots & K(z_n, w_n) \end{vmatrix}.$$

4.5 Solvable Example

Consider the following homogeneous integral equation:

❑ **Example 4.2.** Solve

$$\phi(x) = \lambda \int_0^x dy e^{-(x-y)} \phi(y) + \lambda \int_x^{\infty} dy \phi(y), \quad 0 \le x < \infty. \tag{4.5.1}$$

Solution. This is a Fredholm integral equation of the second kind with the kernel

$$K(x, y) = \begin{cases} e^{-(x-y)} & \text{for} \quad 0 \le y < x < \infty, \\ 1 & \text{for} \quad 0 \le x \le y < \infty. \end{cases} \tag{4.5.2}$$

Note that this kernel (4.5.2) is not square-integrable. Differentiating both sides of Eq. (4.5.1) once, we find after a little algebra

$$e^x \phi'(x) = -\lambda \int_0^x dy e^y \phi(y). \tag{4.5.3}$$

Differentiate both sides of Eq. (4.5.3) once more to obtain the second-order ordinary differential equation of the form

$$\phi''(x) + \phi'(x) + \lambda \phi(x) = 0. \tag{4.5.4}$$

We try a solution of the form

$$\phi(x) = Ce^{\alpha x}. \tag{4.5.5}$$

Substituting Eq. (4.5.5) into Eq. (4.5.4), we obtain $\alpha^2 + \alpha + \lambda = 0$, or

$$\alpha = \frac{-1 \pm \sqrt{1 - 4\lambda}}{2}.$$

In general, we obtain

$$\phi(x) = C_1 e^{\alpha_1 x} + C_2 e^{\alpha_2 x}, \tag{4.5.6}$$

with

$$\alpha_1 = \frac{-1 + \sqrt{1 - 4\lambda}}{2}, \quad \alpha_2 = \frac{-1 - \sqrt{1 - 4\lambda}}{2}. \tag{4.5.7}$$

Now, the expression for $\phi'(x)$ given above, Eq. (4.5.3), indicates that

$$\phi'(0) = 0. \tag{4.5.8}$$

This requires

$$\alpha_1 C_1 + \alpha_2 C_2 = 0 \quad \text{or} \quad C_2 = -\frac{\alpha_1}{\alpha_2} C_1.$$

Hence the solution is

$$\phi(x) = C\left[(e^{\alpha_1 x} / \alpha_1) - (e^{\alpha_2 x} / \alpha_2)\right].$$

However, in order for the integral equation to make sense, we must require the integral

$$\int_x^\infty dy\phi(y)$$

to converge. This requires

$$\mathrm{Re}\,\alpha < 0,$$

which in turn requires

$$\lambda > 0. \tag{4.5.9}$$

Thus, for $\lambda \leq 0$, we have no solution, and for $\lambda > 0$, we have

$$\phi(x) = C\left[(e^{\alpha_1 x}/\alpha_1) - (e^{\alpha_2 x}/\alpha_2)\right], \tag{4.5.10}$$

with α_1 and α_2 given by Eq. (4.5.7).

Note that, in this case, we have a continuous spectrum of eigenvalues ($\lambda > 0$) for which the homogeneous problem has a solution. The reason why the eigenvalue is not discrete is that $K(x, y)$ is not square-integrable.

4.6 Fredholm Integral Equation with a Translation Kernel

Suppose $x \in (-\infty, +\infty)$ and the kernel is *translation invariant*, i.e.,

$$K(x, y) = K(x - y). \tag{4.6.1}$$

Then the inhomogeneous Fredholm integral equation of the second kind is given by

$$\phi(x) = f(x) + \lambda \int_{-\infty}^{+\infty} K(x - y)\phi(y)dy. \tag{4.6.2}$$

Take the *Fourier transform* of both sides of Eq. (4.6.2) to find

$$\hat{\phi}(k) = \hat{f}(k) + \lambda \hat{K}(k)\hat{\phi}(k).$$

Solve for $\hat{\phi}(k)$ to find

$$\hat{\phi}(k) = \frac{\hat{f}(k)}{1 - \lambda \hat{K}(k)}. \tag{4.6.3}$$

Solution $\phi(x)$ is provided by inverting the Fourier transform $\hat{\phi}(k)$ obtained above. It seems so simple, but there are some subtleties involved in the inversion of $\hat{\phi}(k)$. We present some general discussion of the *inversion of the Fourier transform*.

Suppose that the function $F(x)$ has the asymptotic forms

$$F(x) \sim \begin{cases} e^{ax} & \text{as} \quad x \to +\infty \quad (a > 0), \\ e^{bx} & \text{as} \quad x \to -\infty \quad (b > a > 0). \end{cases} \tag{4.6.4}$$

Namely, $F(x)$ grows exponentially as $x \to +\infty$ and decays exponentially as $x \to -\infty$. Then the Fourier transform

$$\hat{F}(k) = \int_{-\infty}^{+\infty} e^{-ikx} F(x) dx \tag{4.6.5}$$

exists as long as

$$-b < \operatorname{Im} k < -a. \tag{4.6.6}$$

This is because the integrand has the magnitude

$$\left| e^{-ikx} F(x) \right| \sim \begin{cases} e^{(k_2+a)x} & \text{as} \quad x \to +\infty, \\ e^{(k_2+b)x} & \text{as} \quad x \to -\infty, \end{cases}$$

where we set

$$k = k_1 + ik_2, \quad \text{with} \quad k_1 \text{ and } k_2 \quad \text{real.}$$

With

$$-b < k_2 < -a,$$

the magnitude of the integrand vanishes exponentially at both ends.

The inverse Fourier transformation then becomes

$$F(x) = \frac{1}{2\pi} \int_{-\infty - i\gamma}^{+\infty - i\gamma} e^{ikx} \hat{F}(k) dk \quad \text{with} \quad a < \gamma < b. \tag{4.6.7}$$

Now if $b = a$ such that $F(x) \sim e^{ax}$ for $|x| \to \infty$ $(a > 0)$, then the inversion contour is on $\gamma = a$ and the Fourier transform exists only for $k_2 = -a$.

Similarly if the function $F(x)$ decays exponentially as $x \to +\infty$ and grows as $x \to -\infty$, we are able to continue defining the Fourier transform and its inverse by going in the upper half plane.

As an example, a function like

$$F(x) = e^{-\alpha|x|}, \tag{4.6.8}$$

which decays exponentially as $x \to \pm\infty$, has a Fourier transform which exists and is analytic in

$$-\alpha < k_2 < \alpha. \tag{4.6.9}$$

With these qualifications, we should be able to invert $\hat{\phi}(k)$ to obtain the solution to the inhomogeneous problem (4.6.2).

Now comes the homogeneous problem,

$$\phi_{\mathrm{H}}(x) = \lambda \int_{-\infty}^{+\infty} K(x-y)\phi_{\mathrm{H}}(y)dy. \tag{4.6.10}$$

By Fourier transforming Eq. (4.6.10), we obtain

$$(1 - \lambda \hat{K}(k))\hat{\phi}_{\mathrm{H}}(k) = 0. \tag{4.6.11}$$

If $1 - \lambda \hat{K}(k)$ has no zeros for all k, then we have

$$\hat{\phi}_{\mathrm{H}}(k) = 0 \quad \Rightarrow \quad \phi_{\mathrm{H}}(x) = 0, \tag{4.6.12}$$

i.e., no nontrivial solution exists for the homogeneous problem. If, on the other hand, $1 - \lambda \hat{K}(k)$ has a zero of order n at $k = \alpha$, $\hat{\phi}_{\mathrm{H}}(k)$ can be allowed to be of the form

$$\hat{\phi}_{\mathrm{H}}(k) = C_1\delta(k-\alpha) + C_2\frac{d}{dk}\delta(k-\alpha) + \cdots + C_n\left(\frac{d}{dk}\right)^{n-1}\delta(k-\alpha).$$

On inversion, we find

$$\phi_{\mathrm{H}}(x) = C_1 e^{i\alpha x} + C_2 x e^{i\alpha x} + \cdots + C_n x^{n-1} e^{i\alpha x} = e^{i\alpha x}\sum_{j=1}^{n} C_j x^{j-1}. \tag{4.6.13}$$

For the homogeneous problem, we choose the inversion contour of the Fourier transform based on the asymptotic behavior of the kernel, a point to be discussed in the following example.

❑ **Example 4.3.** Consider the homogeneous integral equation,

$$\phi_{\mathrm{H}}(x) = \lambda \int_{-\infty}^{+\infty} e^{-|x-y|}\phi_{\mathrm{H}}(y)dy. \tag{4.6.14}$$

Solution. Since the kernel vanishes exponentially as e^{-y} as $y \to \infty$ and as e^{+y} as $y \to -\infty$, we need not require $\phi_{\mathrm{H}}(y)$ to vanish as $y \to \pm\infty$; rather, more generally we may permit

$$\phi_{\mathrm{H}}(y) \to \begin{cases} e^{(1-\varepsilon)y} & \text{as} \quad y \to \infty, \\ e^{(-1+\varepsilon)y} & \text{as} \quad y \to -\infty, \end{cases}$$

and the integral equation still makes sense. So in the Fourier transform, we may allow

$$-1 < k_2 < 1, \tag{4.6.15}$$

and we still have a valid solution.

The Fourier transform of $e^{-\alpha|x|}$ is given by

$$\int_{-\infty}^{+\infty} e^{-ikx} e^{-\alpha|x|} dx = 2\alpha \big/ \left(k^2 + \alpha^2\right). \tag{4.6.16}$$

Taking the Fourier transform of the homogeneous equation with $\alpha = 1$, we find

$$\hat{\phi}_{\mathrm{H}}(k) = (2\lambda/(k^2+1))\hat{\phi}_H(k),$$

from which, we obtain

$$\frac{k^2+1-2\lambda}{k^2+1}\hat{\phi}_{\mathrm{H}}(k) = 0.$$

So there exists no nontrivial solution unless $k = \pm i\sqrt{1-2\lambda}$. By the inversion formula, $\phi_{\mathrm{H}}(x)$ is a superposition of e^{+ikx} terms with amplitude $\hat{\phi}_{\mathrm{H}}(k)$. But $\hat{\phi}_{\mathrm{H}}(k)$ is zero for all but $k = \pm i\sqrt{1-2\lambda}$. Hence we may conclude tentatively

$$\phi_{\mathrm{H}}(x) = C_1 e^{-\sqrt{1-2\lambda}x} + C_2 e^{+\sqrt{1-2\lambda}x}.$$

However, we can at most allow $\phi_{\mathrm{H}}(x)$ to grow as fast as e^x as $x \to \infty$ and as e^{-x} as $x \to -\infty$, as we discussed above. Thus further analysis is in order.

Case (1). $1 - 2\lambda < 0$, or $\lambda > 1/2$.

$\phi_{\mathrm{H}}(x)$ is oscillatory and is given by

$$\phi_{\mathrm{H}}(x) = C_1 e^{-i\sqrt{2\lambda-1}x} + C_2 e^{+i\sqrt{2\lambda-1}x}. \tag{4.6.17a}$$

Case (2). $0 < 1 - 2\lambda < 1$, or $0 < \lambda < 1/2$.

$\phi_{\mathrm{H}}(x)$ grows less fast than $e^{|x|}$ as $|x| \to \infty$:

$$\phi_{\mathrm{H}}(x) = C_1 e^{-\sqrt{1-2\lambda}x} + C_2 e^{+\sqrt{1-2\lambda}x}. \tag{4.6.17b}$$

Case (3). $1 - 2\lambda \geq 1$, or $\lambda \leq 0$.

No acceptable solution for $\phi_{\mathrm{H}}(x)$ exists, since $e^{\pm\sqrt{1-2\lambda}x}$ grows faster than $e^{|x|}$ as $|x| \to \infty$.

Case (4). $\lambda = 1/2$.

$$\phi_{\mathrm{H}}(x) = C_1 + C_2 x. \tag{4.6.17c}$$

Now consider the corresponding inhomogeneous problem.

❑ **Example 4.4.** onsider the inhomogeneous integral equation,

$$\phi(x) = ae^{-\alpha|x|} + \lambda \int_{-\infty}^{+\infty} e^{-|x-y|}\phi(y)dy. \tag{4.6.18}$$

Solution. On taking the Fourier transform of Eq. (4.6.18), we obtain

$$\hat{\phi}(k) = (2a\alpha/(k^2+\alpha^2)) + (2\lambda/(k^2+1))\hat{\phi}(k).$$

Solving for $\hat{\phi}(k)$, we obtain

$$\hat{\phi}(k) = \frac{2a\alpha(k^2+1)}{(k^2+1-2\lambda)(k^2+\alpha^2)}.$$

To invert the latter transform, we note that depending on whether λ is larger or smaller than $1/2$, the poles $k = \pm\sqrt{2\lambda - 1}$ lie on the real or imaginary axis of the complex k plane. What we can do is to choose any contour for the inversion within the strip

$$-\min(\alpha, 1) < k_2 < \min(\alpha, 1) \tag{4.6.19}$$

to get a *particular solution* to our equation and we may then add any multiple of the homogeneous solution when the latter exists. The reason for choosing the strip (4.6.19) instead of

$$-1 < k_2 < 1$$

in this case is that, in order for the Fourier transform of the inhomogeneous term $e^{-\alpha|x|}$ to exist, we must also restrict our attention to

$$-\alpha < k_2 < \alpha.$$

Consider the first three cases given in Example 4.3.

Cases (2) and (3). $\lambda < 1/2$.

In these cases, we have $1 - 2\lambda > 0$. Hence $\hat{\phi}(k)$ has simple poles at $k = \pm i\alpha$ and $k = \pm i\sqrt{1-2\lambda}$. To find a particular solution, use the real k-axis as the integration contour for the inverse Fourier transformation. Then $\phi_P(x)$ is given by

$$\phi_P(x) = \frac{2a\alpha}{2\pi}\int_{-\infty}^{+\infty} dk e^{ikx}\frac{(k^2+1)}{(k^2+1-2\lambda)(k^2+\alpha^2)}.$$

For $x > 0$, we close the contour in the upper half plane to obtain

$$\phi_P(x) = \frac{a}{1-2\lambda-\alpha^2}\left[\left(1-\alpha^2\right)e^{-\alpha x} - \frac{2\lambda\alpha}{\sqrt{1-2\lambda}}e^{-\sqrt{1-2\lambda}x}\right].$$

For $x < 0$, we close the contour in the lower half plane to get an identical result with x replaced with $-x$. Our particular solution $\phi_{\mathrm{P}}(x)$ is given by

$$\phi_{\mathrm{P}}(x) = \frac{a}{1 - 2\lambda - \alpha^2}\left[\left(1 - \alpha^2\right)e^{-\alpha|x|} - \frac{2\lambda\alpha}{\sqrt{1 - 2\lambda}}e^{-\sqrt{1-2\lambda}|x|}\right]. \tag{4.6.20}$$

For Case (3), this is the unique solution because there exists no acceptable homogeneous solution, while for Case (2) we must also add the homogeneous part given by

$$\phi_{\mathrm{H}}(x) = C_1 e^{-\sqrt{1-2\lambda}x} + C_2 e^{+\sqrt{1-2\lambda}x}.$$

Case (1). $\lambda > 1/2$.

In this case, we have $1 - 2\lambda < 0$. Hence $\hat{\phi}(k)$ has simple poles at $k = \pm i\alpha$ and $k = \pm\sqrt{2\lambda - 1}$. To do the inversion for the particular solution, we can take any of the contours (1), (2), (3), or (4) as displayed in Figures 4.1–4.4, or Principal Value contours which are equivalent to half the sum of the first two contours

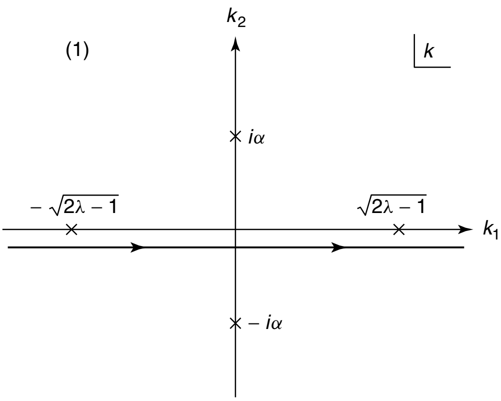

Fig. 4.1 The inversion contour (1) for Case (1).

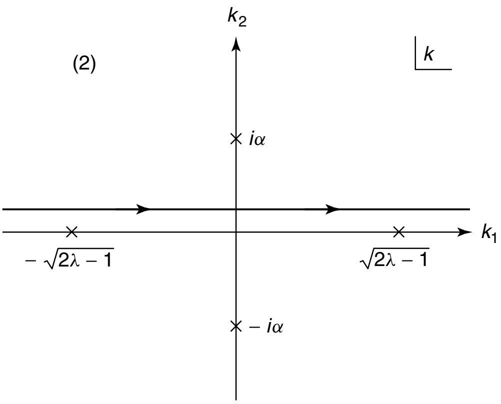

Fig. 4.2 The inversion contour (2) for Case (1).

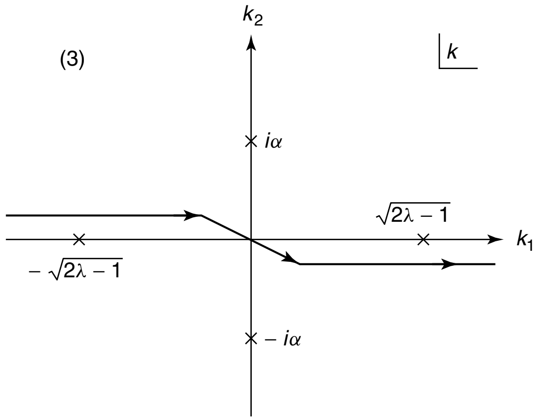

Fig. 4.3 The inversion contour (3) for Case (1).

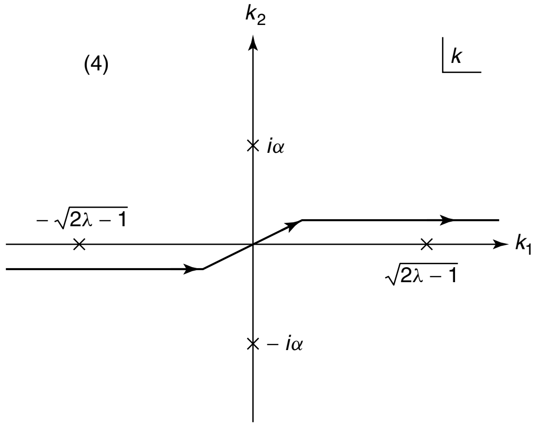

Fig. 4.4 The inversion contour (4) for Case (1).

or half the sum of the latter two contours. Any of these differs by a multiple of the homogeneous solution. Consider a particular choice (4) for the inversion. For $x > 0$, we close the contour in the upper half plane. Then our particular solution $\phi_P(x)$ is given by

$$\phi_P(x) = \frac{a}{1 - 2\lambda - \alpha^2}\left[(1-\alpha^2)e^{-\alpha x} + \frac{2\lambda\alpha i}{\sqrt{2\lambda - 1}}e^{-i\sqrt{2\lambda-1}x}\right].$$

For $x < 0$, we close the contour in the lower half plane to get an identical result with x replaced with $-x$. So, in general, we can write our particular solution with the inversion contour (4) as,

$$\phi_P(x) = \frac{a}{1 - 2\lambda - \alpha^2}\left[\left(1-\alpha^2\right)e^{-\alpha|x|} + \frac{2\lambda\alpha i}{\sqrt{2\lambda - 1}}e^{-i\sqrt{2\lambda-1}|x|}\right], \tag{4.6.21}$$

to which must be added the homogeneous part for Case (1) which reads

$$\phi_{\mathrm{H}}(x) = C_1 e^{-i\sqrt{2\lambda-1}x} + C_2 e^{+i\sqrt{2\lambda-1}x}.$$

4.7 System of Fredholm Integral Equations of the Second Kind

We solve the system of Fredholm integral equations of the second kind,

$$\phi_i(x) - \lambda \int_a^b \sum_{j=1}^{n} K_{ij}(x,y)\phi_j(y)dy = f_i(x), \quad i = 1, 2, \ldots, n, \tag{4.7.1}$$

where the kernels $K_{ij}(x,y)$ are square-integrable. We first extend the basic interval from $[a,b]$ to $[a, a+n(b-a)]$, and set

$$x + (i-1)(b-a) = X < a + i(b-a), \quad y + (j-1)(b-a) = Y < a + j(b-a), \tag{4.7.2}$$

$$\phi(X) = \phi_i(x), \quad K(X,Y) = K_{ij}(x,y), \quad f(X) = f_i(x). \tag{4.7.3}$$

We then obtain the Fredholm integral equation of the second kind,

$$\phi(X) - \lambda \int_a^{a+n(b-a)} K(X,Y)\phi(Y)dY = f(X), \tag{4.7.4}$$

where the kernel $K(X,Y)$ is discontinuous in general but is square-integrable on account of the square-integrability of $K_{ij}(x,y)$. The solution $\phi(X)$ to Eq. (4.7.4) provides the solutions $\phi_i(x)$ to Eq. (4.7.1) with Eqs. (4.7.2) and (4.7.3).

4.8 Problems for Chapter 4

4.1. Calculate $D(\lambda)$ for

(a) $K(x,y) = \begin{cases} xy, & y \le x, \\ 0, & \text{otherwise.} \end{cases}$

(b) $K(x,y) = xy, \quad 0 \le x, y \le 1.$

(c) $K(x,y) = \begin{cases} g(x)h(y), & y \le x, \\ 0, & \text{otherwise.} \end{cases}$

(d) $K(x,y) = g(x)h(y), \quad 0 \le x, y \le 1.$

Find zero of $D(\lambda)$ for each case.

4.2. (due to H. C.). Solve

$$\phi(x) = \lambda \left\{ \int_0^x dy \frac{\phi(y)}{(y+1)^2} \left(\frac{y}{x}\right)^a + \int_x^{+\infty} dy \frac{\phi(y)}{(y+1)^2} \right\}, \quad a > 0.$$

Find all eigenvalues and eigenfunctions.

4.3. (due to H. C.). Solve the Fredholm integral equation of the second kind, given that

$$K(x-y) = e^{-|x-y|}, \quad f(x) = \begin{cases} x, & x > 0, \\ 0, & x < 0. \end{cases}$$

4.4. (due to H. C.). Solve the Fredholm integral equation of the second kind, given that

$$K(x-y) = e^{-|x-y|}, \quad f(x) = x \quad \text{for} \quad -\infty < x < +\infty.$$

4.5. (due to H. C.). Solve

$$\phi(x) + \lambda \int_{-1}^{+1} K(x,y)\phi(y)dy = 1, \quad -1 \le x \le 1, \quad K(x,y) = \sqrt{\frac{1-y^2}{1-x^2}}.$$

Find all eigenvalues of $K(x,y)$. Calculate also $D(\lambda)$ and $D(x,y;\lambda)$.

4.6. (due to H. C.). Solve the Fredholm integral equation of the second kind,

$$\phi(x) = e^{-\frac{x}{2}} + \lambda \int_{-\infty}^{+\infty} \frac{1}{\cosh(x-y)} \phi(y)dy.$$

Hint:

$$\int_{-\infty}^{+\infty} \frac{e^{ikx}}{\cosh x} dx = \frac{\pi}{\cosh(\pi k/2)}.$$

4.7. (due to H. C. and D. M.). Consider the integral equation,

$$\phi(x) = \lambda \int_{-\infty}^{+\infty} \frac{dy}{\sqrt{2\pi}} e^{ixy} \phi(y), \quad -\infty < x < \infty.$$

(a) Show that there are only four eigenvalues of the kernel $(1/\sqrt{2\pi}) \exp[ixy]$. What are these?

(b) Show by an explicit calculation that the functions

$$\phi_n(x) = \exp[-x^2/2]H_n(x),$$

where

$$H_n(x) \equiv (-1)^n \exp[x^2]\frac{d^n}{dx^n}\exp[-x^2],$$

are Hermite polynomials, are eigenfunctions with the corresponding eigenvalues $(i)^n$, $(n = 0, 1, 2, \ldots)$. Why should one expect $\phi_n(x)$ to be Fourier transforms of themselves?

Hint: Think of the Schrödinger equation for the harmonic oscillator.

(c) Using the result in (b) and the fact that $\{\phi_n(x)\}_n$ form a complete set, in some sense, show that any square-integrable solution is of the form

$$\phi(x) = f(x) + C\tilde{f}(x),$$

where $f(x)$ is an arbitrary even or odd, square-integrable function with the Fourier transform $\tilde{f}(k)$, and C is a suitable constant. Evaluate C and relate its values to the eigenvalues found in (a).

(d) From (c), construct a solution by taking $f(x) = \exp[-ax^2/2]$, $a > 0$.

4.8. (due to H. C.). Find an eigenvalue and the corresponding eigenfunction for

$$K(x, y) = \exp[-(ax^2 + 2bxy + cy^2)], \quad -\infty < x, y < \infty, \quad a + c > 0.$$

4.9. Consider the homogeneous integral equation,

$$\phi(x) = \lambda \int_{-\infty}^{\infty} K(x, y)\phi(y)dy, \quad -\infty < x < \infty,$$

where

$$K(x, y) = \frac{1}{\sqrt{1 - t^2}} \exp[\frac{x^2 + y^2}{2}] \exp[-\frac{x^2 + y^2 - 2xyt}{1 - t^2}], \quad t \text{ fixed}, \quad 0 < t < 1.$$

(a) Show directly that $\phi_0(x) = \exp[-x^2/2]$ is an eigenfunction of $K(x, y)$ corresponding to the eigenvalue $\lambda = \lambda_0 = 1/\sqrt{\pi}$.

(b) Let

$$\phi_n(x) = \exp[-x^2/2]H_n(x).$$

Assume that $\phi_n = \lambda_n K\phi_n$. Show that $\phi_{n+1} = \lambda_{n+1}K\phi_{n+1}$ with $\lambda_n = t\lambda_{n+1}$. This means that the original integral equation has eigenvalues $\lambda_n = t^{-n}/\sqrt{\pi}$, with the corresponding eigenfunctions $\phi_n(x)$.

4.10. (due to H. C.). Find the eigenvalues and eigenfunctions of the integral equation,

$$\phi(x) = \lambda \int_0^{\infty} \exp[-xy]\phi(y)dy.$$

Hint: Consider the Mellin transform,

$$\Phi(p) = \int_0^{\infty} x^{ip-\frac{1}{2}}\phi(x)dx \quad \text{with} \quad \phi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x^{-ip-\frac{1}{2}}\Phi(p)dp.$$

4.11. (due to H. C.). Solve

$$\psi(x) = e^{bx} + \lambda \int_0^x \psi(y)dy + 2\lambda \int_x^1 \psi(y)dy.$$

4.12. Solve

$$\phi(x) = \lambda \int_{-1}^{+1} K(x,y)\phi(y)dy - \frac{1}{2}\int_{-1}^{+1} \phi(y)dy \quad \text{with} \quad \phi(\pm 1) = \text{finite},$$

where

$$K(x,y) = \frac{1}{2}\ln\left(\frac{1+x_<}{1-x_>}\right),$$

$$x_< = \frac{1}{2}(x+y) - \frac{1}{2}|x-y| \quad \text{and} \quad x_> = \frac{1}{2}(x+y) + \frac{1}{2}|x-y|.$$

4.13. Solve

$$\phi(x) = \lambda \int_{-\infty}^{+\infty} K(x,y)\phi(y)dy \quad \text{with} \quad \phi(\pm\infty) = \text{finite},$$

where

$$K(x,y) = \sqrt{\frac{\alpha}{\pi}}\left\{\exp\left[\frac{\alpha}{2}(x^2+y^2)\right]\int_{-\infty}^{x_<} \exp[-\alpha\tau^2]d\tau \cdot \int_{x_>}^{+\infty} \exp[-\alpha\tau^2]d\tau\right\},$$

$$x_< = \frac{1}{2}(x+y) - \frac{1}{2}|x-y| \quad \text{and} \quad x_> = \frac{1}{2}(x+y) + \frac{1}{2}|x-y|.$$

4.14. Solve

$$\phi(x) = \lambda \int_0^\infty K(x, y)\phi(y)dy,$$

with

$$|\phi(x)| < \infty \quad \text{for} \quad 0 \le x < \infty,$$

where

$$K(x, y) = \frac{\exp[-\beta |x - y|]}{2\beta xy}.$$

4.15. (due to D. M.) Show that the nontrivial solutions of the homogeneous integral equation,

$$\phi(x) = \lambda \int_{-\pi}^{\pi} \left[\frac{1}{4\pi}(x - y)^2 - \frac{1}{2}|x - y| \right] \phi(y)dy,$$

are $\cos(mx)$ and $\sin(mx)$, where $\lambda = m^2$ and m is any integer.

Hint for Problems 4.12 through 4.15: The kernels change their forms continuously as x passes through y. Differentiate the given integral equations with respect to x and reduce them to the ordinary differential equations.

4.16. (due to D. M.) Solve the inhomogeneous integral equation,

$$\phi(x) = f(x) + \lambda \int_0^\infty \cos(2xy)\phi(y)dy, \quad x \ge 0,$$

where $\lambda^2 \ne 4/\pi$.

Hint: Multiply both sides of the integral equation by $\cos(2x\xi)$ and integrate over x. Use the identity

$$\cos(2xy)\cos(2x\xi) = \frac{1}{2}\left\{\cos[2x(y + \xi)] + \cos[2x(y - \xi)]\right\},$$

and observe that

$$\begin{aligned}\int_0^\infty \cos(\alpha x)dx &= \frac{1}{2}\int_0^\infty (\exp[i\alpha x] + \exp[-i\alpha x])dx \\ &= \frac{1}{2}\int_{-\infty}^\infty \exp[i\alpha x]dx \\ &= \frac{1}{2}\cdot 2\pi\delta(\alpha) = \pi\delta(\alpha).\end{aligned}$$

4.17. (due to D. M.) In the theoretical search for *supergain antennas*, maximizing the directivity in the far field of axially invariant currents $j(\phi)$ that flow along the surface of infinitely long, circular cylinders of radius a leads to the following Fredholm integral equation for the current density $j(\phi)$:

$$j(\phi) = \exp[ika \sin \phi] - \alpha \int_0^{2\pi} \frac{d\phi'}{2\pi} J_0 \left(2ka \sin \frac{\phi - \phi'}{2} \right) j(\phi'), \qquad 0 \leq \phi < 2\pi,$$

where ϕ is the polar angle of the circular cross section, k is a positive wave number, α is a parameter (Lagrange multiplier) that expresses a constraint on the current magnitude, $\alpha \geq 0$, and $J_0(x)$ is the 0th-order *Bessel function of the first kind*.

(a) Determine the eigenvalues of the homogeneous equation.

(b) Solve the given inhomogeneous equation in terms of Fourier series,

$$j(\phi) = \sum_{n=-\infty}^{\infty} f_n \exp[in\phi].$$

Hint: Use the formulas,

$$\exp[ika \sin \phi] = \sum_{n=-\infty}^{\infty} J_n(ka) \exp[in\phi],$$

and

$$J_0(2ka \sin \frac{\phi - \phi'}{2}) = \sum_{m=-\infty}^{\infty} J_m(ka)^2 \exp[im(\phi - \phi')],$$

where $J_n(x)$ is the nth-order *Bessel function of the first kind*. Substitution of Fourier series for $j(\phi)$,

$$j(\phi) = \sum_{n=-\infty}^{\infty} f_n \exp[in\phi],$$

yields the decoupled equation for f_n,

$$f_n = \frac{J_n(ka)}{1 + \alpha J_n(ka)^2}, \qquad n = -\infty, \ldots, \infty.$$

4.18. (due to D. M.) Problem 4.17 corresponds to the circular loop in two dimensions. For the circular disk in two dimensions, we have the following

Fredholm integral equation for the current density $j(\phi)$:

$$j(r,\phi) = \exp[ikr\sin\phi] - \frac{2\alpha}{a^2}\int_0^{2\pi}\frac{d\phi'}{2\pi}\int_0^a r'\,dr' \times J_0(k\sqrt{r^2 + r'^2 - 2rr'\cos(\phi-\phi')})j(r',\phi'),$$

with

$$0 \le r \le a, \quad 0 \le \phi < 2\pi.$$

Solve the given inhomogeneous equation in terms of Fourier series,

$$j(r,\phi) = \sum_{n=-\infty}^{n=\infty} f_n(r)\exp[in\phi].$$

Hint: By using the addition formula

$$J_0(k\sqrt{r^2 + r'^2 - 2rr'\cos(\phi-\phi')}) = \sum_{m=-\infty}^{\infty} J_m(kr)J_m(kr')\exp[im(\phi-\phi')],$$

it is found that $f_n(r)$ satisfy the following integral equation:

$$f_n(r) = \left[1 - \frac{2\alpha}{a^2}\int_0^a r'\,dr' f_n(r')J_n(kr')\right]J_n(kr), \quad n = -\infty,\ldots,\infty.$$

Substitution of

$$f_n(r) = \lambda_n J_n(kr)$$

yields

$$\lambda_n = \left[1 + \frac{2\alpha}{a^2}\int_0^a r'\,dr' J_n(kr')^2\right]^{-1} = \left[1 + \alpha[J_n(ka)^2 - J_{n+1}(ka)J_{n-1}(ka)]\right]^{-1}.$$

4.19. (due to D. M.) Problem 4.17 corresponds to the circular loop in two dimensions. For the circular loop in three dimensions, we have the following Fredholm integral equation for the current density $j(\phi)$:

$$j(\phi) = \exp[ika\sin\phi] - \alpha\int_0^{2\pi}\frac{d\phi'}{2\pi}\mathcal{K}(\phi-\phi')j(\phi'), \quad 0 \le \phi < 2\pi,$$

with

$$\mathcal{K}(\phi) = \frac{\sin w}{w} + \frac{\cos w}{w^2} - \frac{\sin w}{w^3}$$
$$= \frac{1}{4}\int_{-1}^{1}(1+\xi^2)\exp[iw\xi]d\xi,$$

and

$$w = w(\phi) = 2ka\sin\frac{\phi}{2}.$$

Solve the given inhomogeneous equation in terms of Fourier series,

$$j(\phi) = \sum_{n=-\infty}^{n=\infty} f_n \exp[in\phi].$$

Hint: Following the step employed in Problem 4.17, substitute the Fourier series into the integral equation. The decoupled equation for f_n,

$$f_n = \frac{J_n(ka)}{1+\alpha U_n(ka)}, \qquad n = -\infty, \ldots, \infty,$$

results, where

$$U_n(ka) = \int_{-\pi}^{\pi}\frac{d\phi}{2\pi}\mathcal{K}(\phi)\exp[-in\phi]$$
$$= \frac{1}{8\pi}\int_{-1}^{1}d\xi\,(1+\xi^2)\int_{-\pi}^{\pi}d\phi\,\exp[iw(\phi)\xi]\cos(n\phi)$$
$$= \frac{1}{2}\int_{0}^{1}d\xi\,(1+\xi^2)J_{2n}(2ka\xi).$$

The integral for $U_n(ka)$ can be further simplified by the use of *Lommel's function* $S_{\mu,\nu}(x)$ and *Weber's function* $\mathbf{E}_\nu(x)$.

Reference for Problems 4.17, 4.18, and 4.19:

We cite the following article for the Fredholm integral equations of the second kind in the theoretical search for "*supergain antennas.*"

Margetis, D., Fikioris, G. , Myers, J.M., and Wu, T.T. : Phys. Rev. **E58**., 2531, (1998).

We cite the following article for the Fredholm integral equations of the second kind for the two dimensional, highly directive currents on large circular loops.

Margetis, D. and Fikioris, G. : J. Math. Phys. **41**, 6130, (2000).

We can derive the above-stated Fredholm integral equations of the second kind for the localized, monochromatic, and highly directive classical current distributions in two and three dimensions by maximizing the directivity D in the far field while constraining $C = N/T$, where N is the integral of the square of the magnitude of the current density and T is proportional to the total radiated power. This derivation is the application of the calculus of variations. We derive the homogeneous Fredholm integral equations of the second kind and the inhomogeneous Fredholm integral equations of the second kind in their general forms in Section 9.6 of Chapter 9.

4.20. Consider the S-wave scattering off a spherically symmetric potential $U(r)$. The governing Schrödinger equation is given by

$$\frac{d^2}{dr^2}u(r) + k^2 u(r) = U(r)u(r),$$

with

$$u(0) = 0 \quad \text{and} \quad u(r) \sim \frac{\sin(kr+\delta)}{\sin\delta} \quad \text{as} \quad r \to \infty.$$

(a) Convert this differential equation into a Fredholm integral equation of the second kind,

$$u(r) = \exp[-ikr] - \exp[ikr] + \int_0^\infty g(r, r')U(r')u(r'),$$

with Green's function $g(r, r')$ given by

$$g(r, r') = -\frac{1}{2ik}\left\{\exp\left[ik(r + r')\right] - \exp\left[ik\left|r - r'\right|\right]\right\}.$$

(b) Setting

$$K(r, r') = g(r, r')U(r'),$$

rewrite the integral equation above as

$$u(r) = \exp[-ikr] - \exp[ikr] + \int_0^\infty K(r, r')u(r').$$

Apply Fredholm theory for a bounded kernel to this integral equation to obtain a formal solution,

$$u(r) = \exp[-ikr] - \exp[ikr] + \frac{1}{D(k)}\int_0^\infty D(k; r, r')(\exp[-ikr'] - \exp[ikr'])dr',$$

where

$$D(k) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_0^{\infty} dr_1 \cdot \int_0^{\infty} dr_n \begin{vmatrix} K(r_1, r_1) & K(r_1, r_2) & \cdot & K(r_1, r_n) \\ \cdot & & & \\ K(r_n, r_1) & K(r_n, r_2) & \cdot & K(r_n, r_n) \end{vmatrix},$$

and

$$D(k; r, r') = K(r, r') + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_0^{\infty} dr_1 \cdot \int_0^{\infty} dr_n \times \begin{vmatrix} K(r, r') & K(r, r_1) & \cdot & K(r, r_n) \\ K(r_1, r') & K(r_1, r_1) & & K(r_1, r_n) \\ \cdot & & & \\ K(r_n, r') & K(r_n, r_1) & \cdot & K(r_n, r_n) \end{vmatrix}.$$

(c) Obtain the condition on the potential $U(r)$ for this formal solution to converge.

Reference for Problem 4.20:

We cite the following book for the application of Fredholm theory for a bounded kernel to potential scattering problem.

Nishijima, K.: *Relativistic Quantum Mechanics*, Baifuu-kan, Tokyo, (1973), Chapter 4, Section 4.7 (In Japanese).

5
Hilbert–Schmidt Theory of Symmetric Kernel

5.1
Real and Symmetric Matrix

We now would like to examine the case of a *symmetric kernel* (*self-adjoint integral operator*) which is also *square-integrable*. Recalling from our earlier discussions in Chapter 1 that self-adjoint operators can be *diagonalized*, our principal aim is to accomplish the same goal for the case of symmetric kernels.

For this purpose, let us first examine the corresponding problem for an $n \times n$ *real* and *symmetric* matrix A. Suppose A has eigenvalues λ_k and normalized eigenvectors $\vec{v}_k$, i.e.,

$$A\vec{v}_k = \lambda_k \vec{v}_k, \quad k = 1, 2, \ldots, n; \quad \vec{v}_k^{\mathrm{T}} \vec{v}_m = \delta_{km}. \tag{5.1.1}$$

We may thus write

$$A\left[\vec{v}_1, \vec{v}_2, \cdot, \vec{v}_n\right] = \left[\lambda_1 \vec{v}_1, \lambda_2 \vec{v}_2, \cdot, \lambda_n \vec{v}_n\right] = \left[\vec{v}_1, \cdot, \vec{v}_n\right] \begin{bmatrix} \lambda_1 & 0 & & \cdot & 0 \\ 0 & \lambda_2 & & \cdot & \\ & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & & \cdot & 0 & \lambda_n \end{bmatrix}. \tag{5.1.2}$$

Define the matrix S by

$$S = \left[\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\right] \tag{5.1.3a}$$

and consider S^{T},

$$S^{\mathrm{T}} = \begin{bmatrix} \vec{v}_1^{\mathrm{T}} \\ \cdot \\ \cdot \\ \vec{v}_n^{\mathrm{T}} \end{bmatrix}. \tag{5.1.3b}$$

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Then we have

$$S^{\mathrm{T}}S = \begin{bmatrix} \vec{v}_1^{\mathrm{T}} \\ \cdot \\ \cdot \\ \vec{v}_n^{\mathrm{T}} \end{bmatrix} [\vec{v}_1, \ldots, \vec{v}_n] = \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & \\ & & \cdot & 0 \\ 0 & & 0 & 1 \end{bmatrix} = I, \tag{5.1.4a}$$

since we have Eq. (5.1.1). Hence we have

$$S^{\mathrm{T}} = S^{-1}. \tag{5.1.5}$$

Define

$$D = \begin{bmatrix} \lambda_1 & 0 & & 0 \\ 0 & \cdot & & \\ & & \cdot & 0 \\ 0 & & 0 & \lambda_n \end{bmatrix}. \tag{5.1.6}$$

From Eq. (5.1.2), we have

$$AS = SD. \tag{5.1.7a}$$

Hence we have

$$A = SDS^{-1} = SDS^{\mathrm{T}}. \tag{5.1.7b}$$

The above relation can also be written as

$$A = \sum_{k=1}^{n} \lambda_k \vec{v}_k^{\mathrm{T}} \vec{v}_k. \tag{5.1.8}$$

This represents the diagonalization of a symmetric matrix. Equation (5.1.7b) is really convenient for calculation of functions of A. For example, we compute

$$\begin{aligned} A^2 &= \left(SDS^{-1}\right)\left(SDS^{-1}\right) = SD^2S^{-1} = S \begin{bmatrix} \lambda_1^2 & 0 & & 0 \\ 0 & \cdot & & \\ & & \cdot & 0 \\ 0 & & 0 & \lambda_n^2 \end{bmatrix} S^{-1}. \\ e^A &= I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \cdots \\ &= S \begin{bmatrix} e^{\lambda_1} & 0 & & 0 \\ 0 & \cdot & & \\ & & \cdot & 0 \\ 0 & & 0 & e^{\lambda_n} \end{bmatrix} S^{-1}. \end{aligned}$$

$$f(A) = S \begin{bmatrix} f(\lambda_1) & 0 & & 0 \\ 0 & \cdot & & \\ & & \cdot & 0 \\ 0 & & 0 & f(\lambda_n) \end{bmatrix} S^{-1}. \tag{5.1.9}$$

Finally we have

$$\det A = \det SDS^{-1} = \det S \det D \det S^{-1} = \det D = \prod_{k=1}^{n} \lambda_k, \tag{5.1.10a}$$

$$\mathrm{tr}(A) = \mathrm{tr}\left(SDS^{-1}\right) = \mathrm{tr}\left(DS^{-1}S\right) = \mathrm{tr}\,(D) = \sum_{k=1}^{n} \lambda_k. \tag{5.1.10b}$$

5.2 Real and Symmetric Kernel

Symmetric kernels have the property that when transposed they remain the same as the original kernel. We denote the transposed kernel K^{T} by

$$K^{\mathrm{T}}(x, y) = K(y, x), \tag{5.2.1}$$

and note that when the kernel K is symmetric, we have

$$K^{\mathrm{T}}(x, y) = K(y, x) = K(x, y). \tag{5.2.2}$$

The eigenvalues of $K(x, y)$ and eigenvalues of $K^{\mathrm{T}}(x, y)$ are the same. This is because an eigenvalue λ_n is a zero of $D(\lambda)$. From the definition of $D(\lambda)$, we find that, since a determinant remains the same as we exchange its rows and columns,

$$D(\lambda) \text{ for } K(x, y) = D(\lambda) \text{ for } K^{\mathrm{T}}(x, y). \tag{5.2.3}$$

Thus the spectrum of $K(x, y)$ coincides with that of $K^{\mathrm{T}}(x, y)$.

We will need to make use of the orthogonality property held by the eigenfunctions belonging to each eigenvalue. To show this property, we start with the eigenvalue equations,

$$\phi_n(x) = \lambda_n \int_0^h K(x, y)\phi_n(y)dy, \tag{5.2.4}$$

$$\psi_n(x) = \lambda_n \int_0^h K^{\mathrm{T}}(x, y)\psi_n(y)dy, \tag{5.2.5}$$

and from the definition of $K^{\mathrm{T}}(x, y)$,

$$\psi_n(x) = \lambda_n \int_0^h K(y, x)\psi_n(y)dy. \tag{5.2.6}$$

Multiplying by $\psi_m(x)$ in Eq. (5.2.4) and integrating over x, we get

$$\begin{aligned}\int_0^h \psi_m(x)\phi_n(x)dx &= \lambda_n \int_0^h dx\psi_m(x) \int_0^h K(x, y)\phi_n(y)dy \\ &= \frac{\lambda_n}{\lambda_m} \int_0^h \psi_m(y)\phi_n(y)dy.\end{aligned}$$

Then

$$\left(1-\frac{\lambda_n}{\lambda_m}\right)\int_0^h \psi_m(x)\phi_n(x)dx = 0. \tag{5.2.7}$$

If $\lambda_n \neq \lambda_m$, then we have

$$\int_0^h \psi_m(x)\phi_n(x)dx = 0 \quad \text{for} \quad \lambda_n \neq \lambda_m. \tag{5.2.8}$$

In the case of finite matrices, we know that the eigenvalues of a symmetric matrix are real and that the matrix is diagonalizable. Also, the eigenvectors are orthogonal to each other. We shall show that the same statements hold true in the case of *square-integrable symmetric kernels.*

If K is symmetric, then $\psi_n(x) = \phi_n(x)$. Then, by Eq. (5.2.8), the eigenfunctions of a symmetric kernel are orthogonal to each other,

$$\int_0^h \phi_m(x)\phi_n(x)dx = 0 \quad \text{for} \quad \lambda_n \neq \lambda_m. \tag{5.2.9}$$

Furthermore, the eigenvalues of a symmetric kernel must be real. This is seen by supposing that the eigenvalue λ_n is complex. Then we have $\lambda_n \neq \lambda_n^*$. The complex conjugate of Eq. (5.2.4) is given by

$$\phi_n^*(x) = \lambda_n^* \int_0^h K(x,y)\phi_n^*(y)dy, \tag{5.2.10}$$

implying that λ_n^* and $\phi_n^*(x)$ are an eigenvalue and eigenfunction of the kernel $K(x,y)$, respectively. But, Eq. (5.2.9) with $\lambda_n \neq \lambda_n^*$ then requires that

$$\int_0^h \phi_n(x)\phi_n^*(x)dx = \int_0^h \left|\phi_n(x)\right|^2 dx = 0, \tag{5.2.11}$$

implying then that $\phi_n(x) \equiv 0$, which is a contradiction. Thus the eigenvalue must be real, $\lambda_n = \lambda_n^*$, to avoid this contradiction. The *eigenfunctions of a symmetric kernel are orthogonal to each other* and the *eigenvalues are real.*

We now rather boldly expand the symmetric kernel $K(x,y)$ in terms of $\phi_n(x)$,

$$K(x,y) = \sum_n a_n\phi_n(x). \tag{5.2.12}$$

We then normalize the eigenfunctions such that

$$\int_0^h \phi_n(x)\phi_m(x)dx = \delta_{nm} \tag{5.2.13}$$

(even if there is more than one eigenfunction belonging to a certain eigenvalue, we can choose linear combinations of these eigenfunctions to satisfy Eq. (5.2.13)). From the orthogonality (5.2.13), we find

$$a_n = \int_0^h dx \phi_n(x) K(x, y) = \frac{1}{\lambda_n} \phi_n(y), \tag{5.2.14}$$

and thus obtain

$$K(x, y) = \sum_n \frac{\phi_n(y)\phi_n(x)}{\lambda_n}. \tag{5.2.15}$$

There is a problem though. We do not know if the eigenfunctions $\{\phi_n(x)\}_n$ are complete. In fact, we are often sure that the set $\{\phi_n(x)\}_n$ is surely not complete. An example is the kernel in the form of a finite sum of factorized terms. However, the content of the *Hilbert–Schmidt theorem* (which will be proven shortly) is to claim that Eq. (5.2.15) for $K(x, y)$ is valid whether $\{\phi_n(x)\}_n$ is complete or not. The *only conditions* are that $K(x, y)$ be *symmetric* and *square-integrable.*

We calculate the iterated kernel,

$$\begin{aligned} K_2(x, y) &= \int_0^h K(x, z) K(z, y) dz \\ &= \int_0^h \sum_n \frac{\phi_n(x)\phi_n(z)}{\lambda_n} \sum_m \frac{\phi_m(z)\phi_m(y)}{\lambda_m} dz \\ &= \sum_n \sum_m \frac{1}{\lambda_n \lambda_m} \phi_n(x) \delta_{nm} \phi_m(y) = \sum_n \frac{\phi_n(x)\phi_n(y)}{\lambda_n^2}, \end{aligned} \tag{5.2.16}$$

and in general, we obtain

$$K_j(x, y) = \sum_n \frac{\phi_n(x)\phi_n(y)}{\lambda_n^j}, \quad j = 2, 3, \ldots. \tag{5.2.17}$$

Now the definition for the resolvent kernel

$$H(x, y; \lambda) = -K(x, y) - \lambda K_2(x, y) - \cdots - \lambda^j K_{j+1}(x, y) - \cdots$$

becomes

$$\begin{aligned} H(x, y; \lambda) &= -\sum_n \frac{\phi_n(x)\phi_n(y)}{\lambda_n} \left[1 + \frac{\lambda}{\lambda_n} + \frac{\lambda^2}{\lambda_n^2} + \cdots + \frac{\lambda^j}{\lambda_n^j} + \cdots \right] \\ &= -\sum_n \frac{\phi_n(x)\phi_n(y)}{\lambda_n} \frac{1}{1 - \frac{\lambda}{\lambda_n}}, \end{aligned}$$

i.e.,

$$H(x, y; \lambda) = \sum_n \frac{\phi_n(x)\phi_n(y)}{\lambda - \lambda_n}. \tag{5.2.18}$$

This elegant expression explicitly shows the analytic properties of $H(x, y; \lambda)$ in the complex λ plane. We can use this resolvent to solve the *inhomogeneous Fredholm Integral Equation of the second kind with a symmetric and square-integrable kernel.*

$$\begin{aligned}\phi(x) &= f(x) + \lambda \int_0^h K(x, y)\phi(y)dy = f(x) - \lambda \int_0^h H(x, y; \lambda)f(y)dy \\ &= f(x) - \lambda \sum_n \frac{\phi_n(x)}{\lambda - \lambda_n} \int_0^h \phi_n(y)f(y)dy.\end{aligned} \tag{5.2.19}$$

Denoting

$$f_n \equiv \int_0^h \phi_n(y)f(y)dy, \tag{5.2.20}$$

we have the solution to the inhomogeneous equation (5.2.19),

$$\phi(x) = f(x) - \lambda \sum_n \frac{f_n\phi_n(x)}{\lambda - \lambda_n}. \tag{5.2.21}$$

At $\lambda = \lambda_n$, the solution does not exist unless $f_n = 0$, as usual.

As an another application of the eigenfunction expansion (5.2.15), we consider the *Fredholm Integral Equation of the first kind with a symmetric and square-integrable kernel*,

$$f(x) = \int_0^h K(x, y)\phi(y)dy. \tag{5.2.22}$$

Denoting

$$\phi_n \equiv \int_0^h \phi_n(y)\phi(y)dy, \tag{5.2.23}$$

we have

$$f(x) = \sum_n \frac{\phi_n(x)}{\lambda_n}\phi_n. \tag{5.2.24}$$

Immediately we encounter the problem. Equation (5.2.24) states that $f(x)$ is a linear combination of $\phi_n(x)$. In many cases, the set $\{\phi_n(x)\}_n$ is not complete, and thus $f(x)$ is not necessarily representable by a linear superposition of $\{\phi_n(x)\}_n$ and

Eq. (5.2.22) has no solution. If $f(x)$ is representable by a linear superposition of $\{\phi_n(x)\}_n$, it is easy to obtain ϕ_n. From Eqs. (5.2.20) and (5.2.24),

$$f_n = \int_0^h \phi_n(x) f(x) dx = \frac{\phi_n}{\lambda_n}, \tag{5.2.25}$$

$$\phi_n = f_n \lambda_n. \tag{5.2.26}$$

A solution to Eq. (5.2.22) is then given by

$$\phi(x) = \sum_n \phi_n \phi_n(x) = \sum_n \lambda_n f_n \phi_n(x). \tag{5.2.27}$$

If the set $\{\phi_n(x)\}_n$ is not complete, solution (5.2.27) is not unique. We can add to it any linear combination of $\{\psi_i(x)\}_i$ that is orthogonal to $\{\phi_n(x)\}_n$,

$$\phi(x) = \sum_n \lambda_n f_n \phi_n(x) + \sum_i C_i \psi_i(x), \tag{5.2.28}$$

$$\int_0^h \psi_i(x) \phi_n(x) dx = 0 \quad \text{for all } i \text{ and } n. \tag{5.2.29}$$

If the set $\{\phi_n(x)\}_n$ is complete, solution (5.2.27) is the unique solution. It may, however, still diverge since we have λ_n in the numerator, unless f_n vanishes sufficiently rapidly as $n \to \infty$ to ensure the convergence of the series (5.2.27).

We will now prove the Hilbert–Schmidt expansion, (5.2.15), to an extent that everything beautiful about it is exhibited, but to avoid getting too mathematical, we shall not be completely rigorous.

We will outline a plan of the proof. First, note the following lemma.

Lemma: *For a nonzero normed symmetric kernel,*

$$\infty > \|K\| > 0 \quad \textit{and} \quad K(x, y) = K^{\mathrm{T}}(x, y), \tag{5.2.30}$$

there exists at least one eigenvalue λ_1 and one eigenfunction $\phi_1(x)$ (which we normalize to unity).

Once this lemma is established, we can construct a new kernel $\overline{K}(x, y)$ by

$$\overline{K}(x, y) \equiv K(x, y) - \frac{\phi_1(x)\phi_1(y)}{\lambda_1}. \tag{5.2.31}$$

Now $\phi_1(x)$ cannot be an eigenfunction of $\overline{K}(x, y)$ because we have

$$\int_0^h \overline{K}(x, y)\phi_1(y) dy = \int_0^h \left[K(x, y) - \frac{\phi_1(x)\phi_1(y)}{\lambda_1} \right] \phi_1(y) dy$$
$$= \frac{\phi_1(x)}{\lambda_1} - \frac{\phi_1(x)}{\lambda_1} = 0, \tag{5.2.32}$$

which leaves us two possibilities,

(A) $\|\overline{K}\| \equiv 0.$

We have an equality

$$K(x,y) = \frac{\phi_1(x)\phi_1(y)}{\lambda_1}, \tag{5.2.33}$$

except over a set of points x whose measure is zero. The proof for this case is done.

(B) $\|\overline{K}\| \neq 0.$

By the lemma, there exist at least one eigenvalue λ_2 and one eigenfunction $\phi_2(x)$ of a kernel $\overline{K}(x,y)$,

$$\lambda_2 \int_0^h \overline{K}(x,y)\phi_2(y)dy = \phi_2(x). \tag{5.2.34}$$

Namely,

$$\lambda_2 \int_0^h \left[K(x,y) - \frac{\phi_1(x)\phi_1(y)}{\lambda_1} \right] \phi_2(y)dy = \phi_2(x). \tag{5.2.35}$$

We then show that $\phi_2(x)$ and λ_2 are an eigenfunction and eigenvalue of the original kernel $K(x,y)$ orthogonal to $\phi_1(x)$, respectively.

To demonstrate the orthogonality of $\phi_2(x)$ to $\phi_1(x)$, multiply $\phi_1(x)$ in Eq. (5.2.35) and integrate over x:

$$\begin{aligned}\int_0^h \phi_1(x)\phi_2(x)dx &= \lambda_2 \int_0^h \phi_1(x)dx \int_0^h \left[K(x,y) - \frac{\phi_1(x)\phi_1(y)}{\lambda_1} \right] \phi_2(y)dy \\ &= \lambda_2 \int_0^h \left[\frac{1}{\lambda_1}\phi_1(y) - \frac{1}{\lambda_1}\phi_1(y) \right] \phi_2(y)dy = 0.\end{aligned} \tag{5.2.36}$$

From Eq. (5.2.35), we then have

$$\lambda_2 \int_0^h K(x,y)\phi_2(y)dy = \phi_2(x). \tag{5.2.37}$$

Once we find $\phi_2(x)$, we construct a new kernel $\tilde{K}(x,y)$ by

$$\tilde{K}(x,y) \equiv \overline{K}(x,y) - \frac{\phi_2(x)\phi_2(y)}{\lambda_2} = K(x,y) - \sum_{n=1}^{2} \frac{\phi_n(x)\phi_n(y)}{\lambda_n}. \tag{5.2.38}$$

We then repeat the argument for $\tilde{K}(x,y)$.

Ultimately either we find after N steps,

$$K(x,y) = \sum_{n=1}^{N} \frac{\phi_n(x)\phi_n(y)}{\lambda_n}, \tag{5.2.39}$$

or we find the infinite series,

$$K(x,y) \approx \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(y)}{\lambda_n}. \tag{5.2.40}$$

We can show that the *remainder* $R(x,y)$ defined by

$$R(x,y) \equiv K(x,y) - \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(y)}{\lambda_n} \tag{5.2.41}$$

cannot have any eigenfunction. If $\psi(x)$ is the eigenfunction of $R(x,y)$,

$$\lambda_0 \int_0^h R(x,y)\psi(y)dy = \psi(x), \tag{5.2.42}$$

we know that
(1): $\psi(x)$ is distinct from all $\{\phi_n(x)\}_n$,

$$\psi(x) \neq \phi_n(x) \quad \text{for} \quad n = 1, 2, \ldots, \tag{5.2.43}$$

(2): $\psi(x)$ is orthogonal to all $\{\phi_n(x)\}_n$,

$$\int_0^h \psi(x)\phi_n(x)dx = 0 \quad \text{for} \quad n = 1, 2, \ldots. \tag{5.2.44}$$

Substituting definition (5.2.41) of $R(x,y)$ into Eq. (5.2.42) and noting the orthogonality (5.2.44), we find

$$\lambda_0 \int_0^h K(x,y)\psi(y)dy = \psi(x), \tag{5.2.45}$$

which is a contradiction to Eq. (5.2.43). Thus we must have

$$\| R\|^2 = \int_0^h \int_0^h R^2(x,y)dxdy = 0. \tag{5.2.46}$$

Formula (5.2.15) holds in the sense of the mean square convergence.

So we only have to prove the Lemma stated with the condition (5.2.30) and the proof of the Hilbert–Schmidt theorem will be finished. To do so, it is necessary to work with the iterated kernel $K_2(x,y)$, which is also symmetric:

$$K_2(x,y) = \int_0^h K(x,z)K(z,y)dz = \int_0^h K(x,z)K(y,z)dz. \tag{5.2.47}$$

This is because the trace of $K_2(x, y)$ is always positive,

$$\int_0^h K_2(x, x)dx = \int_0^h dx \int_0^h dz K^2(x, z) = \|K\|^2 > 0. \tag{5.2.48}$$

First, we will prove that if $K_2(x, y)$ has an eigenvalue, then $K(x, y)$ has at least one eigenvalue equaling one of the square roots of the former.

Recall the definition of the resolvent kernel of $K(x, y)$,

$$H(x, y; \lambda) \quad = -K(x, y) - \lambda K_2(x, y) - \cdots - \lambda^j K_{j+1}(x, y) - \cdots, \tag{5.2.49}$$

$$H(x, y; -\lambda) = -K(x, y) + \lambda K_2(x, y) - \cdots - (-\lambda)^j K_{j+1}(x, y) - \cdots. \tag{5.2.50}$$

Taking the difference of Eqs. (5.2.49) and (5.2.50), we find

$$\begin{aligned}\frac{1}{2}[H(x, y; \lambda) - H(x, y; -\lambda)] &= -\lambda[K_2(x, y) + \lambda^2 K_4(x, y) + \lambda^4 K_6(x, y) + \cdots] \\ &= \lambda H_2(x, y; \lambda^2),\end{aligned} \tag{5.2.51}$$

which is the resolvent for $K_2(x, y)$. Equality (5.2.51), which is valid for sufficiently small λ where the series expansion in λ is defined, holds for all λ by analytic continuation.

If c is an eigenvalue of $K_2(x, y)$, $H_2(x, y; \lambda^2)$ has a pole at $\lambda^2 = c$. From Eq. (5.2.51), either $H(x, y; \lambda)$ or $H(x, y; -\lambda)$ must have a pole at $\lambda = \pm\sqrt{c}$. This means that at least one of $\pm\sqrt{c}$ is an eigenvalue of $K(x, y)$.

Now we prove that $K_2(x, y)$ has at least one eigenvalue. We have

$$\begin{aligned}\int_0^h D_2(x, x; s)dx \big/ D_2(s) &= \int_0^h H_2(x, x; s)dx \\ &= -(A_2 + sA_4 + s^2 A_6 + \cdots)\end{aligned} \tag{5.2.52}$$

where

$$A_m = \int_0^h K_m(x, x)dx, \quad m = 2, 3, \ldots. \tag{5.2.53}$$

If $K_2(x, y)$ has no eigenvalues, then $D_2(s)$ has no zeros, and series (5.2.52) must be convergent for all values of s. To this end, consider

$$\begin{aligned}A_{m+n} &= \int_0^h dx K_{m+n}(x, x) = \int_0^h dx \int_0^h dz K_m(x, z) K_n(z, x) \\ &= \int_0^h dx \int_0^h dz K_m(x, z) K_n(x, z).\end{aligned} \tag{5.2.54}$$

Applying the Schwarz inequality,

$$\begin{aligned} A_{m+n}^2 &\le \left[\int_0^h dx \int_0^h dz K_m^2(x,z)\right]\left[\int_0^h dx \int_0^h dz K_n^2(x,z)\right] \\ &= \left[\int_0^h dx K_{2m}(x,x)\right]\left[\int_0^h dx K_{2n}(x,x)\right] \\ &= A_{2m}A_{2n}, \end{aligned}$$

i.e., we have

$$A_{m+n}^2 \le A_{2m}A_{2n}. \tag{5.2.55}$$

Setting

$$\begin{cases} m & \to & n-1, \\ n & \to & n+1, \end{cases}$$

in inequality (5.2.55), we have

$$A_{2n}^2 \le A_{2n-2}A_{2n+2}. \tag{5.2.56}$$

Recalling that

$$A_{2m} > 0, \tag{5.2.57}$$

which is precisely the reason why we consider $K_2(x, y)$, we have

$$\frac{A_{2n}}{A_{2n-2}} \le \frac{A_{2n+2}}{A_{2n}}. \tag{5.2.58}$$

Successively we have

$$\frac{A_{2n+2}}{A_{2n}} \ge \frac{A_{2n}}{A_{2n-2}} \ge \frac{A_{2n-2}}{A_{2n-4}} \ge \cdots \ge \frac{A_4}{A_2} \equiv R_1, \tag{5.2.59}$$

so that

$$A_4 = R_1 A_2, \quad A_6 \ge R_1^2 A_2, \quad A_8 \ge R_1^3 A_2,$$

and generally

$$A_{2n} \ge R_1^{n-1} A_2. \tag{5.2.60}$$

Thus we have

$$A_2 + sA_4 + s^2A_6 + s^3A_8 + \cdots \geq A_2(1 + sR_1 + s^2R_1^2 + s^3R_1^3 + \cdots). \qquad (5.2.61)$$

The right-hand side of inequality (5.2.61) surely diverges for those s such that

$$s \geq \frac{1}{R_1} = \frac{A_2}{A_4}. \qquad (5.2.62)$$

Thus

$$\int_0^h H_2(x, x; s)dx$$

is divergent for those s satisfying inequality (5.2.62). Then $K_2(x, y)$ has the eigenvalue s satisfying

$$s \leq \frac{A_2}{A_4},$$

and $K(x, y)$ has the eigenvalue λ_1 satisfying

$$|\lambda_1| \leq \sqrt{\frac{A_2}{A_4}}. \qquad (5.2.63)$$

This completes the proof of the Lemma and finishes the proof of the Hilbert–Schmidt theorem. □

The Hilbert–Schmidt expansion, (5.2.15), can be helpful in many problems where a symmetric and square-integrable kernel is involved.

❑ **Example 5.1.** Solve the integro-differential equation,

$$\frac{\partial}{\partial t}\phi(x, t) = \int_0^h K(x, y)\phi(y, t)dy, \qquad (5.2.64a)$$

with the initial condition

$$\phi(x, 0) = f(x), \qquad (5.2.64b)$$

where $K(x, y)$ is symmetric and square-integrable .

Solution. The Hilbert–Schmidt expansion, (5.2.15), can be applied giving

$$\frac{\partial}{\partial t}\phi(x, t) = \sum_n \frac{\phi_n(x)}{\lambda_n} \int_0^h \phi_n(y)\phi(y, t)dy. \qquad (5.2.65)$$

Defining $A_n(t)$ by

$$A_n(t) \equiv \int_0^h \phi_n(y)\phi(y,t)dy, \tag{5.2.66}$$

and changing the dummy index of summation from n to m in Eq. (5.2.65), we have

$$\frac{\partial}{\partial t}\phi(x,t) = \sum_m \frac{\phi_m(x)}{\lambda_m} A_m(t). \tag{5.2.67}$$

Taking the time derivative of Eq. (5.2.66) yields

$$\frac{d}{dt}A_n(t) = \int_0^h dy\phi_n(y)\frac{\partial}{\partial t}\phi(y,t). \tag{5.2.68}$$

Substituting Eq. (5.2.67) into Eq. (5.2.68) and, noting that the orthogonality of $\{\phi_m(x)\}_m$ means that only the $m = n$ term is left, we get

$$\frac{d}{dt}A_n(t) = \frac{A_n(t)}{\lambda_n}. \tag{5.2.69}$$

The solution to Eq. (5.2.69) is then

$$A_n(t) = A_n(0)\exp\left[\frac{t}{\lambda_n}\right]. \tag{5.2.70}$$

Here we note that

$$A_n(0) = \int_0^h dx\phi_n(x)f(x). \tag{5.2.71}$$

We can now integrate Eq. (5.2.67) from 0 to t, with $A_n(t)$ from Eq. (5.2.70),

$$\int_0^t dt\frac{\partial}{\partial t}\phi(x,t) = \int_0^t dt\sum_n \frac{\phi_n(x)}{\lambda_n}A_n(0)\exp\left[\frac{t}{\lambda_n}\right]. \tag{5.2.72}$$

The left-hand side of Eq. (5.2.72) is now exact, and we obtain

$$\phi(x,t) - \phi(x,0) = \sum_n \frac{\phi_n(x)}{\lambda_n}A_n(0)\left(\exp\left[\frac{t}{\lambda_n}\right]-1\right)\Big/\left(\frac{1}{\lambda_n}\right). \tag{5.2.73}$$

From the initial condition (5.2.64b) and Eq. (5.2.73), we finally get

$$\phi(x,t) = f(x) + \sum_n A_n(0)\left(\exp\left[\frac{t}{\lambda_n}\right]-1\right)\phi_n(x). \tag{5.2.74}$$

As $t \to \infty$, the asymptotic form of $\phi(x, t)$ is given either by

$$\phi(x,t) = f(x) - \sum_n A_n(0)\phi_n(x) \quad \text{if all } \lambda_n < 0, \tag{5.2.75}$$

or by

$$\phi(x,t) = A_i(0)\phi_i(x)\exp\left[\frac{t}{\lambda_i}\right] \quad \text{if} \quad 0 < \lambda_i < \text{all other } \lambda_n. \tag{5.2.76}$$

5.3
Bounds on the Eigenvalues

In the process of proving our lemma in the previous section, we managed to obtain the *upper bound* on the lowest eigenvalue,

$$|\lambda_1| \leq \sqrt{A_2/A_4}.$$

A better upper bound can be obtained as follows. If we call

$$R_2 = A_6/A_4,$$

we note that

$$R_2 \geq R_1,$$

or

$$1/R_2 \leq 1/R_1.$$

Furthermore, we find

$$\begin{aligned} &A_2 + sA_4 + s^2A_6 + s^3A_8 + s^4A_{10} + \cdots \\ &\quad = A_2 + sA_4[1 + s(A_6/A_4) + s^2(A_8/A_4) + \cdots] \\ &\quad \geq A_2 + sA_4[1 + sR_2 + s^2R_2^2 + \cdots], \end{aligned}$$

which diverges if

$$sR_2 > 1.$$

Hence we have singularity for

$$s \leq 1/R_2 \leq 1/R_1.$$

Thus we have an improved upper bound on λ_1,

$$|\lambda_1| \le \sqrt{A_4/A_6} \le \sqrt{A_2/A_4}.$$

So, we have the successively better upper bounds on $|\lambda_1|$,

$$\sqrt{A_2/A_4}, \sqrt{A_4/A_6}, \sqrt{A_6/A_8}, \ldots,$$

for the lowest eigenvalue, each better than the previous one, i.e.,

$$|\lambda_1| \le \cdots \le \sqrt{A_6/A_8} \le \sqrt{A_4/A_6} \le \sqrt{A_2/A_4}. \tag{5.3.1a}$$

Recall also that with a symmetric kernel, we have

$$A_{2m} = \|K_m\|^2 . \tag{5.3.2}$$

The upper bounds (5.3.1a), in terms of the norm of the iterated kernels, become

$$|\lambda_1| \le \cdots \le \frac{\|K_3\|}{\|K_4\|} \le \frac{\|K_2\|}{\|K_3\|} \le \frac{\|K\|}{\|K_2\|}. \tag{5.3.1b}$$

Now consider the question of finding the *lower bounds* for the lowest eigenvalue λ_1. Consider the expansion of the symmetric kernel,

$$K(x, y) \approx \sum_n \phi_n(x)\phi_n(y)/\lambda_n. \tag{5.3.3}$$

This expression is an *equation in the mean*, and hence there is no guarantee that it is true at any point as an exact equation. In particular, on the line $y = x$ which has zero measure in the square $0 \le x, y \le h$, it need not be true. The equality

$$K(x, x) = \sum_n \phi_n(x)\phi_n(x)/\lambda_n$$

need not be true. Hence

$$\int_0^h K(x, x)dx = \sum_n 1/\lambda_n \tag{5.3.4}$$

need not be true. The right-hand side of Eq. (5.3.4) may not converge.

However, for

$$K_2(x, y) = \int_0^h K(x, z)K(z, y)dz = \sum_n \phi_n(x)\phi_n(y)/\lambda_n^2,$$

since we know $K(x, y)$ to be square-integrable,

$$A_2 = \int_0^h K_2(x, x)dx = \sum_n 1/\lambda_n^2 \tag{5.3.5}$$

must converge, and, in general, for $m \geq 2$, we have

$$A_m = \sum_n 1/\lambda_n^m, \quad m = 2, 3, \ldots, \tag{5.3.6}$$

and we know that the right-hand side of Eq. (5.3.6) converges.

Consider now the expansion for A_2, namely

$$A_2 = \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} + \frac{1}{\lambda_3^2} + \cdots = \frac{1}{\lambda_1^2}\left[1 + \left(\frac{\lambda_1}{\lambda_2}\right)^2 + \left(\frac{\lambda_1}{\lambda_3}\right)^2 + \cdots\right] \geq \frac{1}{\lambda_1^2},$$

i.e.,

$$\lambda_1^2 \geq 1/A_2. \tag{5.3.7}$$

Hence we have a lower bound for the eigenvalue λ_1,

$$|\lambda_1| \geq 1/\sqrt{A_2}, \quad \text{or} \quad |\lambda_1| \geq 1/\|K\|. \tag{5.3.8}$$

This is *consistent* with our early discussion of the series solution to the Fredholm integral equation of the second kind for which we concluded that for

$$|\lambda| < 1/\|K\|, \tag{5.3.9}$$

there are no singularities in λ, so that the first eigenvalue $\lambda = \lambda_1$ must satisfy inequality (5.3.8).

We can obtain better lower bounds for the eigenvalue λ_1. Consider A_4,

$$A_4 = \frac{1}{\lambda_1^4} + \frac{1}{\lambda_2^4} + \frac{1}{\lambda_3^4} + \cdots = \frac{1}{\lambda_1^4}\left[1 + \left(\frac{\lambda_1}{\lambda_2}\right)^4 + \left(\frac{\lambda_1}{\lambda_3}\right)^4 + \cdots\right] \geq \frac{1}{\lambda_1^4},$$

i.e.,

$$|\lambda_1| \geq \frac{1}{(A_4)^{1/4}}. \tag{5.3.10}$$

This is an improvement over the previously established lower bound since we know from Eqs. (5.3.1a) and (5.3.8) that

$$1/A_2^{1/2} \leq |\lambda_1| \leq (A_2/A_4)^{1/2},$$

so that

$$A_4 \leq A_2^2,$$

i.e.,

$$1 / (A_4)^{1/4} \geq 1 / (A_2)^{1/2} .$$

Thus $1 / (A_4)^{1/4}$ is a better lower bound than $1 / (A_2)^{1/2}$.

Proceeding in the same way with A_6, A_8,..., we get better and better lower bounds,

$$1 / (A_2)^{1/2} \leq 1 / (A_4)^{1/4} \leq 1 / (A_6)^{1/6} \leq \cdots \leq |\lambda_1| . \tag{5.3.11}$$

Putting both the upper bounds (5.3.1a) and lower bounds (5.3.11) together, we have for the smallest eigenvalue λ_1,

$$\begin{aligned} &\frac{1}{(A_2)^{1/2}} \leq \frac{1}{(A_4)^{1/4}} \leq \frac{1}{(A_6)^{1/6}} \leq \cdots \leq |\lambda_1| \leq \cdots \\ &\quad \leq \left(\frac{A_6}{A_8}\right)^{1/2} \leq \left(\frac{A_4}{A_6}\right)^{1/2} \leq \left(\frac{A_2}{A_4}\right)^{1/2} . \end{aligned} \tag{5.3.12}$$

Strength permitting, we calculate A_6, A_8,..., to obtain better and better upper bounds and lower bounds from Eq. (5.3.12).

5.4 Rayleigh Quotient

Another useful technique for finding the upper bounds for eigenvalues of self-adjoint operators is based on the *Rayleigh quotient*. Consider the self-adjoint integral operator,

$$\tilde{K} = \int_0^h dy K(x, y) \quad \text{with} \quad K(x, y) = K(y, x), \tag{5.4.1}$$

with eigenvalues λ_n and eigenfunctions $\phi_n(x)$,

$$\begin{cases} \tilde{K}\phi_n = (1/\lambda_n)\phi_n, \\ (\phi_n, \phi_m) = \delta_{nm}, \end{cases} \tag{5.4.2}$$

where the eigenvalues are ordered such that

$$|\lambda_1| \leq |\lambda_2| \leq |\lambda_3| \leq \cdots .$$

Consider any given function $g(x)$ such that

$$\|g\| \neq 0. \tag{5.4.3}$$

Consider the series

$$\sum_n b_n \phi_n(x) \quad \text{with} \quad b_n = (\phi_n, g). \tag{5.4.4}$$

This series expansion is the projection of $g(x)$ onto the space spanned by the set $\{\phi_n(x)\}_n$, which may not be complete.

We can easily verify the Bessel inequality, which says

$$\sum_n b_n^2 \leq \|g\|^2. \tag{5.4.5}$$

Proof of the Bessel inequality: Start with

$$\left\| g - \sum_n b_n \phi_n \right\|^2 \geq 0,$$

which implies

$$(g, g) - \sum_m b_m (g, \phi_m) - \sum_n b_n (\phi_n, g) + \sum_n \sum_m b_n b_m (\phi_n, \phi_m) \geq 0.$$

Thus we have $(g, g) - \sum_m b_m^2 \geq 0$, which states $\sum_n b_n^2 \leq \|g\|^2$, completing the proof of the Bessel inequality (5.4.5).

Now, consider the quadratic form $(g, \tilde{K}g)$,

$$(g, \tilde{K}g) = \int_0^h dx g(x) \tilde{K}g(x) = \int_0^h dx \int_0^h dy g(x) K(x, y) g(y). \tag{5.4.6}$$

Substituting the expansion

$$K(x, y) \approx \sum_n \phi_n(x) \phi_n(y) / \lambda_n$$

into Eq. (5.4.6), we obtain

$$(g, \tilde{K}g) = \int_0^h dx \int_0^h dy g(x) \sum_n (\phi_n(x) \phi_n(y) / \lambda_n) g(y) = \sum_n b_n^2 / \lambda_n. \tag{5.4.7}$$

Then, taking the absolute value of the above quadratic form (5.4.7), we obtain

$$\begin{aligned}\left|(g,\tilde{K}g)\right| &= \left|\sum_n b_n^2/\lambda_n\right| \le \left\{\frac{b_1^2}{|\lambda_1|}+\frac{b_2^2}{|\lambda_2|}+\frac{b_3^2}{|\lambda_3|}+\cdots\right\} \\ &= \frac{1}{|\lambda_1|}\left\{b_1^2+\left|\frac{\lambda_1}{\lambda_2}\right|b_2^2+\left|\frac{\lambda_1}{\lambda_3}\right|b_3^2+\cdots\right\} \le \frac{1}{|\lambda_1|}\left\{b_1^2+b_2^2+b_3^2+\cdots\right\} \\ &\le \frac{1}{|\lambda_1|}\left\|g\right\|^2 . \end{aligned} \tag{5.4.8}$$

Hence, from Eq. (5.4.8), the Rayleigh quotient Q, defined by

$$Q \equiv (g,\tilde{K}g)/(g,g), \tag{5.4.9}$$

is bounded above by

$$\left|Q\right| = \left|(g,\tilde{K}g)\right| / \left\|g\right\|^2 \le \frac{1}{|\lambda_1|},$$

i.e., the absolute value of the lowest eigenvalue λ_1 is bounded above by

$$|\lambda_1| \le 1/\left|Q\right| . \tag{5.4.10}$$

To find a good upper bound on $|\lambda_1|$, choose a trial function $g(x)$ with adjustable parameters and obtain the minimum of $1/\left|Q\right|$. Namely, we have

$$|\lambda_1| \le \min\left(1/\left|Q\right|\right), \tag{5.4.11}$$

with Q given by Eq. (5.4.9). □

❑ **Example 5.2.** Find an upper bound on the leading eigenvalue of the symmetric kernel

$$K(x,y) = \begin{cases} (1-x)y, & 0 \le y < x \le 1, \\ (1-y)x, & 0 \le x < y \le 1, \end{cases}$$

using the Rayleigh quotient.

Solution. Consider the trial function $g(x) = ax$ which probably is not very good. We have

$$(g,g) = \int_0^1 dx a^2 x^2 = a^2/3,$$

and

$$
\begin{aligned}
(g, \tilde{K}g) &= \int_0^1 dx \int_0^1 dy g(x) K(x, y) g(y) \\
&= \int_0^1 dx a x \left[\int_0^x dy (1-x) y a y + \int_x^1 dy (1-y) x a y \right] \\
&= a^2 \int_0^1 dx x^2 (1-x^2)/6 = a^2/30.
\end{aligned}
$$

So, the Rayleigh quotient Q is given by $Q = (g, \tilde{K}g)/(g, g) = 1/10$, and we get $\min(1/|Q|) = 10$. Thus, from Eq. (5.4.11), we obtain

$$|\lambda_1| \leq 10,$$

which is a reasonable upper bound. The exact value of λ_1 turns out to be

$$\lambda_1 = \pi^2 \approx 9.8696,$$

so it is not too bad, considering that the eigenfunction for λ_1 turns out to be $A \sin(\pi x)$, not well approximated by ax.

5.5 Completeness of Sturm–Liouville Eigenfunctions

Consider the Sturm–Liouville eigenvalue problem,

$$\frac{d}{dx}[p(x)\frac{d}{dx}\phi(x)] - q(x)\phi(x) = \lambda r(x)\phi(x) \quad \text{on} \quad [0, h], \tag{5.5.1}$$

with $\phi(0) = \phi(h) = 0$, and $p(x) > 0$, $r(x) > 0$, for $x \in [0, h]$. We proved earlier that using Green's function $G(x, y)$ defined by

$$\frac{d}{dx}\left[p(x)\frac{d}{dx}G(x, y)\right] - q(x)G(x, y) = \delta(x - y), \tag{5.5.2}$$

with $G(0, y) = G(h, y) = 0$, Eq. (5.5.1) is equivalent to the integral equation

$$\phi(x) = \lambda \int_0^h G(y, x) r(y) \phi(y) dy. \tag{5.5.3}$$

Since the Sturm–Liouville operator is self-adjoint and symmetric, we have a symmetric Green's function, $G(x, y) = G(y, x)$. Now, define

$$\psi(x) = \sqrt{r(x)}\phi(x), \quad K(x, y) = \sqrt{r(x)}G(x, y)\sqrt{r(y)}. \tag{5.5.4}$$

Then $\psi(x)$ satisfies

$$\psi(x) = \lambda \int_0^h K(x,y)\psi(y)dy, \tag{5.5.5}$$

which has a *symmetric kernel*. Applying the Hilbert–Schmidt theorem, we know that $K(x,y)$ defined above is decomposable in the form

$$K(x,y) \approx \sum_n \psi_n(x)\psi_n(y)/\lambda_n = \sum_n \sqrt{r(x)r(y)}\phi_n(x)\phi_n(y)/\lambda_n, \tag{5.5.6}$$

with λ_n *real* and *discrete*, and the set $\{\psi_n(x)\}_n$ *orthonormal*. Namely,

$$\int_0^h \psi_n(x)\psi_m(x)dx = \int_0^h r(x)\phi_n(x)\phi_m(x)dx = \delta_{nm}. \tag{5.5.7}$$

Note the appearance of the *weight function* $r(x)$ in the middle equation of Eq. (5.5.7).

To prove the *completeness*, we will establish that any function $f(x)$ can be expanded in a series of $\{\psi_n(x)\}_n$ or $\{\phi_n(x)\}_n$. Let us do this for the *differentiable* case (which is *stronger than square-integrable*), i.e., assume that $f(x)$ is differentiable. As such, given any $f(x)$, we can define $g(x)$ by

$$g(x) \equiv Lf(x), \tag{5.5.8}$$

where

$$L = \frac{d}{dx}\left[p(x)\frac{d}{dx}\right] - q(x). \tag{5.5.9}$$

Taking the inner product of both sides of Eq. (5.5.8) with $G(x,y)$, we get $(G(x,y), g(x)) = f(y)$, i.e.,

$$f(x) = \int_0^h G(x,y)g(y)dy = \int_0^h (K(x,y)/\sqrt{r(x)r(y)})g(y)dy. \tag{5.5.10}$$

Substituting expression (5.5.6) for $K(x,y)$ into Eq. (5.5.10), we obtain

$$\begin{aligned} f(x) &= \int_0^h dy \sum_n (\phi_n(x)\phi_n(y)/\lambda_n)g(y) \\ &= \sum_n (\phi_n(x)/\lambda_n)(\phi_n, g) = \sum_n (\beta_n/\lambda_n)\phi_n(x), \end{aligned} \tag{5.5.11}$$

where we set

$$\beta_n \equiv (\phi_n, g) = \int_0^h \phi_n(y)g(y)dy. \tag{5.5.12}$$

Since the above expansion of $f(x)$ in terms of $\phi_n(x)$ is true for any $f(x)$, this demonstrates that the set $\{\phi_n(x)\}_n$ is *complete*.

Actually, in addition, we must require $f(x)$ to satisfy the *homogeneous boundary conditions* in order to avoid boundary terms. Also, we must make sure that the kernel for the Sturm–Liouville eigenvalue problem is *square-integrable*. Since the set $\{\phi_n(x)\}_n$ is complete, we conclude that there must be an infinite number of eigenvalues for Sturm–Liouville system. Also, it is possible to prove the asymptotic results, $\lambda_n = O(n^2)$ as $n \to \infty$.

5.6 Generalization of Hilbert–Schmidt Theory

In this section, we consider the generalization of Hilbert–Schmidt theory.

Direction 1: So far in our discussion of Hilbert–Schmidt theory, we assumed that $K(x, y)$ is real. It is straightforward to extend to the case when $K(x, y)$ is *complex*. We define the norm $\|K\|$ of the kernel $K(x, y)$ by

$$\|K\|^2 = \int_0^h dx \int_0^h dy \left|K(x, y)\right|^2 . \tag{5.6.1}$$

The iteration series solution to the Fredholm integral equation of the second kind converges for $|\lambda| < 1/\|K\|$. Also, the Fredholm theory still remains valid. If $K(x, y)$ is, in addition, *self-adjoint*, i.e., $K(x, y) = K^*(y, x)$, then the Hilbert–Schmidt expansion holds in the form

$$K(x, y) \approx \sum_n \phi_n(x)\phi_n^*(y)/\lambda_n, \tag{5.6.2}$$

where

$$\int_0^h \phi_n^*(x)\phi_m(x)dx = \delta_{nm} \quad \text{and} \quad \lambda_n = \text{real}, \quad n \text{ integer}.$$

Direction 2: We note that in all the discussion so far, the variable x is restricted to a finite basic interval, $x \in [0, h]$. We extend the basic interval $[0, h]$ to $[0, \infty)$. We want to solve the following integral equation:

$$\phi(x) = f(x) + \lambda \int_0^{+\infty} K(x, y)\phi(y)dy, \tag{5.6.3}$$

with

$$\int_0^{+\infty} dx \int_0^{+\infty} dy K^2(x, y) < \infty, \quad \int_0^{+\infty} dx f^2(x) < \infty. \tag{5.6.4}$$

By a change of the independent variable x, it is always possible to transform the interval $[0, \infty)$ of x into $[0, h]$ of t, i.e., $x \in [0, \infty) \Rightarrow t \in [0, h]$. For example, the following transformation will do:

$$x = g(t) = t/(h - t). \tag{5.6.5}$$

Then, writing

$$\tilde{\phi}(t) = \phi(g(t)), \quad \text{etc.,}$$

we have

$$\tilde{\phi}(t) = \tilde{f}(t) + \lambda \int_0^h \tilde{K}(t, t')\tilde{\phi}(t')\, g'(t')dt'.$$

On multiplying by $\sqrt{g'(t)}$ on both sides of the above equation, we have

$$\sqrt{g'(t)}\tilde{\phi}(t) = \sqrt{g'(t)}\tilde{f}(t) + \lambda \int_0^h \sqrt{g'(t)}\tilde{K}(t, t')\sqrt{g'(t')}\sqrt{g'(t')}\tilde{\phi}(t')dt'.$$

Defining $\psi(t)$ by

$$\psi(t) = \sqrt{g'(t)}\tilde{\phi}(t),$$

we obtain

$$\psi(t) = \sqrt{g'(t)}\tilde{f}(t) + \lambda \int_0^h \left[\sqrt{g'(t)}\tilde{K}(t, t')\sqrt{g'(t')}\right] \psi(t')\, dt'. \tag{5.6.6}$$

If the original kernel $K(x, y)$ is symmetric, then the transformed kernel is also symmetric. Furthermore, the transformed kernel $\sqrt{g'(t)}\tilde{K}(t, t')\sqrt{g'(t')}$ and the transformed inhomogeneous term $\sqrt{g'(t)}\tilde{f}(t)$ are square-integrable if $K(x, y)$ and $f(x)$ are square-integrable, since

$$\int_0^h dt \int_0^h dt' g'(t)\tilde{K}^2(t, t')g'(t') = \int_0^{+\infty} dx \int_0^{+\infty} dy K^2(x, y) < \infty, \tag{5.6.7a}$$

and

$$\int_0^h dt g'(t)\tilde{f}^2(t) = \int_0^{+\infty} dx f^2(x) < \infty. \tag{5.6.7b}$$

Thus, under appropriate conditions, the Fredholm theory and the Hilbert–Schmidt theory both apply to Eq. (5.6.3). Similarly, we can extend these theories to the case of infinite range.

Direction 3: As the third generalization, we consider the case where we have *multidimensional independent variables*:

$$\phi(\vec{x}) = f(\vec{x}) + \lambda \int_0^{+\infty} K(\vec{x}, \vec{y}) \phi(\vec{y}) d\vec{y}. \tag{5.6.8}$$

As long as the kernel $K(\vec{x}, \vec{y})$ is square-integrable, i.e.,

$$\int_0^{+\infty} \int_0^{+\infty} K^2(\vec{x}, \vec{y}) d\vec{x} d\vec{y} < \infty, \tag{5.6.9}$$

all the arguments for establishing the Fredholm theory and the Hilbert–Schmidt theory go through.

Direction 4: We will relax *the condition on the square-integrability of the kernel.* When a kernel $K(x, y)$ is not square-integrable, the integral equation is said to be *singular*. Some singular integral equations can be transformed into one with a square-integrable kernel. One way that may work is to try to *symmetrize* them as much as possible. For example, a kernel of the form $H(x, y)$ with $H(x, y)$ bounded can be made square-integrable by symmetrizing it into $H(x, y)/(xy)^{\frac{1}{4}}$. Another way is to *iterate the kernel*. Suppose the kernel is of the form

$$K(x, y) = H(x, y)/|x - y|^{\alpha}, \quad \frac{1}{2} \leq \alpha < 1, \tag{5.6.10}$$

where $H(x, y)$ is bounded. We have the integral equation of the form

$$\phi(x) = f(x) + \lambda \int_0^h K(x, y) \phi(y) dy. \tag{5.6.11}$$

Replacing $\phi(y)$ in the integrand by the right-hand side of Eq. (5.6.11) itself, we obtain

$$\phi(x) = \left[f(x) + \lambda \int_0^h K(x, y) f(y) dy \right] + \lambda^2 \int_0^h K_2(x, y) \phi(y) dy. \tag{5.6.12}$$

The kernel in Eq. (5.6.12) is $K_2(x, y)$, which may be square-integrable since

$$\int_0^h \frac{1}{|x - z|^{\alpha}} \frac{1}{|z - y|^{\alpha}} dz = O\left(1/|x - y|^{2\alpha - 1}\right). \tag{5.6.13}$$

Indeed, for those α such that $\frac{1}{2} \leq \alpha < \frac{3}{4}$, $K_2(x, y)$ is square-integrable. If α is such that $\frac{3}{4} \leq \alpha < 1$, then $K_3(x, y)$, $K_4(x, y)$,. . ., etc., may be square-integrable. In general, when α lies in the interval

$$1 - \frac{1}{2(n - 1)} \leq \alpha < 1 - \frac{1}{2n}, \tag{5.6.14}$$

$K_n(x, y)$ will be square-integrable. Thus, for those α such that

$$\frac{1}{2} \leq \alpha < 1, \tag{5.6.15}$$

we can transform the kernel into a square-integrable kernel by the appropriate number of iterations. However, when $\alpha \geq 1$, we have no hope whatsoever of transforming it into a square-integrable kernel in this way.

For a kernel which cannot be made square-integrable, what properties remain valid? Does the Fredholm theory hold? Is the spectrum of the eigenvalues discrete? Does the Hilbert–Schmidt expansion hold? The example below gives us some insight into these questions.

❑ **Example 5.3.** Suppose we want to solve the homogeneous equation,

$$\phi(x) = \lambda \int_0^{+\infty} e^{-|x-y|}\phi(y)dy. \tag{5.6.16}$$

The kernel in the above equation is symmetric, but not square-integrable; yet, this equation can be solved in the closed form.

Solution. Writing out Eq. (5.6.16) explicitly, we have

$$\phi(x) = \lambda \int_0^{x} e^{-(x-y)}\phi(y)dy + \lambda \int_x^{+\infty} e^{-(y-x)}\phi(y)dy.$$

Multiplying both sides by e^{-x}, we have

$$e^{-x}\phi(x) = \lambda e^{-2x} \int_0^{x} e^{y}\phi(y)dy + \lambda \int_x^{+\infty} e^{-y}\phi(y)dy. \tag{5.6.17}$$

Differentiating the above equation with respect to x, we obtain

$$e^{-x}(-\phi(x) + \phi'(x)) = -2\lambda e^{-2x} \int_0^{x} e^{y}\phi(y)dy.$$

Multiplying both sides by e^{2x}, we have

$$e^{x}(-\phi(x) + \phi'(x)) = -2\lambda \int_0^{x} e^{y}\phi(y)dy. \tag{5.6.18}$$

Differentiating the above equation with respect to x and canceling the factor e^x we obtain

$$\phi''(x) + (2\lambda - 1)\phi(x) = 0.$$

The solution to the above equation is given by

(i) $1 - 2\lambda > 0$,

$$\phi(x) = C_1 e^{\sqrt{1-2\lambda}x} + C_2 e^{-\sqrt{1-2\lambda}x},$$

and,

(ii) $1 - 2\lambda < 0$,

$$\phi(x) = C_1' e^{i\sqrt{2\lambda-1}x} + C_2' e^{-i\sqrt{2\lambda-1}x}.$$

These solutions satisfy Eq. (5.6.16) only if

$$\phi'(0) = \phi(0),$$

which follows from the once differentiated equation (5.6.18). Thus we have

$$\sqrt{1-2\lambda}(C_1 - C_2) = C_1 + C_2.$$

In order to satisfy Eq. (5.6.17), we must require that the integral

$$\int_x^{+\infty} e^{-y}\phi(y)dy \tag{5.6.19}$$

converge. If $\frac{1}{2} > \lambda > 0$, $\phi(x)$ grows exponentially but the integral in Eq. (5.6.19) converges. If $\lambda > \frac{1}{2}$, $\phi(x)$ oscillates and the integral in Eq. (5.6.19) converges. If $\lambda < 0$, however, the integral in Eq. (5.6.19) diverges and no solution exists.

In summary, the solution exists for $\lambda > 0$, and no solution exists for $\lambda < 0$. We note that

(1) the spectrum is not discrete.

(2) the eigenfunctions for $\lambda > \frac{1}{2}$ alone constitute a complete set (very much like a Fourier sine or cosine expansion). Thus not all of the eigenfunctions are necessary to represent an $\mathbb{L}_2$ function.

Direction 5: As the last generalization of the theorem, we shall retain the *square-integrability of the kernel* but consider the case of the *nonsymmetric kernel*. Since this generalization is not always possible, we shall illustrate the point by presenting one example.

❑ **Example 5.4.** We consider the following Fredholm integral equation of the second kind:

$$\phi(x) = f(x) + \lambda \int_0^1 K(x,y)\phi(y)dy, \tag{5.6.20}$$

where the kernel is nonsymmetric,

$$K(x,y) = \begin{cases} 2, & 0 \le y < x \le 1, \\ 1, & 0 \le x < y \le 1, \end{cases} \tag{5.6.21}$$

but is square-integrable.

Solution. We first consider the *homogeneous equation*:

$$\phi(x) = \lambda \int_0^x 2\phi(y)dy + \lambda \int_x^1 \phi(y)dy = \lambda \int_0^1 \phi(y)dy + \lambda \int_0^x \phi(y)dy. \tag{5.6.22}$$

Differentiating Eq. (5.6.22) with respect to x, we obtain

$$\phi^{'}(x) = \lambda\phi(x). \tag{5.6.23}$$

From this, we obtain

$$\phi(x) = C\exp[\lambda x], \quad 0 \le x \le 1, \quad C \ne 0. \tag{5.6.24}$$

From Eq. (5.6.22), we get the boundary conditions

$$\begin{cases} \phi(0) = \lambda \int_0^1 \phi(y)dy, \\ \phi(1) = 2\lambda \int_0^1 \phi(y)dy, \end{cases} \Rightarrow \phi(1) = 2\phi(0). \tag{5.6.25}$$

Hence, we require that

$$C\exp[\lambda] = 2C \quad \Rightarrow \quad \exp[\lambda] = 2 = \exp[\ln(2) + i2n\pi], \quad n \text{ integer}.$$

Thus we should have the eigenvalues

$$\lambda = \lambda_n = \ln(2) + i2n\pi, \quad n = 0, \pm 1, \pm 2, \ldots. \tag{5.6.26}$$

The corresponding eigenfunctions are

$$\phi_n(x) = C_n \exp[\lambda_n x], \quad C_n \text{ real}.$$

Finally,

$$\phi_n(x) = C_n \exp[\{\ln(2) + i2n\pi\}x], \quad n \text{ integer}. \tag{5.6.27}$$

Clearly, the kernel is nonsymmetric, $K(x,y) \ne K(y,x)$. The transposed kernel $K^{\mathrm{T}}(x,y)$ is given by

$$K^{\mathrm{T}}(x,y) = K(y,x) = \begin{cases} 2, & 0 \le x < y \le 1, \\ 1, & 0 \le y < x \le 1. \end{cases} \tag{5.6.28}$$

We next consider the *homogeneous equation for the transposed kernel*: $K^{\mathrm{T}}(x,y)$.

$$\psi(x)=\lambda\int_0^x\psi(y)dy+2\lambda\int_x^1\psi(y)dy=2\lambda\int_0^1\psi(y)dy-\lambda\int_0^x\psi(y)dy. \quad (5.6.29)$$

Differentiating Eq. (5.6.29) with respect to x, we obtain

$$\psi'(x)=-\lambda\psi(x). \quad (5.6.30)$$

From this, we obtain

$$\psi(x)=F\exp[-\lambda x],\quad 0\leq x\leq 1,\quad F\neq 0. \quad (5.6.31)$$

From Eq. (5.6.29), we get the boundary conditions

$$\begin{cases}\psi(0)=2\lambda\int_0^1\psi(y)dy,\\ \psi(1)=\lambda\int_0^1\psi(y)dy,\end{cases}\Rightarrow\psi(1)=\frac{1}{2}\psi(0). \quad (5.6.32)$$

Hence, we require that

$$F\exp[-\lambda]=\frac{1}{2}F\quad\Rightarrow\quad\exp[\lambda]=2=\exp[\ln(2)+i2n\pi],\quad n \text{ integer}.$$

Thus we should have the same eigenvalues,

$$\lambda=\lambda_n=\ln(2)+i2n\pi,\quad n=0,\pm1,\pm2,\ldots. \quad (5.6.33)$$

The corresponding eigenfunctions are $\psi_n(x)=F_n\exp[-\lambda_n x]$ with F_n real. Finally,

$$\psi_n(x)=F_n\exp[-\{\ln(2)+i2n\pi\}x],\quad n \text{ integer}. \quad (5.6.34)$$

These $\psi_n(x)$ are the solution to the transposed problem. For $n\neq m$, we have

$$\int_0^1\phi_n(x)\psi_m(x)dx=C_nF_m\int_0^1\exp[i2\pi(n-m)x]dx=0,\quad n\neq m.$$

The spectral representation of the kernel $K(x,y)$ is given by

$$K(x,y)=\sum_{n=-\infty}^{\infty}\frac{\phi_n(x)\psi_n(y)}{\lambda_n}=\sum_{n=-\infty}^{\infty}\frac{\exp\left[\{\ln(2)+i2n\pi\}(x-y)\right]}{\ln(2)+i2n\pi}, \quad (5.6.35)$$
$$C_n=F_n=1,$$

which follows from the following orthogonality:

$$\int_0^1\phi_n(x)\psi_m(x)dx=\delta_{nm}. \quad (5.6.36)$$

In establishing Eq. (5.6.35), we first write

$$R(x,y) \equiv K(x,y) - \sum_{n=1}^{\infty} \frac{\phi_n(x)\psi_n(y)}{\lambda_n}, \tag{5.6.37}$$

and demonstrate the fact that the *remainder* $R(x, y)$ cannot have any eigenfunction by exhausting all of the eigenfunctions. By an explicit solution, we know already that the kernel has at least one eigenvalue.

Crucial to this generalization is that the original integral equation and the transposed integral equation have the same eigenvalues and that the eigenfunctions of the transposed kernel are orthogonal to the eigenfunctions of the original kernel. This last generalization is not always possible for the general nonsymmetric kernel.

5.7 Generalization of the Sturm–Liouville System

In Section 5.5, we have shown that, if $p(x) > 0$ and $r(x) > 0$, the eigenvalue equation

$$\frac{d}{dx}\left[p(x)\frac{d}{dx}\phi(x)\right] - q(x)\phi(x) = \lambda r(x)\phi(x) \quad \text{where} \quad x \in [0, h], \tag{5.7.1}$$

with appropriate boundary conditions has the eigenfunctions which form a complete set $\{\phi_n(x)\}_n$ belonging to the discrete eigenvalues λ_n. In this section, we shall relax the conditions on $p(x)$ and $r(x)$. In particular, we shall consider the case in which $p(x)$ has simple or double zeros at the end points, which therefore, may be regular singular points of the differential equation (5.7.1).

Let L_x be a second-order differential operator,

$$L_x \equiv a_0(x)\frac{d^2}{dx^2} + a_1(x)\frac{d}{dx} + a_2(x), \quad \text{where} \quad x \in [0, h], \tag{5.7.2}$$

which is, in general, *non self-adjoint*. As a matter of fact, we can always transform a second-order differential operator L_x into a self-adjoint form by multiplying $p(x)/a_0(x)$ on L_x, with

$$p(x) \equiv \exp\left[\int^x \frac{a_1(y)}{a_0(y)}dy\right].$$

However, it is instructive to see what happens when L_x is non-self-adjoint. So, we shall not transform the differential operator L_x, (5.7.2), into a self-adjoint form. Let us assume that certain boundary conditions at $x = 0$ and $x = h$ are given.

Consider Green's functions $G(x, y)$ and $G(x, y; \lambda)$ defined by

$$\begin{cases} L_x G(x,y) = \delta(x-y), \\ (L_x - \lambda)G(x,y;\lambda) = \delta(x-y), \\ G(x,y;\lambda=0) = G(x,y). \end{cases} \tag{5.7.3}$$

We would like to find a representation of $G(x, y; \lambda)$ in a form similar to $H(x, y; \lambda)$ given by Eq. (5.2.9). Symbolically we write $G(x, y; \lambda)$ as

$$G(x, y; \lambda) = (L_x - \lambda)^{-1}. \tag{5.7.4}$$

Since the defining equation of $G(x, y; \lambda)$ depends on λ analytically, we expect $G(x, y; \lambda)$ to be an analytic function of λ by the *Poincaré theorem*. There are two possible exceptions: (1) at a regular singular point, the indicial equation yields an exponent which, considered as a function λ, may have branch cuts; (2) for some value of λ, it may be impossible to match the discontinuity at $x = y$.

To elaborate on the second point (2), let ϕ_1 and ϕ_2 be the solution of

$$(L_x - \lambda)\phi_i(x; \lambda) = 0 \quad (i = 1, 2). \tag{5.7.5}$$

We can construct $G(x, y; \lambda)$ to be

$$G(x, y; \lambda) \propto \begin{cases} \phi_1(x; \lambda)\phi_2(y; \lambda), & 0 \le x \le y, \\ \phi_2(x; \lambda)\phi_1(y; \lambda), & y < x \le h, \end{cases} \tag{5.7.6}$$

where $\phi_1(x; \lambda)$ satisfies the boundary condition at $x = 0$, and $\phi_2(x; \lambda)$ satisfies the boundary condition at $x = h$. The constant of the proportionality of Eq. (5.7.6) is given by

$$C / W(\phi_1(y; \lambda), \phi_2(y; \lambda)). \tag{5.7.7}$$

When the Wronskian becomes zero as a function of λ, $G(x, y; \lambda)$ develops a singularity. It may be a pole or a branch point in λ. However, the vanishing of the Wronskian $W(\phi_1(y; \lambda), \phi_2(y; \lambda))$ implies that $\phi_2(x; \lambda)$ is proportional to $\phi_1(x; \lambda)$ for such λ; namely we have an eigenfunction of L_x. Thus the singularities of $G(x, y; \lambda)$ as a function of λ are associated with the eigenfunctions of L_x. Hence we shall treat $G(x, y; \lambda)$ as an analytic function of λ, except at poles located at $\lambda = \lambda_i$ ($i = 1, \ldots, n, \ldots$) and at a branch point located at $\lambda = \lambda_B$, from which a branch cut is extended to $-\infty$. Assuming that $G(x, y; \lambda)$ behaves as

$$G(x, y; \lambda) = O\left(\frac{1}{\lambda}\right) \quad \text{as} \quad |\lambda| \to \infty, \tag{5.7.8a}$$

we obtain

$$\lim_{R\to\infty} \frac{1}{2\pi i} \oint_{C_R} \frac{G(x, y; \lambda')}{\lambda' - \lambda} d\lambda' = 0, \tag{5.7.8b}$$

where C_R is the circle of radius R, centered at the origin of the complex λ plane.

Invoking the Cauchy's Residue theorem, we have

$$\lim_{R\to\infty} \frac{1}{2\pi i}\oint_{C_R} \frac{G(x,y;\lambda^{'})}{\lambda^{'}-\lambda}d\lambda^{'} = G(x,y;\lambda) + \sum_n \frac{R_n(x,y)}{\lambda_n-\lambda} - \frac{1}{2\pi i}\int_{-\infty}^{\lambda_B} \frac{G(x,y;\lambda^{'}+i\varepsilon) - G(x,y;\lambda^{'}-i\varepsilon)}{\lambda^{'}-\lambda}d\lambda^{'}, \tag{5.7.9}$$

where

$$R_n(x,y) = \operatorname{Res} G(x,y;\lambda^{'})\Big|_{\lambda^{'}=\lambda_n}. \tag{5.7.10}$$

Combining Eqs. (5.7.8b) and (5.7.9), we obtain

$$G(x,y;\lambda) = -\sum_n \frac{R_n(x,y)}{\lambda_n-\lambda} + \frac{1}{2\pi i}\int_{-\infty}^{\lambda_B} \frac{G(x,y;\lambda'+i\varepsilon) - G(x,y;\lambda'-i\varepsilon)}{\lambda'-\lambda}d\lambda'. \tag{5.7.11}$$

Let us concentrate on the first term in Eq. (5.7.11). Multiplying the second equation in Eq. (5.7.3) by $(\lambda-\lambda_n)$ and letting $\lambda \to \lambda_n$,

$$\lim_{\lambda\to\lambda_n} (\lambda-\lambda_n)\,(L_x-\lambda)\,G(x,y;\lambda) = \lim_{\lambda\to\lambda_n} (\lambda-\lambda_n)\,\delta(x-y),$$

from which, we obtain

$$(L_x-\lambda_n)\,R_n(x,y) = 0. \tag{5.7.12}$$

Thus we obtain

$$R_n(x,y) \propto \psi_n(y)\phi_n(x), \tag{5.7.13a}$$

where $\phi_n(x)$ is the eigenfunction of L_x belonging to the eigenvalue λ_n (by assuming that the eigenvalue is not degenerate),

$$(L_x-\lambda_n)\,\phi_n(x) = 0. \tag{5.7.14}$$

We claim that $\psi_n(x)$ is the eigenfunction of L_x^{T}, belonging to the same eigenvalue λ_n,

$$(L_x^{\mathrm{T}}-\lambda_n)\psi_n(x) = 0, \tag{5.7.15}$$

where L_x^{T} is defined by

$$L_x^{\mathrm{T}} \equiv \frac{d^2}{dx^2}a_0(x) - \frac{d}{dx}a_1(x) + a_2(x). \tag{5.7.16}$$

Suppose we want to solve the following equation:

$$(L_x^{\mathrm{T}} - \lambda)h(x) = f(x). \tag{5.7.17}$$

We construct the following expression:

$$\int_0^h dx G(x, y; \lambda)(L_x^{\mathrm{T}} - \lambda)h(x) = \int_0^h dx G(x, y; \lambda) f(x), \tag{5.7.18}$$

and perform the integral by parts on the left-hand side of Eq. (5.7.18). We obtain

$$\int_0^h dx \left[(L_x - \lambda)G(x, y; \lambda)\right] h(x) = \int_0^h dx G(x, y; \lambda) f(x). \tag{5.7.19}$$

The expression in the square bracket on the left-hand side of Eq. (5.7.19) is $\delta(x - y)$, and we have (after exchanging the variables x and y)

$$h(x) = \int_0^h dy G(y, x; \lambda) f(y). \tag{5.7.20}$$

Operating $(L_x^{\mathrm{T}} - \lambda)$ on both sides of Eq. (5.7.20), recalling Eq. (5.7.17), we have

$$f(x) = (L_x^{\mathrm{T}} - \lambda)h(x) = \int_0^h dy (L_x^{\mathrm{T}} - \lambda) G(y, x; \lambda) f(y). \tag{5.7.21}$$

This is true if and only if

$$(L_x^{\mathrm{T}} - \lambda) G(y, x; \lambda) = \delta(x - y). \tag{5.7.22}$$

Multiplying by $(\lambda - \lambda_n)$ on both sides of Eq. (5.7.22), and letting $\lambda \to \lambda_n$,

$$\lim_{\lambda \to \lambda_n} (\lambda - \lambda_n)(L_x^{\mathrm{T}} - \lambda) G(y, x; \lambda) = \lim_{\lambda \to \lambda_n} (\lambda - \lambda_n)\delta(x - y), \tag{5.7.23}$$

from which, we obtain

$$(L_x^{\mathrm{T}} - \lambda_n) R_n(y, x) = 0. \tag{5.7.24}$$

Since we know from Eq. (5.7.13a)

$$R_n(y, x) \propto \psi_n(x)\phi_n(y), \tag{5.7.13b}$$

Eq. (5.7.24) indeed demonstrates that $\psi_n(x)$ is the eigenfunction of L_x^{T}, belonging to the eigenvalue λ_n as claimed in Eq. (5.7.15). Thus $R_n(x, y)$ is a product of the eigenfunctions of L_x and L_x^{T}, i.e., a product of $\phi_n(x)$ and $\psi_n(y)$. Incidentally, this also proves that the eigenvalues of L_x^{T} are the same as the eigenvalues of L_x.

Let us now analyze the second term in Eq. (5.7.11),

$$\frac{1}{2\pi i}\int_{-\infty}^{\lambda_B} \frac{G(x,y;\lambda'+i\varepsilon)-G(x,y;\lambda'-i\varepsilon)}{\lambda'-\lambda} d\lambda'.$$

In the limit $\varepsilon \to 0$, we have, from Eq. (5.7.3), that

$$(L_x-\lambda)G(x,y;\lambda+i\varepsilon)=\delta(x-y), \tag{5.7.25a}$$

$$(L_x-\lambda)G(x,y;\lambda-i\varepsilon)=\delta(x-y). \tag{5.7.25b}$$

Taking the difference of the two expressions above, we have

$$(L_x-\lambda)\left[G(x,y;\lambda+i\varepsilon)-G(x,y;\lambda-i\varepsilon)\right]=0. \tag{5.7.26}$$

Thus we conclude

$$G(x,y;\lambda+i\varepsilon)-G(x,y;\lambda-i\varepsilon) \propto \psi_\lambda(y)\phi_\lambda(x). \tag{5.7.27}$$

Hence we finally obtain, by choosing proper normalization for ψ_n and ϕ_n,

$$G(x,y;\lambda)=+\sum_n \psi_n(y)\phi_n(x)/(\lambda_n-\lambda)+\int_{-\infty}^{\lambda_B} d\lambda' \psi_{\lambda'}(y)\phi_{\lambda'}(x)/(\lambda'-\lambda). \tag{5.7.28}$$

The first term in Eq. (5.7.28) represents a discrete contribution to $G(x,y;\lambda)$ from the *poles* at $\lambda=\lambda_n$, while the second term represents a continuum contribution from the *branch cut* starting at $\lambda=\lambda_B$ and extending to $-\infty$ along the negative real axis. Equation (5.7.28) is the generalization of the formula, Eq. (5.2.9), for the resolvent kernel $H(x,y;\lambda)$. Equation (5.7.28) is consistent with the assumption (5.7.8a). Setting $\lambda=0$ in Eq. (5.7.28), we obtain

$$G(x,y)=\sum_n \psi_n(y)\phi_n(x)/\lambda_n+\int_{-\infty}^{\lambda_B} d\lambda' \psi_{\lambda'}(y)\phi_{\lambda'}(x)/\lambda', \tag{5.7.29}$$

which is the generalization of the formula, Eq. (5.2.7), for the kernel $K(x,y)$.

We now anticipate that the completeness of the eigenfunctions will hold with minor modification to take care of the fact that L_x is non-self-adjoint. In order to see this, we operate $(L_x-\lambda)$ on $G(x,y;\lambda)$ in Eq. (5.7.28):

$$\begin{aligned}(L_x-\lambda)G(x,y;\lambda) &= (L_x-\lambda)\sum_n \psi_n(y)\phi_n(x)/(\lambda_n-\lambda)\\ &\quad + (L_x-\lambda)\int_{-\infty}^{\lambda_B} d\lambda' \psi_{\lambda'}(y)\phi_{\lambda'}(x)/(\lambda'-\lambda),\\ \Rightarrow \delta(x-y) &= \sum_n \psi_n(y)\phi_n(x)+\int_{-\infty}^{\lambda_B} d\lambda' \psi_{\lambda'}(y)\phi_{\lambda'}(x).\end{aligned} \tag{5.7.30}$$

This is a statement of the completeness of the eigenfunctions; the discrete eigenfunctions $\{\phi_n(x), \psi_n(y)\}$ and the continuum eigenfunctions $\{\phi_{\lambda'}(x), \psi_{\lambda'}(y)\}$ together form a complete set. We anticipate that the orthogonality of the eigenfunctions will survive with minor modification.

We first consider the following integral:

$$\int_0^h \psi_n(x) L_x \phi_m(x) dx = \lambda_m \int_0^h \psi_n(x)\phi_m(x) dx$$
$$= \int_0^h (L_x^{\mathrm{T}} \psi_n(x))\phi_m(x) dx = \lambda_n \int_0^h \psi_n(x)\phi_m(x) dx,$$

from which, we obtain

$$(\lambda_n - \lambda_m) \int_0^h \psi_n(x)\phi_m(x) dx = 0$$
$$\Rightarrow \int_0^h \psi_n(x)\phi_m(x) dx = 0 \quad \text{when} \quad \lambda_n \neq \lambda_m. \tag{5.7.31}$$

The eigenfunctions $\{\phi_m(x), \psi_n(x)\}$ belonging to the distinct eigenvalues are orthogonal to each other. Secondly, we multiply $\psi_n(x)$ on the completeness relation (5.7.30) and integrate over x. Since $\lambda_n \neq \lambda'$, we have by Eq. (5.7.31)

$$\int_0^h dx \psi_n(x) \delta(x - y) = \sum_m \psi_m(y) \int_0^h dx \psi_n(x)\phi_m(x),$$
$$\Rightarrow \psi_n(y) = \sum_m \psi_m(y) \int_0^h \psi_n(x)\phi_m(x) dx. \tag{5.7.32}$$

Then we must have

$$\int_0^h \psi_n(x)\phi_m(x) dx = \delta_{mn}. \tag{5.7.33}$$

Thirdly, by multiplying $\psi_{\lambda''}(x)$ on the completeness relation (5.7.30) and integrating over x, since $\lambda'' \neq \lambda_n$, we have by Eq. (5.7.32)

$$\psi_{\lambda''}(y) = \int_{-\infty}^{\lambda_B} d\lambda' \psi_{\lambda'}(y) \int_0^h dx \psi_{\lambda''}(x)\phi_{\lambda'}(x).$$

Then we must have

$$\int_0^h \psi_{\lambda''}(x)\phi_{\lambda'}(x) dx = \delta(\lambda' - \lambda''). \tag{5.7.34}$$

Thus the discrete eigenfunctions $\{\phi_m(x), \psi_n(x)\}$ and the continuum eigenfunctions $\{\phi_{\lambda'}(x), \psi_{\lambda''}(x)\}$ are normalized, respectively, as Eqs. (5.7.33) and (5.7.34).

5.8 Problems for Chapter 5

5.1. (due to H. C.). Find an upper bound and a lower bound for the first eigenvalue of

$$K(x, y) = \begin{cases} (1 - x)y, & 0 \le y \le x \le 1, \\ (1 - y)x, & 0 \le x \le y \le 1. \end{cases}$$

5.2. (due to H. C.). Consider the Bessel equation

$$(xu')' + \lambda xu = 0,$$

with the boundary conditions

$$u'(0) = u(1) = 0.$$

Transform this differential equation to an integral equation and find approximately the lowest eigenvalue.

5.3. Obtain an upper limit for the lowest eigenvalue of

$$\nabla^2 u + \lambda r u = 0,$$

where, in three dimensions,

$$0 < r < a \quad \text{and} \quad u = 0 \quad \text{on} \quad r = a.$$

5.4. (due to H. C.). Consider the Gaussian kernel $K(x, y)$ given by

$$K(x, y) = e^{-x^2 - y^2}, \quad -\infty < x, y < +\infty.$$

(a) Find the eigenvalues and the eigenfunctions of this kernel.

(b) Verify the Hilbert–Schmidt expansion of this kernel.

(c) By calculating A_2 and A_4, we obtain the exact lowest eigenvalue.

(d) Solve the integro-differential equation

$$\frac{\partial}{\partial t}\phi(x, t) = \int_{-\infty}^{+\infty} K(x, y)\phi(y, t)dy, \quad \text{with} \quad \phi(x, 0) = f(x).$$

5.5. Show that the boundary condition of the Sturm–Liouville system can be replaced with

$$\alpha_1\phi(0) + \alpha_2\phi^{'}(0) = 0 \quad \text{and} \quad \beta_1\phi(h) + \beta_2\phi^{'}(h) = 0$$

where $\alpha_1, \alpha_2, \beta_1$, and β_2 are some constants and the corresponding boundary condition on $G(x, y)$ is replaced accordingly.

5.6. Verify the Hilbert–Schmidt expansion for the case of *Direction 1* in Section 5.6, when the kernel $K(x, y)$ is *self-adjoint* and *square-integrable*.

5.7. (due to H. C.). Reproduce all the results of Section 5.7 with Green's function $G(x, y; \lambda)$ defined by

$$(L_x - \lambda r(x))G(x, y; \lambda) = \delta(x - y),$$

with the weight function

$$r(x) > 0 \quad \text{on} \quad x \in [0, h].$$

5.8. (due to H. C.). Consider the eigenvalue problem of the fourth-order ordinary differential equation of the form

$$\left(\frac{d^4}{dx^4} + 1\right)\phi(x) = -\lambda x\phi(x), \quad 0 < x < 1,$$

with the boundary conditions

$$\phi(0) = \phi^{'}(0) = 0,$$

$$\phi(1) = \phi^{'}(1) = 0.$$

Do the eigenfunctions form a complete set?

Hint: Check whether the differential operator L_x defined by

$$L_x \equiv -\left(\frac{d^4}{dx^4} + 1\right)$$

is self-adjoint or not under the specified boundary conditions.

5.9. (due to D. M.) Show that, if $\tilde{\lambda}$ is an eigenvalue of the symmetric kernel $K(x, y)$, the inhomogeneous Fredholm integral equation of the second kind,

$$\phi(x) = f(x) + \tilde{\lambda}\int_a^b K(x, y)\phi(y)dy, \quad a \leq x \leq b,$$

has no solution, unless the inhomogeneous term $f(x)$ is orthogonal to all of the eigenfunctions $\phi(x)$ corresponding to the eigenvalue $\tilde{\lambda}$.

Hint: You may suppose that $\{\lambda_n\}$ is the sequence of the eigenvalues of the symmetric kernel $K(x, y)$ (ordered by the increasing magnitude), with the corresponding eigenfunctions $\{\phi_n(x)\}$. You may assume that the eigenvalue $\tilde{\lambda}$ has the multiplicity 1. The extension to the higher multiplicity k, $k \geq 2$, is immediate.

5.10. (due to D. M.) Consider the integral equation

$$\phi(x) = f(x) + \lambda \int_0^1 \sin^2 \left[\pi(x - y)\right] \phi(y) dy, \quad 0 \leq x \leq 1.$$

(a) Solve the homogeneous equation by setting

$$f(x) = 0.$$

Determine all the eigenfunctions and eigenvalues. What is the spectral representation of the kernel?

Hint: Express the kernel

$$\sin^2[\pi(x - y)],$$

which is translationally invariant, periodic, and symmetric, in terms of the powers of

$$\exp[\pi(x - y)].$$

(b) Find the resolvent kernel of this equation.

(c) Is there a solution to the given inhomogeneous integral equation when

$$f(x) = \exp[im\pi x], \quad m \text{ integer},$$

and $\lambda = 2$?

5.11. (due to D. M.) Consider the kernel of the Fredholm integral equation of the second kind, which is given by

$$K(x, y) = \begin{cases} 3, & 0 \leq y < x \leq 1, \\ 2, & 0 \leq x < y \leq 1. \end{cases}$$

(a) Find the eigenfunctions $\phi_n(x)$ and the corresponding eigenvalues λ_n of the kernel.

(b) Is $K(x, y)$ symmetric? Determine the transposed kernel $K^T(x, y)$, and find its eigenfunctions $\psi_n(x)$ and the corresponding eigenvalues λ_n.

(c) Show by an explicit calculation that any $\phi_n(x)$ is orthogonal to any $\psi_m(x)$ if $m \neq n$.

(d) Derive the spectral representation of $K(x, y)$ in terms of $\phi_n(x)$ and $\psi_n(x)$.

5.12. (due to D. M.) Consider the Fredholm integral equation of the second kind,

$$\phi(x) = f(x) + \lambda \int_0^1 \left[\frac{1}{2}(x+y) - \frac{1}{2}\left|x-y\right| \right] \phi(y)dy, \quad 0 \le x \le 1.$$

(a) Find all nontrivial solutions $\phi_n(x)$ and corresponding eigenvalues λ_n for $f(x) \equiv 0$.

Hint: Obtain a differential equation for $\phi(x)$ with the suitable conditions for $\phi(x)$ and $\phi(x)'$.

(b) For the original inhomogeneous equation ($f(x) \neq 0$), will the iteration series converge?

(c) Evaluate the series $\sum_n \lambda_n^{-2}$ by using an appropriate integral.

5.13. If $\left|h\right| < 1$, find the nontrivial solutions of the homogeneous integral equation,

$$\phi(x) = \frac{\lambda}{2\pi} \int_{-\pi}^{\pi} \frac{1-h^2}{1-2h\cos(x-y)+h^2} \phi(y)dy.$$

Evaluate the corresponding values of the parameter λ.

Hint: The kernel of this equation is translationally invariant. Write $\cos(x-y)$ as a sum of two exponentials, express the kernel in terms of the complex variable $\zeta = \exp[i(y-x)]$, use the partial fractions, and then expand each fraction in powers of ζ.

5.14. If $\left|h\right| < 1$, find the solution of the integral equation,

$$f(x) = \frac{\lambda}{2\pi} \int_{-\pi}^{\pi} \frac{1-h^2}{1-2h\cos(x-y)+h^2} \phi(y)dy,$$

where $f(x)$ is the periodic and square-integrable known function.

Hint: Write $\cos(x-y)$ as a sum of two exponentials and express the kernel in terms of the complex variable $\zeta = \exp[i(y-x)]$.

5.15. Find the nontrivial solutions of the homogeneous integral equation,

$$\phi(x) = \lambda \int_{-\pi}^{\pi} \left[\frac{1}{4\pi}(x-y)^2 - \frac{1}{2}\left|x-y\right| \right] \phi(y)dy.$$

Evaluate the corresponding values of the parameter λ. Find the Hilbert–Schmidt expansion of the symmetric kernel displayed above.

5.16. (a) Transform the following integral equation to the integral equation with the symmetric kernel:

$$\phi(x) = \lambda \int_0^1 xy^2 \phi(y) dy.$$

(b) Solve the derived integral equation with the symmetric kernel.

5.17. (a) Transform the following integral equation to the integral equation with the symmetric kernel:

$$\phi(x) - \lambda \int_0^1 xy^2 \phi(y) dy = x + 3.$$

(b) Solve the derived integral equation with the symmetric kernel.

5.18. (a) Transform the following integral equation to the integral equation with the symmetric kernel.

$$\int_0^1 xy^2 \phi(y) dy = 2x.$$

(b) Solve the derived integral equation with the symmetric kernel.

5.19. Obtain the eigenfunctions of the following kernel and establish the Hilbert–Schmidt expansion of the kernel,

$$K(x, y) = 1, \quad 0 < x, y < 1.$$

5.20. Obtain the eigenfunctions of the following kernel and establish the Hilbert–Schmidt expansion of the kernel,

$$K(x, y) = \sin x \sin y, \quad 0 < x, y < 2\pi.$$

5.21. Obtain the eigenfunctions of the following kernel and establish the Hilbert–Schmidt expansion of the kernel,

$$K(x, y) = x + y, \quad 0 < x, y < 1.$$

5.22. Obtain the eigenfunctions of the following kernel and establish the Hilbert–Schmidt expansion of the kernel,

$$K(x, y) = \exp[x + y], \quad 0 < x, y < \ln 2.$$

5.23. Obtain the eigenfunctions of the following kernel and establish the Hilbert–Schmidt expansion of the kernel,

$$K(x, y) = xy + x^2 y^2, \quad 0 < x, y < 1.$$

6
Singular Integral Equations of the Cauchy Type

6.1
Hilbert Problem

Suppose that rather than the discontinuity $f^{+}(x) - f^{-}(x)$ across a branch cut, the ratio of the values on either side is known. That is, suppose that we wish to find $H(z)$ which has branch cut on $[a, b]$ with

$$H^{+}(x) = R(x)H^{-}(x) \quad \text{on} \quad x \in [a, b]. \tag{6.1.1}$$

Solution. Take the logarithm of both sides of Eq. (6.1.1) to obtain

$$\ln H^{+}(x) - \ln H^{-}(x) = \ln R(x) \quad \text{on} \quad x \in [a, b]. \tag{6.1.2}$$

Define

$$h(z) \equiv \ln H(z) \quad \text{and} \quad r(x) \equiv \ln R(x). \tag{6.1.3}$$

Then we have

$$h^{+}(x) - h^{-}(x) = r(x) \quad \text{on} \quad x \in [a, b]. \tag{6.1.4}$$

Hence we can apply the discontinuity formula (1.8.5) to determine $h(z)$:

$$h(z) = \frac{1}{2\pi i} \int_a^b \frac{r(x)}{x - z} dx + g(z) \tag{6.1.5}$$

where $g(z)$ is an arbitrary function with no cut on $[a, b]$, i.e.,

$$H(z) = G(z) \exp\left[\frac{1}{2\pi i} \int_a^b \frac{\ln R(x)}{x - z} dx\right]. \tag{6.1.6}$$

We check this result. Supposing $G(z)$ to be continuous across the cut, we have from this result that

$$H^{+}(x) = G(x) \exp\left[\frac{1}{2\pi i} \mathrm{P} \int_a^b \frac{\ln R(y)}{y - x} dy + \frac{1}{2} \ln R(x)\right], \tag{6.1.7a}$$

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$$H^-(x) = G(x)\exp\left[\frac{1}{2\pi i}\mathrm{P}\int_a^b \frac{\ln R(y)}{y-x}dy - \frac{1}{2}\ln R(x)\right]. \tag{6.1.7b}$$

Here, we used the Plemelj formulas (1.8.14a) and (1.8.14b) to take the limit as $z \to x$ from above and below. Dividing $H^+(x)$ by $H^-(x)$, we obtain

$$H^+(x)/H^-(x) = \exp\left(\ln R(x)\right) = R(x),$$

as expected.

Thus, in general, $G(z)$ may be of any form as long as it is continuous across the cut. In particular, we can choose $G(z)$ so as to make $H(z)$ have any desired behavior as $z \to a$ or b, by taking it of the form

$$G(z) = (a-z)^n(b-z)^m, \tag{6.1.8}$$

wherein $G(z) = \exp(g(z))$ has no branch points on $[a, b]$. In particular, we can always take $G(z)$ of the form above with n and m integers, and choose n and m to obtain a desired behavior in $H(z)$ as $z \to a$, $z \to b$, and $z \to \infty$.

We now address the inhomogeneous Hilbert problem.

Inhomogeneous Hilbert problem:
Obtain $\Phi(z)$ which has a branch cut on $[a, b]$ with

$$\Phi^+(x) = R(x)\Phi^-(x) + f(x) \quad \text{on} \quad x \in [a, b]. \tag{6.1.9}$$

Solution. To solve the inhomogeneous problem, we begin with the solution to the corresponding *homogeneous Hilbert problem*,

$$H^+(x) = R(x)H^-(x) \quad \text{on} \quad x \in [a, b], \tag{6.1.10}$$

whose solution, we know, is given by

$$H(z) = G(z)\exp\left[\frac{1}{2\pi i}\int_a^b \frac{\ln R(x)}{x-z}dx\right]. \tag{6.1.11}$$

We then seek for a solution of the form

$$\Phi(z) = H(z)K(z); \tag{6.1.12}$$

hence $K(z)$ must satisfy

$$H^+(x)K^+(x) = R(x)H^-(x)K^-(x) + f(x) \tag{6.1.13}$$

or, from Eq. (6.1.10), $H^+(x)K^+(x) = H^+(x)K^-(x) + f(x)$. Therefore, we have

$$K^+(x) - K^-(x) = f(x)/H^+(x). \tag{6.1.14}$$

The problem reduces to an ordinary application of the discontinuity theorem provided that $H(z)$ is chosen so that $f(x)/H^+(x)$ is less singular than a pole at both end points. With such a choice for $H(z)$, we then get

$$K(z) = \frac{1}{2\pi i}\int_a^b \frac{f(x)}{H^+(x)(x-z)}dx + L(z), \tag{6.1.15}$$

where $L(z)$ has no branch points on $[a, b]$. We thus find the *solution to the original inhomogeneous problem* to be

$$\Phi(z) = H(z)\,K(z), \tag{6.1.16}$$

i.e.,

$$\Phi(z) = G(z)\exp\left[\frac{1}{2\pi i}\int_a^b \frac{\ln R(x)}{x-z}dx\right]\left\{\frac{1}{2\pi i}\int_a^b \frac{f(x)}{H^+(x)(x-z)}dx + L(z)\right\}, \tag{6.1.17}$$

with $L(z)$ arbitrary and $G(z)$ chosen such that $f(x)/H^+(x)$ is integrable on $[a, b]$. Neither $L(z)$ nor $G(z)$ can have branch points on $[a, b]$.

❑ **Example 6.1.** Find $\Phi(z)$ satisfying

$$\Phi^+(x) = -\Phi^-(x) + f(x) \quad \text{on} \quad x \in [a, b]. \tag{6.1.18}$$

Solution.
Step 1. Solve the corresponding *homogeneous problem* first:

$$H^+(x) = -H^-(x). \tag{6.1.19}$$

Taking the logarithm on both sides we obtain

$$\ln H^+(x) = \ln H^-(x) + \ln(-1).$$

Setting

$$h(z) = \ln H(z),$$

we have

$$h^+(x) - h^-(x) = i\pi. \tag{6.1.20}$$

Then we can solve for $h(z)$ with the use of the discontinuity formula as

$$h(z) = \frac{1}{2\pi i}\int_a^b \frac{i\pi}{x-z}dx + g(z) = \frac{1}{2}\ln\left(\frac{b-z}{a-z}\right) + g(z),$$

hence we obtain the homogeneous solution to be

$$H(z) = G(z)\exp\left[\frac{1}{2}\ln\left(\frac{b-z}{a-z}\right)\right] = G(z)\sqrt{\frac{b-z}{a-z}}. \tag{6.1.21}$$

Suppose we choose

$$G(z) = 1/(b-z),$$

so that

$$H(z) = 1/\sqrt{(b-z)(a-z)}. \tag{6.1.22}$$

Therefore,

$$1/H(z) = \sqrt{(b-z)(a-z)}$$

does not blow up at $z = a$ or b. So, it is a good choice, since it does not make $f(x)/H^+(x)$ blow up any faster than $f(x)$ itself. Hence we take Eq. (6.1.22) for $H(z)$. Take that branch of $H(z)$ for which on the upper bank of the cut, we have

$$H^+(x) = 1/i\sqrt{(b-x)(x-a)},$$

where we have now the usual real square root. To do this, we set

$$z = a + r_1 e^{i\theta_1} \quad \text{and} \quad z = b + r_2 e^{i\theta_2},$$

so that

$$\sqrt{(b-z)(z-a)} = \sqrt{r_1 r_2}\exp(i\frac{\theta_1+\theta_2}{2}).$$

On $x \in [a, b]$, we have $\sqrt{r_1 r_2} = \sqrt{(b-x)(x-a)}$ which requires on the upper bank,

$$\exp(i\frac{\theta_1+\theta_2}{2}) = i.$$

We take on the upper bank $\theta_1 = 0$ and $\theta_2 = \pi$. Thus we have

$$H^+(x) = 1/i\sqrt{(b-x)(x-a)} \quad \text{and} \quad H^-(x) = -1/i\sqrt{(b-x)(x-a)}. \tag{6.1.23}$$

Step 2. Now look at the *inhomogeneous problem*:

$$\Phi^+(x) = -\Phi^-(x) + f(x). \tag{6.1.24}$$

Seek for the solution of the form

$$\Phi(z) = H(z)\,K(z)\,. \tag{6.1.25}$$

Then we have

$$H^+(x)K^+(x) = -H^-(x)K^-(x) + f(x) = H^+(x)K^-(x) + f(x),$$

where Eq. (6.1.19) has been used. Dividing both sides of the above equation by $H^+(x)$, we have

$$K^+(x) - K^-(x) = f(x)/H^+(x) = i\sqrt{(b-x)(x-a)}f(x). \tag{6.1.26}$$

By applying the discontinuity formula, we obtain

$$K(z) = \frac{1}{2\pi i}\int_a^b \frac{i\sqrt{(b-x)(x-a)}f(x)}{x-z}dx + g(z), \tag{6.1.27}$$

where $g(z)$ is an arbitrary function with no cut on $[a, b]$. Thus the final solution is given by

$$\begin{aligned}\Phi(z) &= H(z)K(z)\\ &= \frac{1}{\sqrt{(b-z)(a-z)}}\left\{\frac{1}{2\pi}\int_a^b \frac{\sqrt{(b-x)(x-a)}f(x)}{x-z}dx + g(z)\right\},\end{aligned}$$

or equivalently

$$\Phi(z) = H(z)\left\{\frac{1}{2\pi i}\int_a^b \frac{f(x)}{H^+(x)(x-z)}dx + g(z)\right\}. \tag{6.1.28}$$

6.2 Cauchy Integral Equation of the First Kind

Consider now the *inhomogeneous Cauchy integral equation of the first kind*:

$$\frac{1}{\pi i}\mathrm{P}\int_a^b \frac{\phi(y)}{y-x}dy = f(x), \quad a < x < b. \tag{6.2.1}$$

Define

$$\Phi(z) \equiv \frac{1}{2\pi i}\int_a^b \frac{\phi(y)}{y-z}dy \quad \text{for} \quad z \text{ not on } [a, b]. \tag{6.2.2}$$

As z approaches the branch cut from above and below, we find

$$\Phi^{+}(x) = \frac{1}{2\pi i}\mathrm{P}\int_a^b \frac{\phi(y)}{y-x}dy + \frac{1}{2}\phi(x), \tag{6.2.3a}$$

$$\Phi^{-}(x) = \frac{1}{2\pi i}\mathrm{P}\int_a^b \frac{\phi(y)}{y-x}dy - \frac{1}{2}\phi(x). \tag{6.2.3b}$$

Adding (6.2.3a) and (6.2.3b) and subtracting (6.2.3b) from (6.2.3a) results in

$$\Phi^{+}(x) + \Phi^{-}(x) = \frac{1}{\pi i}\mathrm{P}\int_a^b \frac{\phi(y)}{y-x}dy = f(x), \tag{6.2.4}$$

$$\Phi^{+}(x) - \Phi^{-}(x) = \phi(x). \tag{6.2.5}$$

Our strategy is the following. Solve Eq. (6.2.4) (which is the inhomogeneous Hilbert problem) to find $\Phi(z)$. Then use Eq. (6.2.5) to obtain $\phi(x)$. We remark that $\Phi(z)$ defined above is analytic in $\mathbb{C} - [a, b]$, so it can have no other singularities. Also note that it behaves as A/z as $|z| \to \infty$.

Solution.

$$\Phi^{+}(x) = -\Phi^{-}(x) + f(x), \quad a < x < b.$$

This is the inhomogeneous Hilbert problem that we solved in Section 6.2. We know

$$\Phi(z) = H(z)\left\{\frac{1}{2\pi i}\int_a^b \frac{f(x)}{H^{+}(x)(x-z)}dx + g(z)\right\}. \tag{6.2.6}$$

However, we can say something about the form of $g(z)$ in this case by examining the behavior of $\Phi(z)$ as $|z| \to \infty$. By our original definition of $\Phi(z)$, we have

$$\Phi(z) \sim A/z \quad \text{as} \quad |z| \to \infty, \tag{6.2.7a}$$

with

$$A = -\frac{1}{2\pi i}\int_a^b \phi(y)dy. \tag{6.2.8}$$

If we examine the above solution, upon noting that

$$H(z) \sim 1/z \quad \text{as} \quad |z| \to \infty,$$

we find that it has the asymptotic form

$$\Phi(z) \sim O(1/z^2) + g(z)/z \quad \text{as} \quad |z| \to \infty. \tag{6.2.7b}$$

Now, since $\Phi(z)$ is analytic everywhere away from the branch cut $[a, b]$, we conclude that $g(z)H(z)$ must also be analytic away from the cut $[a, b]$. Other singularities of $g(z)$ on $[a, b]$ can also be excluded. Hence $g(z)$ must be entire. Comparing the asymptotic forms (6.2.7a) and (6.2.7b), we conclude $g(z) \to A$ as $|z| \to \infty$. Therefore, by Liouville's theorem, we must have $g(z) = A$ identically. Therefore, we have

$$\Phi(z) = H(z)\left\{\frac{1}{2\pi i}\int_a^b \frac{f(x)}{H^+(x)(x-z)}dx + A\right\}. \tag{6.2.9}$$

Thus we have

$$\Phi^+(x) = H^+(x)\left\{\frac{1}{2\pi i}\mathrm{P}\int_a^b \frac{f(y)}{H^+(y)(y-x)}dy + \frac{1}{2}\frac{f(x)}{H^+(x)} + A\right\}, \tag{6.2.10a}$$

$$\Phi^-(x) = -H^+(x)\left\{\frac{1}{2\pi i}\mathrm{P}\int_a^b \frac{f(y)}{H^+(y)(y-x)}dy - \frac{1}{2}\frac{f(x)}{H^+(x)} + A\right\}. \tag{6.2.10b}$$

Hence $\phi(x)$ is given by

$$\phi(x) = \Phi^+(x) - \Phi^-(x) = H^+(x)\frac{1}{\pi i}\mathrm{P}\int_a^b \frac{f(y)}{H^+(y)(y-x)}dy + 2AH^+(x). \tag{6.2.11}$$

Note that

$$H^+(x) = 1/\sqrt{(b-x)(a-x)} = 1/i\sqrt{(b-x)(x-a)}. \tag{6.2.12}$$

The second term on the right-hand side of $\phi(x)$ turns out to be the solution of the homogeneous problem. So A is arbitrary. In order to understand this point more completely, it makes sense to examine the homogeneous problem separately.

Consider the *homogeneous Cauchy integral equation of the first kind*:

$$\frac{1}{\pi i}\mathrm{P}\int_a^b \frac{\phi(y)}{y-x}dy = 0. \tag{6.2.13}$$

Define

$$\Phi(z) \equiv \frac{1}{2\pi i}\int_a^b \frac{\phi(y)}{y-z}dy.$$

$\Phi(z)$ is analytic everywhere away from $[a, b]$, and it has the asymptotic behavior

$$\Phi(z) \sim A/z \quad \text{as} \quad |z| \to \infty,$$

with

$$A = -\frac{1}{2\pi i}\int_a^b \phi(y)dy. \tag{6.2.14}$$

We have

$$\Phi^+(x) = \frac{1}{2\pi i}\mathrm{P}\int_a^b \frac{\phi(y)}{y-x}dy + \frac{1}{2}\phi(x),$$

$$\Phi^-(x) = \frac{1}{2\pi i}\mathrm{P}\int_a^b \frac{\phi(y)}{y-x}dy - \frac{1}{2}\phi(x).$$

From these, we obtain

$$\Phi^+(x) + \Phi^-(x) = 0, \tag{6.2.15 a}$$

$$\Phi^+(x) - \Phi^-(x) = \phi(x). \tag{6.2.15b}$$

Solve

$$\Phi^+(x) = -\Phi^-(x),$$

which is the homogeneous Hilbert problem whose solution $\Phi(z)$ we already know from Section 6.2. Namely, we have

$$\Phi(z) = G(z)\sqrt{\frac{b-z}{a-z}}, \tag{6.2.16}$$

with $G(z)$ having no branch cuts on $[a, b]$. But in this case, we can determine the form of $G(z)$.

(i) We know that $\Phi(z)$ is analytic away from $[a, b]$, so $G(z)$ can only have singularities on $[a, b]$, but has no branch cuts there.

(ii) We know

$$\Phi(z) \sim A/z \quad \text{as} \quad |z| \to \infty,$$

so $G(z)$ must also behave as

$$G(z) \sim A/z \quad \text{as} \quad |z| \to \infty,$$

taking that branch of $\sqrt{(b-z)/(a-z)}$ which goes to $+1$ as $|z| \to \infty$.

(iii) $\Phi(z)$ can only be singular at the end points and then not as bad as a pole, since it is of the form

$$\frac{1}{2\pi i}\int_a^b \frac{\phi(y)}{y-z}dy.$$

So, $G(z)$ may only be singular at the end points a or b.

The last condition (iii), together with the asymptotic form of $G(z)$ as $|z| \to \infty$, suggests functions of the form

$$\frac{(a-z)^n}{(b-z)^{n+1}} \quad \text{or} \quad \frac{(b-z)^n}{(a-z)^{n+1}}.$$

The only choice for which the resulting $\Phi(z)$ does not blow up as bad as a pole at the two end points is

$$G(z) = A/(b-z) \quad \text{with} \quad A = \text{arbitrary constant.} \tag{6.2.17}$$

Therefore, the form of $\Phi(z)$ is determined to be

$$\Phi\,(z) = A/\sqrt{(b-z)(a-z)}. \tag{6.2.18}$$

Then the homogeneous solution is given by

$$\phi\,(x) = \Phi^+(x) - \Phi^-(x) = 2\Phi^+(x) = 2A/\sqrt{(b-x)(a-x)} = 2AH^+(x), \tag{6.2.19}$$

where $H^+(x)$ is given by Eq. (6.2.12). To verify that any A works, recall Eq. (6.2.14),

$$A = -\frac{1}{2\pi i}\int_a^b \phi(y)dy.$$

Substituting $\phi(y)$ just obtained, Eq. (6.2.19), into the above expression for A, we have

$$A = -\frac{2A}{2\pi i}\int_a^b \frac{dy}{\sqrt{(b-y)(a-y)}} = \frac{A}{\pi}\int_a^b \frac{dy}{\sqrt{(b-y)(y-a)}}$$
$$= \frac{A}{\pi}\int_{-1}^{+1} \frac{d\xi}{\sqrt{1-\xi^2}} = A,$$

which is true for any A.

6.3 Cauchy Integral Equation of the Second Kind

We now consider the *inhomogeneous Cauchy integral equation of the second kind,*

$$\phi(x) = f(x) + \frac{\lambda}{\pi}\mathrm{P}\int_0^1 \frac{\phi(y)}{y-x}dy, \quad 0 < x < 1. \tag{6.3.1}$$

Define $\Phi(z)$

$$\Phi(z) \equiv \frac{1}{2\pi i}\int_0^1 \frac{\phi(y)}{y-z}dy. \tag{6.3.2}$$

The boundary values of $\Phi(z)$ as z approaches the cut $[0, 1]$ from above and below are given, respectively, by

$$\Phi^+(x) = \frac{1}{2\pi i} \mathrm{P} \int_0^1 \frac{\phi(y)}{y - x} dy + \frac{1}{2}\phi(x), \quad 0 < x < 1,$$

$$\Phi^-(x) = \frac{1}{2\pi i} \mathrm{P} \int_0^1 \frac{\phi(y)}{y - x} dy - \frac{1}{2}\phi(x), \quad 0 < x < 1,$$

so that

$$\Phi^+(x) - \Phi^-(x) = \phi(x), \quad 0 < x < 1, \tag{6.3.3}$$

$$\Phi^+(x) + \Phi^-(x) = \frac{1}{\pi i} \mathrm{P} \int_0^1 \frac{\phi(y)}{y - x} dy, \quad 0 < x < 1. \tag{6.3.4}$$

Equation (6.3.1) now reads

$$\Phi^+(x) - \Phi^-(x) = f(x) + i\lambda(\Phi^+(x) + \Phi^-(x)),$$

or

$$\Phi^+(x) - \frac{1 + i\lambda}{1 - i\lambda}\Phi^-(x) = \frac{1}{1 - i\lambda} f(x). \tag{6.3.5}$$

We recognize this as the *inhomogeneous Hilbert problem*.

Consider the case $\lambda > 0$ and set λ equal to

$$\lambda = \tan \pi\gamma, \quad 0 < \gamma \leq \frac{1}{2}. \tag{6.3.6}$$

Then we have

$$\frac{1 + i\lambda}{1 - i\lambda} = e^{2\pi i\gamma}.$$

First, we solve a homogeneous problem:

$$H^+(x) = H^-(x)e^{2\pi i\gamma}, \quad 0 < x < 1. \tag{6.3.7}$$

In terms of $h(z)$ defined by

$$H(z) = e^{h(z)},$$

we have

$$h^+(x) - h^-(x) = 2\pi i\gamma.$$

Hence we obtain

$$h(z) = \frac{1}{2\pi i}\int_0^1 \frac{2\pi i\gamma}{y - z}dz = \gamma \ln\left(\frac{1-z}{-z}\right),$$

and

$$H(z) = \left(\frac{1-z}{-z}\right)^{\gamma}, \quad 0 < \gamma \leq 1/2. \tag{6.3.8}$$

Since we can add any function with no cut on $[0, 1]$ to $h(z)$, we can multiply any function with no cut on $[0, 1]$ onto $H(z)$. By multiplying the above equation by $e^{i\pi\gamma}/(1-z)$, we choose

$$H(z) = 1/[(1-z)^{1-\gamma}z^{\gamma}], \quad 0 < \gamma \leq 1/2. \tag{6.3.9}$$

Returning to the inhomogeneous Hilbert problem, Eq. (6.3.5), we write

$$\Phi(z) = H(z)G(z). \tag{6.3.10}$$

Then Eq. (6.3.5) reads

$$H^+(x)G^+(x) - H^+(x)G^-(x) = \frac{1}{1-i\lambda}f(x).$$

Dividing through the above equation by $H^+(x)$, we have

$$G^+(x) - G^-(x) = \frac{1}{1-i\lambda}f(x)/H^+(x) = \frac{1}{1-i\lambda}f(x)(1-x)^{1-\gamma}x^{\gamma}. \tag{6.3.11}$$

Since, for our choice of $H(z)$, Eq. (6.3.9), we have

$$1/H^+(0) = 1/H^+(1) = 0,$$

we did not bring in extra singular behavior at $x = 0$ and 1; rather we made $f(x)/H^+(x)$ better behaved than $f(x)$ at $x = 0$ and 1. By the discontinuity formula, we have

$$G(z) = \frac{1}{2\pi i}\frac{1}{1-i\lambda}\int_0^1 [f(y)(1-y)^{1-\gamma}y^{\gamma}/(y-z)]dy + g(z). \tag{6.3.12}$$

The integral on the right-hand side of Eq. (6.3.12) takes care of the discontinuity of $G(z)$ across the cut on $[0, 1]$ so that $g(z)$ does not have any cut on $[0, 1]$. We know furthermore that

(1) $G(z)$ is analytic in the cut z plane.

(2) As $z \to 0$, $G(z)$ is bounded by

$$\left|G(z)\right| < \left|\frac{1}{z^{\alpha}} \cdot z^{\gamma}\right|,$$

and similarly for $z \to 1$.

(3) As $|z| \to \infty$, $G(z)$ is bounded by constant. Thus we know from Eq. (6.3.12) that $g(z)$ is analytic everywhere and

$$g(z) \sim \text{constant} \quad \text{as} \quad |z| \to \infty.$$

By Liouville's theorem, we obtain

$$g(z) = \text{constant} = k. \tag{6.3.13}$$

Hence we obtain

$$\Phi(z) = \frac{1}{(1-z)^{1-\gamma}z^{\gamma}}\frac{1}{2\pi i}\frac{1}{1-i\lambda}\int_0^1 \frac{(1-y)^{1-\gamma}y^{\gamma}}{y-z}f(y)dy + \frac{k}{(1-z)^{1-\gamma}z^{\gamma}}. \tag{6.3.14}$$

Then by choosing the branch of $H(z)$ as indicated in Figure 6.1, we have from the Plemelj formula

$$\Phi^{+}(x) = \frac{1}{2\pi i}\frac{1}{1-i\lambda}\frac{1}{(1-x)^{1-\gamma}x^{\gamma}}\mathrm{P}\int_0^1 \frac{(1-y)^{1-\gamma}y^{\gamma}}{y-x}f(y)dy + \frac{1}{2}\frac{1}{1-i\lambda}f(x) + \frac{k}{(1-x)^{1-\gamma}x^{\gamma}}, \tag{6.3.15a}$$

$$\Phi^{-}(x) = \frac{1}{2\pi i}\frac{1}{1-i\lambda}\frac{e^{-2\pi i\gamma}}{(1-x)^{1-\gamma}x^{\gamma}}\mathrm{P}\int_0^1 \frac{(1-y)^{1-\gamma}y^{\gamma}}{y-x}f(y)dy - \frac{1}{2}\frac{e^{-2\pi i\gamma}}{1-i\lambda}f(x) + \frac{ke^{-2\pi i\gamma}}{(1-x)^{1-\gamma}x^{\gamma}}. \tag{6.3.15b}$$

By Eq. (6.3.3),

$$\begin{aligned}\phi(x) &= \Phi^{+}(x) - \Phi^{-}(x)\\ &= \frac{1}{2\pi i}\frac{1-e^{-2\pi i\gamma}}{1-i\lambda}\frac{1}{(1-x)^{1-\gamma}x^{\gamma}}\mathrm{P}\int_0^1 \frac{(1-y)^{1-\gamma}y^{\gamma}}{y-x}f(y)dy\\ &\quad + \frac{1}{2}\frac{1+e^{-2\pi i\gamma}}{1-i\lambda}f(x) + \frac{c}{(1-x)^{1-\gamma}x^{\gamma}}.\end{aligned} \tag{6.3.16}$$

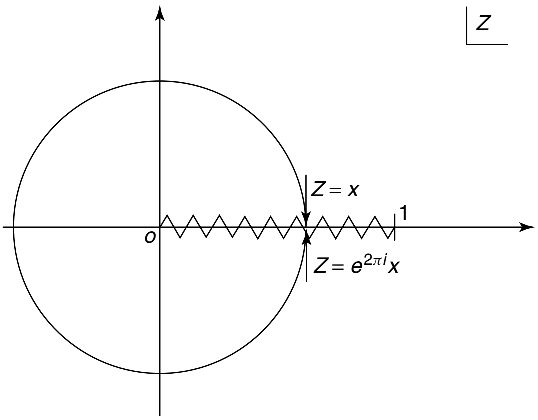

Fig. 6.1 The branch of $H(z)$ chosen for Eqs. (6.3.15a) and (6.3.15b).

Since we have

$$1 - e^{-2\pi i\gamma} = \frac{2i\lambda}{1+i\lambda}, \quad 1 + e^{-2\pi i\gamma} = \frac{2}{1+i\lambda},$$

we finally obtain

$$\phi(x) = \frac{1}{\pi}\frac{\lambda}{1+\lambda^2}\frac{1}{(1-x)^{1-\gamma}x^{\gamma}}\mathrm{P}\int_0^1 \frac{(1-y)^{1-\gamma}y^{\gamma}}{y-x}f(y)dy$$
$$+\frac{1}{1+\lambda^2}f(x) + \frac{c}{(1-x)^{1-\gamma}x^{\gamma}}, \tag{6.3.17}$$

where c is an arbitrary constant. We note that solution (6.3.17) involves one arbitrary constant and the spectrum of the eigenvalue λ of Eq. (6.3.1) with $f(x) \equiv 0$ is continuous. It is noted that the homogeneous solution of Eq. (6.3.1) with $f(x) \equiv 0$ comes from the entire function $g(z)$.

6.4
Carleman Integral Equation

Consider the *inhomogeneous Carleman integral equation*

$$a(x)\phi(x) = f(x) + \lambda\mathrm{P}\int_{-1}^{+1}\frac{\phi(y)}{y-x}dy, \quad -1 < x < 1, \tag{6.4.1}$$

which has a *Cauchy kernel* and is a *generalization of the Cauchy integral equation of the second kind*. Assume that λ is real and that $f(x)$ and $a(x)$ are prescribed real functions. Without loss of generality, we can take $\lambda > 0$ (otherwise, we just change the sign of $a(x)$ and $f(x)$).

As usual, define

$$\Phi(z) \equiv \frac{1}{2\pi i}\int_{-1}^{+1} \frac{\phi(y)}{y-z}dy. \tag{6.4.2}$$

Then $\Phi(z)$ is analytic in $\mathbb{C}-[-1,1]$. The asymptotic behavior of $\Phi(z)$ as $|z|\to\infty$ is given by $\Phi(z) \sim A/z$ as $|z|\to\infty$, with A given by

$$A = -\frac{1}{2\pi i}\int_{-1}^{+1} \phi(y)dy.$$

The end point behavior of $\Phi(z)$ is that $\Phi(z)$ is less singular than a pole as $z \to \pm 1$.

The boundary values of $\Phi(z)$ as z approaches the cut from above and below are given by Plemelj formulas, namely

$$\Phi^{+}(x) = \frac{1}{2\pi i}\mathrm{P}\int_{-1}^{+1} \frac{\phi(y)}{y-x}dy + \frac{1}{2}\phi(x) \quad \text{for} \quad -1 < x < 1, \tag{6.4.3a}$$

$$\Phi^{-}(x) = \frac{1}{2\pi i}\mathrm{P}\int_{-1}^{+1} \frac{\phi(y)}{y-x}dy - \frac{1}{2}\phi(x) \quad \text{for} \quad -1 < x < 1. \tag{6.4.3b}$$

Adding (6.4.3a) and (6.4.3b), and subtracting (6.4.3b) from (6.4.3.a), we obtain

$$\Phi^{+}(x) + \Phi^{-}(x) = \frac{1}{\pi i}P\int_{-1}^{+1} \frac{\phi(y)}{y-x}dy, \quad \Phi^{+}(x) - \Phi^{-}(x) = \phi(x). \tag{6.4.4}$$

Using the Carleman integral equation (6.4.1), we rewrite Eq. (6.4.4) as

$$\left[a(x) - \lambda\pi i\right]\Phi^{+}(x) = \left[a(x) + \lambda\pi i\right]\Phi^{-}(x) + f(x),$$

which is an *inhomogeneous Hilbert problem,*

$$\Phi^{+}(x) = R(x)\Phi^{-}(x) + \frac{f(x)}{a(x) - \lambda\pi i}, \tag{6.4.5a}$$

with

$$R(x) \equiv \frac{a(x) + \lambda\pi i}{a(x) - \lambda\pi i}. \tag{6.4.5b}$$

Since we need to take $\ln R(x)$, we represent $R(x)$ in the polar form. Recalling that

$$x + iy = \sqrt{x^2 + y^2}e^{i\tan^{-1}(y/x)},$$

we have

$$R(x) = \exp\left[2i\tan^{-1}\left(\frac{\lambda\pi}{a(x)}\right)\right]. \tag{6.4.6}$$

Since the function $a(x) + \lambda\pi i$ is always in the upper half plane, $\tan^{-1}(\frac{\lambda\pi}{a(x)})$ is in the range $(0, \pi)$. Define

$$\theta(x) \equiv \tan^{-1}\left(\frac{\lambda\pi}{a(x)}\right). \tag{6.4.7}$$

Then we have

$$R(x) = e^{2i\theta(x)} \quad \text{with} \quad 0 < \theta(x) < \pi,$$

and we obtain

$$\Phi^+(x) = e^{2i\theta(x)}\Phi^-(x) + \frac{f(x)}{a(x) - \lambda\pi i}. \tag{6.4.8}$$

Homogeneous problem: We first solve the homogeneous problem. Namely, we solve

$$H^+(x) = e^{2i\theta(x)}H^-(x). \tag{6.4.9}$$

Taking the logarithm of both sides, we have

$$\ln H^+(x) - \ln H^-(x) = 2i\theta(x).$$

By the now familiar method of solving the homogeneous Hilbert problem, we have

$$\ln H(z) = \frac{1}{\pi}\int_{-1}^{+1}\frac{\theta(x)dx}{x - z} + g(z),$$

or

$$H(z) = G(z)\exp\left[\frac{1}{\pi}\int_{-1}^{+1}\frac{\theta(x)dx}{x - z}\right]. \tag{6.4.10}$$

Behavior at the end points: Since $\theta(x)$ is bounded at the end points, the integral in the square bracket above has logarithmic singularities as $z \to \pm 1$:

$$\frac{1}{\pi}\int_{-1}^{+1}\frac{\theta(x)dx}{x - z} \sim \frac{-\theta(-1)}{\pi}\ln(-1 - z) \quad \text{as} \quad z \to -1,$$

$$\exp\left[\frac{1}{\pi}\int_{-1}^{+1}\frac{\theta(x)dx}{x - z}\right] \sim (-1 - z)^{-\theta(-1)/\pi} \quad \text{as} \quad z \to -1, \tag{6.4.11a}$$

and

$$\frac{1}{\pi}\int_{-1}^{+1}\frac{\theta(x)dx}{x - z} \sim \frac{\theta(1)}{\pi}\ln(1 - z) \quad \text{as} \quad z \to +1,$$

$$\exp\left[\frac{1}{\pi}\int_{-1}^{+1}\frac{\theta(x)dx}{x-z}\right] \sim (1-z)^{\theta(1)/\pi} \quad \text{as} \quad z \to +1. \tag{6.4.11b}$$

Since $0 < \theta(x) < \pi$, we have $0 < \frac{\theta(1)}{\pi} < 1$, and $-1 < \frac{-\theta(-1)}{\pi} < 0$. So, disregarding $G(z)$, $H(z)$ could be singular as $z \to -1$ so that $1/H(z) \to 0$ as $z \to -1$, but $H(z)$ could be zero as $z \to 1$ so that $1/H(z)$ becomes singular as $z \to +1$. Since we wish to ensure that $1/H^+(x)$ is not singular in order to facilitate the solution of the inhomogeneous problem, choose $G(z)$ so that $1/H(z)$ is not singular as $z \to +1$. A good choice for $G(z)$ is $G(z) = 1/(1-z)$.

Then we obtain

$$H(z) = \frac{1}{1-z}\exp\left[\frac{1}{\pi}\int_{-1}^{+1}\frac{\theta(x)dx}{x-z}\right]. \tag{6.4.12}$$

With this choice,

$$H(z) \sim 1/(-1-z)^{\theta(-1)/\pi} \quad \text{as} \quad z \to -1 \quad \left(0 < \frac{\theta(-1)}{\pi} < 1\right), \tag{6.4.13a}$$

$$H(z) \sim 1/(1-z)^{1-\theta(1)/\pi} \quad \text{as} \quad z \to +1 \quad \left(0 < 1 - \frac{\theta(1)}{\pi} < 1\right), \tag{6.4.13b}$$

so that $1/H^+(x)$ is not singular as $x \to \pm 1$.

Inhomogeneous problem: We now solve the inhomogeneous problem:

$$\Phi^+(x) = e^{2i\theta(x)}\Phi^-(x) + \frac{f(x)}{a(x)-\lambda\pi i}. \tag{6.4.14}$$

We seek the solution of the form

$$\Phi(z) = H(z)K(z). \tag{6.4.15}$$

We obtain

$$H^+(x)K^+(x) = H^+(x)K^-(x) + \frac{f(x)}{a(x)-\lambda\pi i}.$$

We then have

$$K^+(x) - K^-(x) = f(x)/\left[H^+(x)(a(x)-\lambda\pi i)\right], \tag{6.4.16}$$

from which we immediately obtain, by the discontinuity theorem,

$$K(z) = \frac{1}{2\pi i}\int_{-1}^{+1}\frac{f(x)}{H^+(x)(a(x)-\lambda\pi i)(x-z)}dx + L(z). \tag{6.4.17}$$

Hence we obtain

$$\Phi(z) = H(z)\left[\frac{1}{2\pi i}\int_{-1}^{+1}\frac{f(x)}{H^{+}(x)(a(x)-\lambda\pi i)(x-z)}dx\right] + H(z)L(z). \qquad (6.4.18)$$

Determination of $L(z)$ **from its behavior at infinity**: We know that $\Phi(z)$ is analytic in $\mathbb{C}-[-1,1]$. Our choice of $H(z)$ is also analytic in $\mathbb{C}-[-1,1]$. So, $L(z)$ is analytic in $\mathbb{C}-[-1,1]$. Furthermore, $L(z)$ can have no branch cut in $[-1,1]$, so it can at worst have poles on $[-1,1]$. But the poles cannot occur inside $(-1,1)$ since $\Phi(z)$ has no poles there. Hence at worst $L(z)$ has poles at the end points $z\to\pm 1$. However, we know that $\Phi(z)$ is less singular than a pole as $z\to\pm 1$. This is true of the first term in (6.4.18), which can be shown to be as singular as $f(x)$ at the end points (the contribution from $H(z)$ and $1/H^{+}(x)$ canceling). Hence $H(z)L(z)$ must be less singular than a pole as $z\to\pm 1$. This implies that $L(z)$ cannot have poles at ± 1. Therefore, $L(z)$ is entire. We know

$$\Phi(z)\sim A/z, \quad H(z)\sim -1/z \quad \text{as} \quad |z|\to\infty.$$

From our solution above, we have

$$\Phi(z)\sim O(1/z^2) - L(z)/z \quad \text{as} \quad |z|\to\infty.$$

Thus we conclude that $L(z)\sim -A$ as $|z|\to\infty$. By Liouville's theorem, we must have

$$L(z) = -A. \qquad (6.4.19)$$

Then our solution for $\Phi(z)$ is given by

$$\Phi(z) = H(z)\left[\frac{1}{2\pi i}\int_{-1}^{+1}\frac{f(x)}{H^{+}(x)(a(x)-\lambda\pi i)(x-z)}dx\right] - AH(z), \qquad (6.4.20)$$

with

$$H(z) = \frac{1}{1-z}\exp\left[\frac{1}{\pi}\int_{-1}^{+1}\frac{\theta(x)dx}{x-z}\right]. \qquad (6.4.21)$$

With that choice for $H(z)$, $H^{+}(x)$ can be chosen as

$$H^{+}(x) = \frac{1}{1-x}\exp\left[\frac{1}{\pi}\mathrm{P}\int_{-1}^{+1}\frac{\theta(y)}{y-x}dy + i\theta(x)\right]. \qquad (6.4.22)$$

To obtain $\phi(x)$, we use

$$\Phi^{+}(x) - \Phi^{-}(x) = \phi(x).$$

From $\Phi(z)$ obtained above, by the Plemelj formula, we have

$$\Phi^+(x) = H^+(x)\left[\frac{1}{2\pi i}\mathrm{P}\int_{-1}^{+1}\frac{f(y)dy}{H^+(y)(a(y)-\lambda\pi i)(y-x)} + \frac{1}{2}\frac{f(x)}{H^+(x)(a(x)-\lambda\pi i)}\right]$$
$$-AH^+(x), \tag{6.4.23a}$$

$$\Phi^-(x) = H^-(x)\left[\frac{1}{2\pi i}\mathrm{P}\int_{-1}^{+1}\frac{f(y)dy}{H^+(y)(a(y)-\lambda\pi i)(y-x)} - \frac{1}{2}\frac{f(x)}{H^+(x)(a(x)-\lambda\pi i)}\right]$$
$$-AH^-(x). \tag{6.4.23b}$$

Then our solution is given by

$$\phi(x) = H^+(x)\left[1-e^{-2i\theta(x)}\right]\left[\frac{1}{2\pi i}\mathrm{P}\int_{-1}^{+1}\frac{f(y)dy}{H^+(y)(a(y)-\lambda\pi i)(y-x)}\right]$$

$$+\left[1+e^{-2i\theta(x)}\right]\frac{f(x)}{2(a(x)-\lambda\pi i)} - AH^+(x)\left[1-e^{-2i\theta(x)}\right]. \tag{6.4.24}$$

Simplification: We set

$$H^+(x) = \frac{1}{1-x}\exp\left[\frac{1}{\pi}\mathrm{P}\int_{-1}^{+1}\frac{\theta(y)dy}{y-x}\right]e^{i\theta(x)} \equiv B(x)e^{i\theta(x)}, \tag{6.4.25}$$

where we define

$$B(x) \equiv \frac{1}{1-x}\exp\left[\frac{1}{\pi}\mathrm{P}\int_{-1}^{+1}\frac{\theta(y)dy}{y-x}\right]. \tag{6.4.26}$$

Also, using the definition of $\theta(x)$, Eq. (6.4.7), we note

$$e^{i\theta(x)}(1-e^{-2i\theta(x)}) = 2\pi i\lambda\Big/\sqrt{a^2(x)+\lambda^2\pi^2},$$

$$e^{i\theta(x)}(a(x)-\lambda\pi i) = \sqrt{a^2(x)+\lambda^2\pi^2},$$

$$\frac{1+e^{-2i\theta(x)}}{2(a(x)-\lambda\pi i)} = \frac{a(x)}{a^2(x)+\lambda^2\pi^2}.$$

Thus the final form of the solution is given by

$$\phi(x) = \frac{\lambda B(x)}{\sqrt{a^2(x)+\lambda^2\pi^2}}\mathrm{P}\int_{-1}^{+1}\frac{f(y)dy}{B(y)\sqrt{a^2(y)+\lambda^2\pi^2}(y-x)} + \frac{a(x)f(x)}{a^2(x)+\lambda^2\pi^2}$$
$$+C\frac{B(x)}{\sqrt{a^2(x)+\lambda^2\pi^2}}, \tag{6.4.27}$$

with $B(x)$ defined by Eq. (6.4.26).

We remark that when $f(x) = 0$, i.e., for the *homogeneous Carleman integral equation*,

$$a(x)\phi(x) = \lambda \mathrm{P} \int_{-1}^{+1} \frac{\phi(y)}{y - x} dy,$$

the homogeneous solution is, by setting $f(x) = f(y) \equiv 0$ in Eq. (6.4.27),

$$\phi_{\mathrm{H}}(x) = C \frac{B(x)}{\sqrt{a^2(x) + \lambda^2 \pi^2}},$$

being well defined for all $\lambda > 0$. Thus there is a *continuous spectrum of eigenvalues* for the homogeneous Carleman integral equation.

Setting $a(x) \equiv 1$, solution (6.4.27) to the inhomogeneous Carleman integral equation of the second kind reduces to solution (6.3.17) to the inhomogeneous Cauchy integral equation of the second kind, because the latter is the special case ($a(x) \equiv 1$) of the former.

6.5
Dispersion Relations

Dispersion relations in classical electrodynamics are closely related to the Cauchy integral equations.

Suppose that the complex function $f(z)$ is analytic in the upper-half plane, $\mathrm{Im}\, z > 0$, and that $f(z) \to 0$ as $|z| \to \infty$. We let its boundary value on the real x-axis in the complex z plane to be given by

$$F(x) = \lim_{\varepsilon \to 0} f(x + i\varepsilon), \quad \text{with} \quad \varepsilon = \text{positive infinitesimal}. \tag{6.5.1}$$

From the Cauchy integral formula,

$$f(z) = \frac{1}{2\pi i} \int_{C_+} \frac{f(\varsigma)}{\varsigma - z} d\varsigma,$$

where C_+ is the infinite semicircle in the upper half plane, we immediately obtain

$$f(z) = \frac{1}{2\pi i} \int_{\text{real axis}} \frac{f(\varsigma)}{\varsigma - z} d\varsigma. \tag{6.5.2}$$

From Eqs. (6.5.1) and (6.5.2), we obtain

$$2\pi i F(x) = \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \frac{f(x')}{x' - x - i\varepsilon} dx' = \mathrm{P} \int_{-\infty}^{\infty} \frac{F(x')}{x' - x} dx' + \pi i F(x).$$

Thus we have

$$F(x) = \frac{1}{\pi i} \mathrm{P} \int_{-\infty}^{\infty} \frac{F(x')}{x' - x} dx'. \tag{6.5.3}$$

Equating the real part and the imaginary part of Eq. (6.5.3), we obtain the following dispersion relations:

$$\mathrm{Re}\, F(x) = \frac{1}{\pi} \mathrm{P} \int_{-\infty}^{\infty} \frac{\mathrm{Im}\, F(x')}{x' - x} dx', \quad \mathrm{Im}\, F(x) = -\frac{1}{\pi} \mathrm{P} \int_{-\infty}^{\infty} \frac{\mathrm{Re}\, F(x')}{x' - x} dx'. \tag{6.5.4}$$

Suppose that $f(z)$ has an even symmetry,

$$f(-z) = f^*(z^*). \tag{6.5.5}$$

Then we write

$$\begin{aligned}
F(x) &= \frac{1}{\pi i} \mathrm{P} \int_{-\infty}^{\infty} \frac{F(x')}{x' - x} dx' = \frac{1}{\pi i} \mathrm{P} \int_{\infty}^{0} \frac{F^*(x')}{x' + x} dx' + \frac{1}{\pi i} \mathrm{P} \int_{0}^{\infty} \frac{F(x')}{x' - x} dx' \\
&= -\frac{1}{\pi i} \mathrm{P} \int_{0}^{\infty} \frac{\mathrm{Re}\, F(x') - i\, \mathrm{Im}\, F(x')}{x' + x} dx' \\
&\quad + \frac{1}{\pi i} \mathrm{P} \int_{0}^{\infty} \frac{\mathrm{Re}\, F(x') + i\, \mathrm{Im}\, F(x')}{x' - x} dx' \\
&= \frac{i}{\pi} \mathrm{P} \int_{0}^{\infty} \left(\frac{\mathrm{Re}\, F(x')}{x' + x} - \frac{\mathrm{Re}\, F(x')}{x' - x} \right) dx' \\
&\quad + \frac{1}{\pi} \mathrm{P} \int_{0}^{\infty} \left(\frac{\mathrm{Im}\, F(x')}{x' + x} + \frac{\mathrm{Im}\, F(x')}{x' - x} \right) dx' \\
&= \frac{-2i}{\pi} \mathrm{P} \int_{0}^{\infty} \frac{x\, \mathrm{Re}\, F(x')}{x'^2 - x^2} dx' + \frac{2}{\pi} \mathrm{P} \int_{0}^{\infty} \frac{x'\, \mathrm{Im}\, F(x')}{x'^2 - x^2} dx'.
\end{aligned}$$

Thus, we obtain the following dispersion relations:

$$\mathrm{Re}\, F(x) = \frac{2}{\pi} \mathrm{P} \int_{0}^{\infty} \frac{x'\, \mathrm{Im}\, F(x')}{x'^2 - x^2} dx', \quad \mathrm{Im}\, F(x) = -\frac{2}{\pi} \mathrm{P} \int_{0}^{\infty} \frac{x\, \mathrm{Re}\, F(x')}{x'^2 - x^2} dx'. \tag{6.5.6}$$

Suppose that $f(z)$ has an odd symmetry,

$$f(-z) = -f^*(z^*). \tag{6.5.7}$$

Then we write

$$F(x) = \frac{1}{\pi i} \mathrm{P} \int_{-\infty}^{\infty} \frac{F(x')}{x' - x} dx' = -\frac{1}{\pi i} \mathrm{P} \int_{\infty}^{0} \frac{F^*(x')}{x' + x} dx' + \frac{1}{\pi i} \mathrm{P} \int_{0}^{\infty} \frac{F(x')}{x' - x} dx'$$

$$= \frac{1}{\pi i} \mathrm{P} \int_{0}^{\infty} \frac{\mathrm{Re}\, F(x') - i\, \mathrm{Im}\, F(x')}{x' + x} dx' + \frac{1}{\pi i} \mathrm{P} \int_{0}^{\infty} \frac{\mathrm{Re}\, F(x') + i\, \mathrm{Im}\, F(x')}{x' - x} dx'$$

$$= \frac{-i}{\pi} \mathrm{P} \int_{0}^{\infty} \left(\frac{\mathrm{Re}\, F(x')}{x'+x} + \frac{\mathrm{Re}\, F(x')}{x'-x} \right) dx' - \frac{1}{\pi} \mathrm{P} \int_{0}^{\infty} \left(\frac{\mathrm{Im}\, F(x')}{x'+x} - \frac{\mathrm{Im}\, F(x')}{x'-x} \right) dx'$$

$$= \frac{-2i}{\pi} \mathrm{P} \int_{0}^{\infty} \frac{x'\, \mathrm{Re}\, F(x')}{x'^2 - x^2} dx' + \frac{2}{\pi} \mathrm{P} \int_{0}^{\infty} \frac{x\, \mathrm{Im}\, F(x')}{x'^2 - x^2} dx'.$$

Thus, we obtain the following dispersion relations:

$$\mathrm{Re}\, F(x) = \frac{2}{\pi} \mathrm{P} \int_{0}^{\infty} \frac{x\, \mathrm{Im}\, F(x')}{x'^2 - x^2} dx', \quad \mathrm{Im}\, F(x) = -\frac{2}{\pi} \mathrm{P} \int_{0}^{\infty} \frac{x'\, \mathrm{Re}\, F(x')}{x'^2 - x^2} dx'. \tag{6.5.8}$$

These dispersion relations were derived by H.A. Kramers in 1927 and R. de L. Kronig in 1926 independently for the X-ray dispersion and the optical dispersion. Kramers–Kronig dispersion relations are of very general validity which only depend on the assumption of the causality. The analyticity of $f(z)$ assumed at the outset is identical to the requirement of the causality.

In the mid-1950s, these dispersion relations were derived from quantum field theory and applied to strong interaction physics, where the requirement of the causality and the unitarity of the S matrix are mandatory.

The application of the covariant perturbation theory to strong interaction physics was hopeless due to the large coupling constant, despite the fact that the pseudoscalar meson theory is renormalizable by power counting.

For some time, the dispersion theoretic approach to strong interaction physics was the only realistic approach which provided many sum rules. To cite a few, we have the Goldberger–Treiman relation, the Goldberger–Miyazawa–Oehme formula, and the Adler–Weisberger sum rule.

In the dispersion theoretic approach to strong interaction physics, the experimentally observed data were directly used in the sum rules.

The situation changed dramatically in the early 1970s when the quantum field theory of strong interaction physics (quantum chromodynamics, *QCD* in short) was invented with the use of the asymptotically free non-Abelian gauge field theory.

We now present two examples of the integral equation in the dispersion theory in quantum mechanics.

❑ **Example 6.2.** Solve

$$F(x) = \frac{1}{\pi} \int \frac{\left| F(x') \right|^2 h(x')}{x' - x - i\varepsilon} dx', \tag{6.5.9}$$

where $h(x)$ is a given real function.

Solution. We set

$$f(z) = \frac{1}{\pi} \int \frac{g(x')}{x' - z} dx' \quad \text{with} \quad g(x) = \left|F(x)\right|^2 h(x). \tag{6.5.10}$$

Since $g(x)$ is real, it is the imaginary part of $F(x)$,

$$\operatorname{Im} F(x) = \left|F(x)\right|^2 h(x). \tag{6.5.11}$$

Assuming that $f(z)$ never vanishes anywhere, we set

$$\phi(z) = \frac{1}{f(z)}. \tag{6.5.12}$$

We compute the discontinuity

$$\phi(x + i\varepsilon) - \phi(x - i\varepsilon) = 2i \operatorname{Im} \phi(x + i\varepsilon) = -2i \frac{\operatorname{Im} f(x + i\varepsilon)}{\left|f(x + i\varepsilon)\right|^2} = -2ih(x). \tag{6.5.13}$$

Thus we have

$$\phi(z) = -\frac{1}{\pi} \int \frac{h(x')}{x' - z} dx', \tag{6.5.14}$$

and

$$F(x) = f(x + i\varepsilon) = \frac{1}{\phi(x + i\varepsilon)} = -\left[\frac{1}{\pi} \int \frac{h(x')}{x' - x - i\varepsilon} dx'\right]^{-1}, \tag{6.5.15}$$

provided that $\phi(z)$ never vanishes anywhere.

This example originates from the unitarity of the elastic scattering amplitude in the potential scattering problem in quantum mechanics. The unitarity of the elastic scattering amplitude often requires $F(x)$ to be of the form

$$F(x) = \frac{\exp[i\delta(x)] \sin \delta(x)}{h(x)} \quad \text{with} \quad \delta(x) \text{ and } h(x) \quad \text{real}, \tag{6.5.16}$$

where

$$\operatorname{Im} F(x) = \left|F(x)\right|^2 h(x). \tag{6.5.17}$$

In the three-dimensional scattering problem in quantum mechanics, we can obtain the dispersion relations for the scattering amplitude in the forward direction. We shall now address this problem for the spherically symmetric potential in three dimensions.

❑ **Example 6.3.** Potential scattering from the spherically symmetric potential in three dimensions in quantum mechanics.
The Schrödinger equation for the spherically symmetric potential in three dimensions is given by

$$\vec{\nabla}^2\psi(\vec{r}) + k^2\psi(\vec{r}) - U(\vec{r})\psi(\vec{r}) = 0, \tag{6.5.18}$$

where $U(\vec{r})$ is the potential $V(\vec{r})$ multiplied by $2m/\hbar^2$. Using Green's function $\exp[ikR]/4\pi R$ for the Helmholtz equation, we obtain

$$\psi(\vec{r}) = \exp[i\vec{k}_0\vec{r}] - \frac{1}{4\pi}\int \frac{\exp[ik\left|\vec{r}-\vec{r}_1\right|]}{\left|\vec{r}-\vec{r}_1\right|}U(\vec{r}_1)\psi(\vec{r}_1)d^3\vec{r}_1, \tag{6.5.19}$$

where $\exp[i\vec{k}_0\vec{r}]$ is the incident wave with $\vec{k}_0$ along the z-axis in the spherical polar coordinate. In the region with $|\vec{r}| \gg 1$, the scattering amplitude $f(\theta, k)$ is obtained as

$$f(\theta, k) = -\frac{1}{4\pi}\int \exp[-i\vec{k}_s\vec{r}_1]U(\vec{r}_1)\psi(\vec{r}_1)d^3\vec{r}_1, \tag{6.5.20}$$

where θ is the polar angle between $\vec{r}$ and the z-axis, and $\vec{k}_s$ is in the direction of $\vec{r}$ with magnitude $|\vec{k}_s| = k$.

Solving Eq. (6.5.18) for $\psi(\vec{r})$ iteratively for the case of the weak potential, we have

$$f(\theta, k) = \sum f_n(\theta, k),$$

where

$$f_n(\theta, k) = (\frac{-1}{4\pi})^n \int\cdots\int d^3\vec{r}_1\cdots d^3\vec{r}_n \frac{U(\vec{r}_1)U(\vec{r}_1)\cdots U(\vec{r}_1)}{\left|\vec{r}-\vec{r}_1\right|\left|\vec{r}_1-\vec{r}_2\right|\cdots\left|\vec{r}_{n-1}-\vec{r}_n\right|}$$

$$\times \exp[-i\vec{k}_s\vec{r}_1 + ik\left|\vec{r}_1-\vec{r}_2\right| + \cdots + ik\left|\vec{r}_{n-1}-\vec{r}_n\right| + i\vec{k}_0\vec{r}_n], \tag{6.5.21}$$

which has the k-dependence only in the exponent. In the case of the forward scattering ($\theta = 0$), we have $\vec{k}_s = \vec{k}_0$. The exponent of Eq. (6.5.21) becomes

$$i\vec{k}_0(\vec{r}_n - \vec{r}_1) + ik\left|\vec{r}_1-\vec{r}_2\right| + ik\left|\vec{r}_2-\vec{r}_3\right| + \cdots + ik\left|\vec{r}_{n-1}-\vec{r}_n\right|.$$

Since the sum $|\vec{r}_1 - \vec{r}_2| + \left|\vec{r}_2-\vec{r}_3\right| + \cdots + \left|\vec{r}_{n-1}-\vec{r}_n\right|$ is greater than $\left|\vec{r}_n-\vec{r}_1\right|$, we have the exponent of Eq. (6.5.21) as

$$ik \times (\text{positive number}).$$

Thus $f_n(0,k)$ is analytic in the upper half plane, $\operatorname{Im} k > 0$. In order to apply the dispersion relation to $f(0,k)$, we must define $f_n(0,k)$ for the real negative k. From Eq. (6.5.21), we have

$$f_n(0,-k) = f_n^*(0,k) \quad \text{with } k \text{ real and positive.} \tag{6.5.22}$$

We note that

$$f_1(0,0) = -\frac{1}{4\pi}\int U(\vec{r}_1)d^3\vec{r}_1 = f_1(0,k)$$

is a real constant so that we can invoke the dispersion relation,

$$\operatorname{Re} F(x) = \frac{2}{\pi}\mathrm{P}\int_0^\infty \frac{x' \operatorname{Im} F(x')}{x'^2 - x^2}dx', \quad \operatorname{Im} F(x) = -\frac{2}{\pi}\mathrm{P}\int_0^\infty \frac{x \operatorname{Re} F(x')}{x'^2 - x^2}dx', \tag{6.5.6}$$

to the present case with minor modification,

$$F(k) = f(0,k) - f_1(0,k) \quad \text{with} \quad k \text{ real and positive.} \tag{6.5.23}$$

Thus we have

$$\operatorname{Re}[f(0,k) - f_1(0,0)] = \frac{2}{\pi}\mathrm{P}\int_0^\infty \frac{k' \operatorname{Im} f(0,k')}{k'^2 - k^2}dk'. \tag{6.5.24}$$

We tacitly assumed the square-integrability of $[f(0,k) - f_1(0,0)]$ in the present discussion. If this assumption fails, we may replace

$$[f(0,k) - f_1(0,0)] \quad \text{with} \quad [f(0,k) - f_1(0,0)]/k^2$$

where the latter is assumed to be bounded at $k = 0$ and square-integrable. We obtain the dispersion relation,

$$\operatorname{Re}[f(0,k) - f_1(0,0)] = \frac{2k^2}{\pi}\mathrm{P}\int_0^\infty \frac{\operatorname{Im} f(0,k')}{k'(k'^2 - k^2)}dk'. \tag{6.5.25}$$

We shall now replace the incident plane wave in Eq. (6.5.19) with the incident spherical wave originating from the point on the negative z-axis, $\vec{r} = \vec{r}_0$,

$$\psi_0(\vec{r}) = \frac{r_0}{|\vec{r} - \vec{r}_0|}\exp[ik|\vec{r} - \vec{r}_0| - ikr_0],$$

and consider the scattered wave. The exponent in Eq. (6.5.21) gets replaced with

$$-i\vec{k}_s\vec{r}_1 + ik|\vec{r}_1 - \vec{r}_2| + \cdots + ik|\vec{r}_{n-1} - \vec{r}_n| + ik|\vec{r}_n - \vec{r}_0| - ikr_0.$$

In the case of the forward scattering, we have $\vec{k}_s = \vec{k}_0$ and $\vec{k}_s$ points in the positive z-direction. Since we have $kr_0 = -\vec{k}_0\vec{r}_0$, the exponent written out above becomes

$$i\vec{k}_0(\vec{r}_0 - \vec{r}_1) + ik\{|\vec{r}_1 - \vec{r}_2| + \cdots + |\vec{r}_{n-1} - \vec{r}_n| + |\vec{r}_n - \vec{r}_0|\},$$

so that we have the exponent as

$$ik \times (\text{positive number}).$$

Since we have $\vec{r}_0$ on the z-axis, we have

$$\frac{\exp[ik|\vec{r} - \vec{r}_0|]}{|\vec{r} - \vec{r}_0|} = ik\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)j_l(kr)h_l^{(1)}(kr_0), \quad r_0 > r. \tag{6.5.26}$$

The total wave must assume the form

$$ik\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)h_l^{(1)}(kr_0)r_0\exp[-ikr_0](j_l(kr) + i\exp[i\delta_l]\sin\delta_l h_l^{(1)}(kr)),$$

and the corresponding forward scattering amplitude $f'(0,k)$ becomes

$$\begin{aligned} f'(0,k) &= \sum_{l=0}^{\infty}(2l+1)h_l^{(1)}(kr_0)r_0\exp[-ikr_0]\cdot i\exp[i\delta_l]\sin\delta_l\cdot i^{-l} \\ &= \frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)(\exp[i\delta_l]-1)q_l(kr_0). \end{aligned} \tag{6.5.27}$$

Here

$$q_l(kr_0) = kr_0\cdot h_l^{(1)}(kr_0)\cdot i^{-l+1}\exp[-ikr_0] = \sum_{m=0}^{l}\frac{i^m(l+m)!}{m!(l-m)!}\frac{1}{(2kr_0)^m}. \tag{6.5.28}$$

We note that $f'(0,k)$ is a function of r_0. Since r_0 is arbitrary, we can obtain many relations from the dispersion relation for $f'(0,k)$ by equating the same power of r_0. Picking up the $1/r_0^m$ term, we obtain

$$f_m(k) = \frac{1}{2ik^{2m+1}}\sum_{l=m}^{\infty}(2l+1)\frac{(l+m)!}{(l-m)!}(\exp[i\delta_l]-1). \tag{6.5.29}$$

Using the identity

$$\sum_{m=l}^{l'}\frac{(-1)^{m-l}(l'+m)!}{(m-l)!(m+l+1)!(l'-m)!} = \begin{cases} 1/(2l+1), & \text{for} \quad l' = l, \\ 0, & \text{for} \quad l' > l, \end{cases}$$

we invert Eq. (6.5.29) for $\exp[i\delta_l] - 1$, obtaining

$$\frac{\exp[i\delta_l] - 1}{2i} = \sum_{m=l}^{\infty} \frac{(-1)^{m-l} k^{2m+1}}{(m-l)!(m+l+1)!} f_m(k).$$

The dispersion relation for the phase shift δ_l becomes

$$\begin{aligned} \operatorname{Im}\exp[2i\delta_l] &= \sum_{m=l}^{\infty} \frac{(-1)^{m-l} k^{2m+1}}{(m-l)!(m+l+1)!} \\ &\times \left[\frac{2}{\pi} \mathrm{P} \int \frac{\sum_{l'=m}^{\infty} [(2l'+1)(l'+m)!/(l'-m)!] \operatorname{Re}(1 - \exp[i\delta_{l'}(k')])}{k'^2 - k^2} dk' + C_m \right], \end{aligned} \tag{6.5.30}$$

where

$$C_m = \frac{1}{2} \frac{d^m}{dk^m} \operatorname{Im}\exp[2i\delta_m(k)] \bigg|_{k=0} .$$

We refer the reader to the following book for the mathematical details of the dispersion relations in classical electrodynamics, quantum mechanics and relativistic quantum field theory.

Goldberger, M.L. and Watson, K.M.: *Collision Theory*, John Wiley & Sons, New York, (1964). Chapter 10 and Appendix G.2.

6.6
Problems for Chapter 6

6.1. Solve

$$\frac{1}{\pi i} \mathrm{P} \int_{-1}^{+1} \frac{1}{y - x} \phi(y) dy = 0, \quad -1 \leq x \leq 1.$$

6.2. (due to H. C.). Solve

$$\phi(x) = \frac{\tan x}{\pi} \mathrm{P} \int_0^1 \frac{1}{y - x} \phi(y) dy + f(x), \quad 0 \leq x \leq 1.$$

6.3. (due to H. C.). Solve

$$\phi(x) = \frac{\tan(x^3)}{\pi} \mathrm{P} \int_0^1 \frac{1}{y - x} \phi(y) dy + f(x), \quad 0 \leq x \leq 1.$$

6.4. (due to H. C.). Solve

$$\int_0^x \frac{1}{\sqrt{x-y}}\phi(y)dy + A\int_x^1 \frac{1}{\sqrt{y-x}}\phi(y)dy = 1,$$

where

$$0 \le x \le 1.$$

6.5. (due to H. C.). Solve

$$\lambda \mathrm{P}\int_0^{+\infty} \frac{1}{y-x}\phi(y)dy = \phi(x), \quad 0 \le x < \infty.$$

Find the eigenvalues and the corresponding eigenfunctions.

6.6. (due to H. C.). Solve

$$\mathrm{P}\int_{-1}^{+1} \frac{1+\alpha(y-x)}{y-x}\phi(y)dy = 0, \quad -1 \le x \le 1.$$

6.7. (due to H. C.). Obtain the eigenvalues and the eigenfunctions of

$$\phi(x) = \frac{\lambda}{\pi}\mathrm{P}\int_{-1}^{+1} \frac{1}{y-x}\phi(y)dy, \quad -1 \le x \le 1.$$

6.8. (due to H. C.). Solve

$$\sqrt{x}\psi(x) = \frac{1}{\pi}\mathrm{P}\int_0^{+\infty} \frac{1}{y-x}\psi(y)dy, \quad 0 \le x < \infty.$$

6.9. (due to H. C.). Solve

$$\frac{1}{\pi}\mathrm{P}\int_{-\infty}^{+\infty} \frac{1}{y-x}\phi(y)dy = f(x), \quad -\infty < x < \infty.$$

6.10. (due to H. C.). Solve

$$\frac{1}{\pi}\mathrm{P}\int_0^{+\infty} \frac{1}{y-x}\phi(y)dy = f(x), \quad 0 \le x < \infty.$$

6.11. (due to H. C.). Solve

$$\int_0^1 \ln\left|x-y\right|\phi(y)dy = 1, \quad 0 \le x \le 1.$$

6.12. Solve

$$\frac{\Gamma(x)}{B(x)} + \frac{1}{\pi} \mathrm{P} \int_{-a}^{+a} \frac{\Gamma'(y)}{x-y} dy = f(x)$$

with $\Gamma(x)$, $B(x)$, and $f(x)$ even and

$$\Gamma(-a) = \Gamma(a) = 0.$$

Hint: The unknown $\Gamma(x)$ is the circulation of the thin wing in the Prandtl's thin wing theory.

Kondo, J. : *Integral Equations*, Kodansha Ltd., Tokyo, (1991), p. 412.

6.13. (due to H. C.). Prove that

$$\mathrm{P} \int_0^{\infty} \frac{dy}{\sqrt{y}(y-x)} = 0 \quad \text{for} \quad x > 0.$$

6.14. (due to H. C.). Prove that

$$\mathrm{P} \int_0^1 \frac{dy}{\sqrt{y(1-y)}(y-x)} = 0, \quad 0 < x < 1.$$

6.15. (due to H. C.). Prove that

$$\frac{1}{\pi} \mathrm{P} \int_0^1 \frac{\sqrt{y(1-y)}}{y-x} dy = \frac{1}{2} - x, \quad 0 < x < 1.$$

6.16. (due to H. C.). Prove that

$$\frac{1}{\pi} \mathrm{P} \int_0^1 \frac{\ln y}{\sqrt{y(1-y)}(y-x)} dy = \frac{\pi}{\sqrt{x(1-x)}} - \int_1^{\infty} \frac{1}{\sqrt{y(y-1)}(y-x)} dy$$

$$0 \le x \le 1.$$

6.17. Solve the integral equation of the Cauchy type,

$$F(x) = \frac{1}{\pi} \int \frac{F(x')h^*(x')}{x'-x-i\varepsilon} dx',$$

where $h(x)$ is given by

$$h(x) = \exp[i\delta(x)] \sin \delta(x),$$

and $\delta(x)$ is a given real function.

Hint: Consider $f(z)$ defined by

$$f(z) = \frac{1}{\pi} \int \frac{F(x')h^*(x')}{x' - z} dx',$$

and compute the discontinuity across the real x-axis. Also use the identity

$$1 - 2ih^*(x) = \exp[-2i\delta(x)].$$

The function $h(x)$ originates from the phase shift analysis of the potential scattering problem in quantum mechanics.

7
Wiener–Hopf Method and Wiener–Hopf Integral Equation

7.1
The Wiener–Hopf Method for Partial Differential Equations

In Sections 6.3, 6.4, and 6.5, we reduced the singular integral equations of Cauchy type and their variants to the inhomogeneous Hilbert problem through the introduction of the function $\Phi(z)$ appropriately defined.

Suppose now we are given one linear equation involving two unknown functions, $\phi_-(k)$ and $\psi_+(k)$, in the complex k plane,

$$\phi_-(k) = \psi_+(k) + F(k), \tag{7.1.1}$$

where $\phi_-(k)$ is analytic in the lower half plane ($\operatorname{Im} k < \tau_-$) and $\psi_+(k)$ is analytic in the upper half plane ($\operatorname{Im} k \geq \tau_+$). Can we solve Eq. (7.1.1) for $\phi_-(k)$ and $\psi_+(k)$? As long as $\phi_-(k)$ and $\psi_+(k)$ have a common region of analyticity as in Figure 7.1, namely

$$\tau_+ \leq \tau_-, \tag{7.1.2}$$

we can solve for $\phi_-(k)$ and $\psi_+(k)$. In the most stringent case, the common region of analyticity can be an arc below which (excluding the arc) $\phi_-(k)$ is analytic and above which (including the arc) $\psi_+(k)$ is analytic.

We proceed to split $F(k)$ into a sum of two functions, one analytic in the upper half plane and the other analytic in the lower half plane,

$$F(k) = F_+(k) + F_-(k). \tag{7.1.3}$$

This *sum splitting* can be carried out either by inspection or by the general method utilizing the Cauchy integral formula to be discussed in Section 7.3. Once the sum splitting is accomplished, we write Eq. (7.1.1) in the following form:

$$\phi_-(k) - F_-(k) = \psi_+(k) + F_+(k) \equiv G(k). \tag{7.1.4}$$

We immediately note that $G(k)$ is *entire* in k. If the asymptotic behaviors of $F_\pm(k)$ as $|k| \to \infty$ are such that

$$F_\pm(k) \to 0 \quad \text{as} \quad |k| \to \infty, \tag{7.1.5}$$

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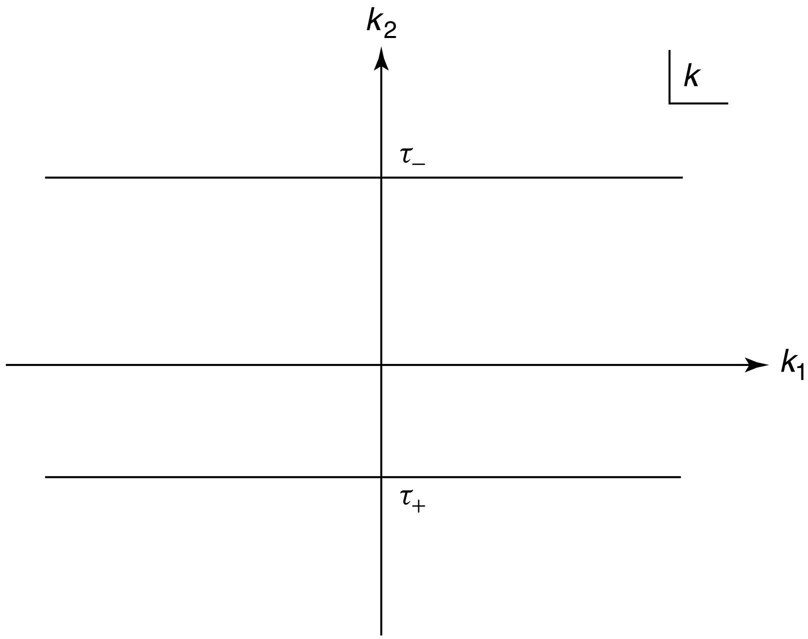

Fig. 7.1 The common region of analyticity for $\phi_-(k)$ and $\psi_+(k)$ in the complex k plane. $\psi_+(k)$ and $F_+(k)$ are analytic in the upper half plane, $\text{Im}\, k \geq \tau_+$. $\phi_-(k)$ and $F_-(k)$ are analytic in the lower half plane, $\text{Im}\, k < \tau_-$. The common region of analyticity for $\phi_-(k)$ and $\psi_+(k)$ is inside the strip, $\tau_+ \leq \text{Im}\, k < \tau_-$ in the complex k plane.

and on some physical ground,

$$\phi_-(k), \psi_+(k) \to 0 \quad \text{as} \quad |k| \to \infty, \tag{7.1.6}$$

then the entire function $G(k)$ must vanish by Liouville's theorem. Thus we obtain

$$\phi_-(k) = F_-(k), \tag{7.1.7a}$$

$$\psi_+(k) = -F_+(k). \tag{7.1.7b}$$

We call this method the *Wiener–Hopf method*.

In the following examples, we apply this method to the *mixed boundary value problem* of the partial differential equation.

❑ **Example 7.1.** Find the solution to the Laplace equation in the half-plane.

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi(x, y) = 0, \quad y \geq 0, \quad -\infty < x < \infty, \tag{7.1.8}$$

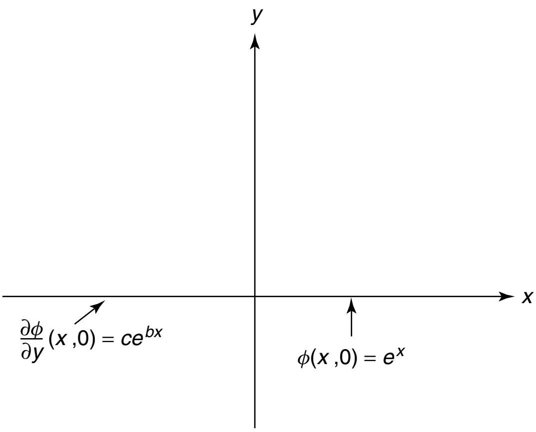

Fig. 7.2 Boundary conditions of $\phi(x, y)$ for the Laplace equation (7.1.8) in the half plane, $y \geq 0$ and $-\infty < x < \infty$.

subject to the boundary condition on $y = 0$, as displayed in Figure 7.2,

$$\phi(x, 0) = e^{-x}, \quad x > 0, \tag{7.1.9a}$$

$$\phi_y(x, 0) = ce^{bx}, \quad b > 0, \quad x < 0. \tag{7.1.9b}$$

We further assume

$$\phi(x, y) \to 0 \quad \text{as} \quad x^2 + y^2 \to \infty. \tag{7.1.9c}$$

Solution. Since $-\infty < x < \infty$, we may take the *Fourier transform* with respect to x, i.e.,

$$\hat{\phi}(k, y) \equiv \int_{-\infty}^{+\infty} dx e^{-ikx} \phi(x, y), \quad \phi(x, y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk e^{ikx} \hat{\phi}(k, y). \tag{7.1.10}$$

So, if we can obtain $\hat{\phi}(k, y)$, we will be done. Taking the Fourier transform of the partial differential equation (7.1.8) with respect to x, we have

$$\left(-k^2 + \frac{\partial^2}{\partial y^2}\right) \hat{\phi}(k, y) = 0,$$

i.e.,

$$\frac{\partial^2}{\partial y2} \hat{\phi}(k, y) = k^2 \hat{\phi}(k, y), \tag{7.1.11}$$

from which we obtain

$$\hat{\phi}(k, y) = C_1(k) e^{ky} + C_2(k) e^{-ky}.$$

The boundary condition at infinity, $\hat{\phi}(k, y) \to 0$ as $y \to +\infty$, implies

$$\begin{cases} C_1(k) = 0 & \text{for} \quad k > 0, \\ C_2(k) = 0 & \text{for} \quad k < 0. \end{cases}$$

We can write more generally

$$\hat{\phi}(k, y) = A(k)e^{-|k|y}. \tag{7.1.12}$$

To obtain $A(k)$, we need to apply the boundary conditions at $y = 0$.

Take the Fourier transform of $\phi(x, 0)$ to find

$$\begin{aligned} \hat{\phi}(k, 0) &\equiv \int_{-\infty}^{+\infty} dx e^{-ikx}\phi(x, 0) \\ &= \int_{-\infty}^{0} dx e^{-ikx}\phi(x, 0) + \int_{0}^{+\infty} dx e^{-ikx} e^{-x}. \end{aligned} \tag{7.1.13}$$

Here the first term is unknown for $x < 0$, while the second term is equal to $1/(1 + ik)$. Recall that

$$\left|e^{-ikx}\right| = \left|e^{-i(k_1+ik_2)x}\right| = e^{k_2 x} \quad \text{with} \quad k = k_1 + ik_2,$$

which vanishes in the upper half plane ($k_2 > 0$) for $x < 0$. So, define

$$\phi_+(k) \equiv \int_{-\infty}^{0} dx e^{-ikx}\phi(x, 0), \tag{7.1.14}$$

which is a + function. (Assuming that $\phi(x, 0) \sim O(e^{bx})$ as $x \to -\infty$, $\phi_+(k)$ is analytic for $k_2 > -b$.) So, we have from Eqs. (7.1.12) and (7.1.13)

$$\hat{\phi}(k, 0) = \phi_+(k) + \frac{1}{1 + ik} \quad \text{or} \quad A(k) = \phi_+(k) + \frac{1}{1 + ik}. \tag{7.1.15}$$

Also, take the Fourier transform of $\partial\phi(x, 0)/\partial y$ to find

$$\begin{aligned} \frac{\partial\hat{\phi}}{\partial y}(k, 0) &\equiv \int_{-\infty}^{+\infty} dx e^{-ikx} \frac{\partial\phi(x, 0)}{\partial y} \\ &= \int_{-\infty}^{0} dx e^{-ikx} c e^{bx} + \int_{0}^{+\infty} dx e^{-ikx} \frac{\partial\phi(x, 0)}{\partial y}, \end{aligned} \tag{7.1.16}$$

where the first term is equal to $c/(b - ik)$, and the second term is unknown for $x > 0$. So, define

$$\psi_-(k) \equiv \int_{0}^{+\infty} dx e^{-ikx} \frac{\partial\phi(x, 0)}{\partial y}, \tag{7.1.17}$$

which is a $-$ function. Assuming that $\partial\phi(x,0)/\partial y \sim O(e^{-x})$ as $x \to \infty$, $\psi_-(k)$ is analytic for $k_2 < 1$. Thus we obtain

$$-\left|k\right| A(k) = \frac{c}{b - ik} + \psi_-(k),$$

or we have

$$A(k) = \frac{-c}{\left|k\right| (b - ik)} - \frac{\psi_-(k)}{\left|k\right|}. \tag{7.1.18}$$

Equating the two expressions (7.1.15) and (7.1.18) for $A(k)$ (assuming that they are both valid in some common region, say $k_2 = 0$), we get

$$\phi_+(k) + \frac{1}{1 + ik} = \frac{-c}{\left|k\right| (b - ik)} - \frac{\psi_-(k)}{\left|k\right|}. \tag{7.1.19}$$

Now, the function $\left|k\right|$ is not analytic and so we cannot proceed with the Wiener–Hopf method unless we express $\left|k\right|$ in a suitable form. One such form, often suitable for application, is

$$\left|k\right| = \lim_{\varepsilon \to 0^+} (k^2 + \varepsilon^2)^{1/2} = \lim_{\varepsilon \to 0^+} (k + i\varepsilon)^{1/2}(k - i\varepsilon)^{1/2}, \tag{7.1.20}$$

where, in the last expression, $(k + i\varepsilon)^{1/2}$ is a $+$ function and $(k - i\varepsilon)^{1/2}$ is a $-$ function, as displayed in Figure 7.3.

We can verify that on the real axis,

$$\sqrt{k^2 + \varepsilon^2} = \left|k\right| \quad \text{as} \quad \varepsilon \to 0^+. \tag{7.1.21}$$

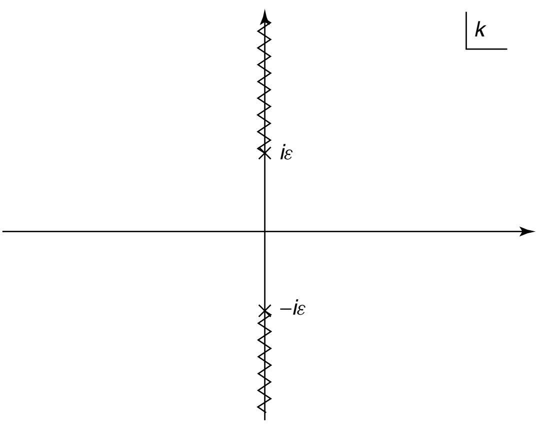

Fig. 7.3 Factorization of $\left|k\right|$ into a $+$ function and a $-$ function. The points, $k = \pm i\varepsilon$, are the branch points of $\sqrt{k^2 + \varepsilon^2}$, from which the branch cuts are extended to $\pm i\infty$ along the imaginary k-axis in the complex k plane.

We thus have

$$\phi_+(k) - \frac{i}{k-i} = \frac{-ci}{(k+ib)(k+i\varepsilon)^{1/2}(k-i\varepsilon)^{1/2}} - \frac{\psi_-(k)}{(k+i\varepsilon)^{1/2}(k-i\varepsilon)^{1/2}}. \tag{7.1.22}$$

Since $(k+i\varepsilon)^{1/2}$ is a + function (i.e., is analytic in the upper half plane), we multiply the whole Eq. (7.1.22) by $(k+i\varepsilon)^{1/2}$ to get

$$(k+i\varepsilon)^{1/2}\phi_+(k) - \frac{i(k+i\varepsilon)^{1/2}}{k-i} = \frac{-ci}{(k+ib)(k-i\varepsilon)^{1/2}} - \frac{\psi_-(k)}{(k-i\varepsilon)^{1/2}}, \tag{7.1.23}$$

where the first term on the left-hand side is a + function and the second term on the right-hand side is a − function. We shall decompose the remaining terms into a sum of the + function and the − function. Consider the second term of the left-hand side of Eq. (7.1.23). The numerator is a + function but the denominator is a − function. We rewrite

$$-\frac{i(k+i\varepsilon)^{1/2}}{k-i} = \frac{-i(k+i\varepsilon)^{1/2} + i(i+i\varepsilon)^{1/2}}{k-i} - \frac{i(i+i\varepsilon)^{1/2}}{k-i},$$

where in the first term on the right-hand side, we removed the singularity at $k = i$ by making the numerator vanish at $k = i$ so that it is a + function. The second term on the right-hand side has a pole at $k = i$ so that it is a − function. Similarly, the first term on the right-hand side of Eq. (7.1.23) is rewritten in the following form:

$$\frac{-ci}{(k+ib)(k-i\varepsilon)^{1/2}} = \frac{-ci}{k+ib}\left[\frac{1}{(k-i\varepsilon)^{1/2}} - \frac{1}{(-ib-i\varepsilon)^{1/2}}\right] - \frac{ci}{(k+ib)(-ib-i\varepsilon)^{1/2}},$$

where the first term on the right-hand side is no longer singular at $k = -ib$, but has a branch point at $k = i\varepsilon$, so it is a − function. The second term on the right-hand side has a pole at $k = -ib$, so it is a + function.

Collecting all this, we have the following equation:

$$\begin{aligned}(k+i\varepsilon)^{1/2}\phi_+(k) &+ \frac{-i(k+i\varepsilon)^{1/2} + i(i+i\varepsilon)^{1/2}}{k-i} + \frac{ci}{(k+ib)(-ib-i\varepsilon)^{1/2}} \\ &= -\frac{\psi_-(k)}{(k-i\varepsilon)^{1/2}} - \frac{ci}{(k+ib)}\left[\frac{1}{(k-i\varepsilon)^{1/2}} - \frac{1}{(-ib-i\varepsilon)^{1/2}}\right] \\ &\quad + \frac{i(i+i\varepsilon)^{1/2}}{k-i}.\end{aligned} \tag{7.1.24}$$

Here each term on the left-hand side is a + function while each term on the right-hand side is a − function. Applying the Wiener–Hopf method, and noting that the left-hand side and the right-hand side both go to 0 as $|k| \to \infty$, we have

$$\phi_+(k) = \frac{i}{k-i}\left[1 - \frac{(i+i\varepsilon)^{1/2}}{(k+i\varepsilon)^{1/2}}\right] - \frac{ci}{(k+ib)(-ib-i\varepsilon)^{1/2}(k+i\varepsilon)^{1/2}}.$$

We can simplify a bit:

$$(i+i\varepsilon)^{1/2} = (1+\varepsilon)^{1/2}e^{i\pi/4} = e^{i\pi/4} \quad \text{as} \quad \varepsilon \to 0^+.$$

Similarly,

$$(-ib-i\varepsilon)^{1/2} = \sqrt{b+\varepsilon}e^{-i\pi/4} = \sqrt{b}e^{-i\pi/4} \quad \text{as} \quad \varepsilon \to 0^+.$$

Hence we have

$$\phi_+(k) = \frac{i}{k-i}\left[1 - \frac{e^{i\pi/4}}{(k+i\varepsilon)^{1/2}}\right] - \frac{(c/\sqrt{b})ie^{i\pi/4}}{(k+ib)(k+i\varepsilon)^{1/2}}. \tag{7.1.25}$$

So finally, we obtain $A(k)$ from Eqs. (7.1.15) and (7.1.25) as

$$A(k) = \frac{-ie^{i\pi/4}}{(k+i\varepsilon)^{1/2}}\left[\frac{c}{\sqrt{b}(k+ib)} + \frac{1}{k-i}\right], \tag{7.1.26}$$

where we note

$$(k+i\varepsilon)^{1/2} = \begin{cases} \sqrt{k} & \text{for} \quad k>0, \\ \sqrt{|k|}e^{i\pi/2} = i\sqrt{|k|} & \text{for} \quad k<0. \end{cases}$$

As such,

$$\hat{\phi}(k,y) = A(k)e^{-|k|y}$$

can be inverted with respect to k to obtain $\phi(x,y)$.

In this example, we carried out the sum splitting by inspection (or by brute force). We now discuss another example.

❑ **Example 7.2.** Sommerfeld diffraction problem.
Solve the wave equation in two space dimension:

$$\frac{\partial^2}{\partial t^2}u(x,y,t) = c^2\nabla^2 u(x,y,t) \tag{7.1.27}$$

with the boundary condition

$$\frac{\partial}{\partial y}u(x,y,t) = 0 \quad \text{at} \quad y=0 \quad \text{for} \quad x<0. \tag{7.1.28}$$

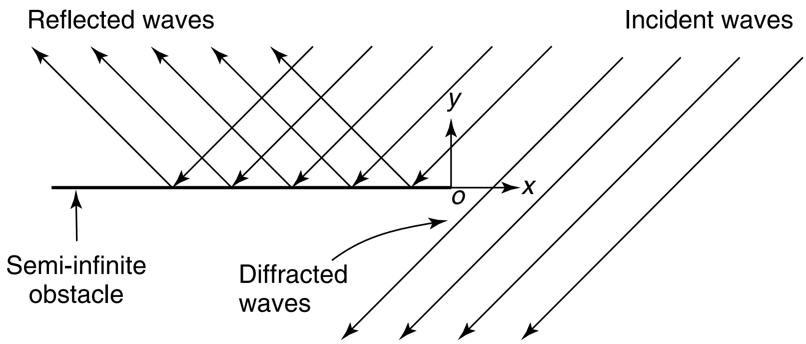

Fig. 7.4 Incident wave, reflected wave, and diffracted wave in the *Sommerfeld diffraction problem*.

The incident, reflected, and diffracted waves are drawn in Figure 7.4.

Solution. We look for the solution of the form

$$u(x, y, t) = \phi(x, y)e^{-i\omega t}. \tag{7.1.29}$$

Setting

$$p \equiv \omega/c, \tag{7.1.30}$$

the wave equation assumes the following form:

$$\nabla^2\phi + p^2\phi = 0. \tag{7.1.31}$$

Letting the forcing increase exponentially in time (so it is absent as $t \to -\infty$), we must have $\mathrm{Im}\,\omega > 0$, which requires

$$p = p_1 + i\varepsilon, \quad \varepsilon \to 0^+. \tag{7.1.32}$$

So, $\phi(x, y)$ satisfies

$$\begin{cases} \nabla^2\phi + p^2\phi = 0, \\ \partial\phi/\partial y = 0 \qquad \text{at} \quad y = 0 \quad \text{for} \quad x < 0. \end{cases} \tag{7.1.33}$$

Consider the incident waves,

$$u_{\text{inc}} = e^{i(\vec{p}\cdot\vec{x}-\omega t)} = \phi_{\text{inc}}e^{-i\omega t} \quad \text{with} \quad |\vec{p}| = p,$$

so that u_{inc} also satisfies the wave equation. The u_{inc} is a plane wave moving in the direction of $\vec{p}$. We take

$$\vec{p} = -p\cos\theta \cdot \vec{e}_x - p\sin\theta \cdot \vec{e}_y,$$

and assume $0 < \theta < \pi/2$. Then $\phi_{\mathrm{inc}}(x, y)$ is given by

$$\phi_{\mathrm{inc}}(x, y) = e^{-ip(x\cos\theta + y\sin\theta)}. \tag{7.1.34}$$

We seek a *disturbance solution*

$$\psi(x, y) \equiv \phi(x, y) - \phi_{\mathrm{inc}}(x, y). \tag{7.1.35}$$

Thus the governing equation and the boundary condition for $\psi(x, y)$ become

$$\nabla^2\psi + p^2\psi = 0, \tag{7.1.36}$$

subject to

$$\frac{\partial\psi(x,0)}{\partial y} = -\frac{\partial\phi_{\mathrm{inc}}(x,0)}{\partial y} = ip\sin\theta\cdot e^{-ip(x\cos\theta)} \quad \text{for} \quad x < 0 \quad \text{at} \quad y = 0. \tag{7.1.37}$$

Asymptotic behavior: We replace p by $p_1 + i\varepsilon$,

$$p = p_1 + i\varepsilon. \tag{7.1.38}$$

In the *reflection region,* we have

$$\psi(x, y) \sim e^{-ip(x\cos\theta - y\sin\theta)} \quad \text{for} \quad y > 0, \quad x \to -\infty.$$

We note the change of sign in front of y. Near $y = 0^+$, we have $\psi(x, 0^+) \sim e^{-ipx\cos\theta}$ as $x \to -\infty$, or $\left|\psi(x, 0^+)\right| \sim e^{\varepsilon x\cos\theta}$ as $x \to -\infty$. In the *shadow region* $(y < 0, x \to -\infty)$, $\phi(-\infty, y) = 0 \quad \Rightarrow \quad \psi = -\phi_{\mathrm{inc}}$, so that $\psi(x, 0^-) \sim -e^{-ipx\cos\theta}$, or $\left|\psi(x, 0^-)\right| \sim e^{\varepsilon x\cos\theta}$ as $x \to -\infty$.

In summary,

$$\left|\psi(x, y)\right| \sim e^{\varepsilon x\cos\theta} \quad \text{as} \quad x \to -\infty, \quad y = 0^{\pm}. \tag{7.1.39}$$

Although $\psi(x, y)$ might be discontinuous across the obstacle, its normal derivative is continuous as given by the boundary condition. On the other side, as $x \to +\infty$, both $\psi(x, 0)$ and $\partial\psi(x, 0)/\partial x$ are continuous, but the asymptotic behavior at infinity is obtained by approximating the effect of the leading edge of the obstacle as a delta function,

$$\nabla^2\psi + p^2\psi = -4\pi\delta(x)\delta(y). \tag{7.1.40}$$

From Eq. (7.1.40), we obtain

$$\psi(x, y) = \pi i H_0^{(1)}(pr), \quad r = \sqrt{x^2 + y^2}, \tag{7.1.41}$$

where $H_0^{(1)}(pr)$ is the 0th order *Hankel function of the first kind*. It behaves as

$$H_0^{(1)}(pr) \sim \frac{1}{\sqrt{r}} \exp(ipr) \quad \text{as} \quad r \to \infty,$$

which implies that

$$\left|\psi(x,0)\right| \sim \frac{1}{\sqrt{x}} e^{-\varepsilon x} \quad \text{as} \quad x \to \infty \quad \text{near} \quad y = 0. \tag{7.1.42}$$

Now, we try to solve the equation for $\psi(x, y)$ using the Fourier transforms,

$$\hat{\psi}(k, y) \equiv \int_{-\infty}^{+\infty} dx e^{-ikx} \psi(x, y), \tag{7.1.43a}$$

$$\psi(x, y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk e^{ikx} \hat{\psi}(k, y). \tag{7.1.43b}$$

We take the Fourier transform of the partial differential equation,

$$\nabla^2 \psi + p^2 \psi = 0,$$

resulting in the form

$$\frac{\partial^2}{\partial y^2} \hat{\psi}(k, y) = (k^2 - p^2) \hat{\psi}(k, y). \tag{7.1.44}$$

Consider the branch

$$(k^2 - p^2)^{1/2} = \lim_{\varepsilon \to 0^+} [(k - p_1 - i\varepsilon)(k + p_1 + i\varepsilon)]^{1/2}, \tag{7.1.45}$$

where the branch cuts are drawn in Figure 7.5.
Then we have

$$[k^2 - (p_1 + i\varepsilon)^2]^{1/2} = \sqrt{r_1 r_2} e^{i(\phi_1 + \phi_2)/2},$$

so that

$$\text{Re}[k^2 - (p_1 + i\varepsilon)^2]^{1/2} = \sqrt{r_1 r_2} \cos\left(\frac{\phi_1 + \phi_2}{2}\right).$$

On the real axis above, $-\pi < \phi_1 + \phi_2 < 0$, so that

$$0 < \cos\left(\frac{\phi_1 + \phi_2}{2}\right) < 1.$$

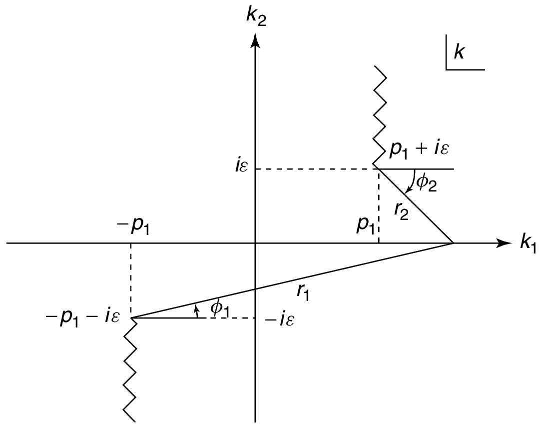

Fig. 7.5 Branch cuts of $(k^2 - p^2)^{1/2}$ in the complex k plane. The points, $k = \pm(p_1 + i\varepsilon)$, are the branch points of $\sqrt{k^2 - p^2}$, from which the branch cuts are extended to $\pm(p_1 + i\infty)$.

Thus the branch chosen has

$$\mathrm{Re}[k^2 - (p_1 + i\varepsilon)^2]^{1/2} > 0 \tag{7.1.46}$$

on the whole real axis. We can then write the solution as

$$\frac{\partial^2}{\partial y^2}\hat{\psi}(k, y) = [k^2 - (p_1 + i\varepsilon)^2]\hat{\psi}(k, y) \tag{7.1.47}$$

as

$$\hat{\psi}(k, y) = \begin{cases} A(k)\exp(-\sqrt{k^2 - (p_1 + i\varepsilon)^2}y), & y > 0, \\ B(k)\exp(+\sqrt{k^2 - (p_1 + i\varepsilon)^2}y), & y < 0. \end{cases}$$

But since $\psi(x, y)$ is not continuous across $y = 0$ for $x < 0$, the amplitudes $A(k)$ and $B(k)$ need not be identical. However, we know that $\partial\psi(x, y)/\partial y$ is continuous across $y = 0$ for all x. We must therefore have

$$\frac{\partial}{\partial y}\hat{\psi}(k, 0^+) = \frac{\partial}{\partial y}\hat{\psi}(k, 0^-),$$

from which we obtain

$$B(k) = -A(k).$$

Thus we have

$$\hat{\psi}(k, y) = \begin{cases} A(k)\exp(-\sqrt{k^2 - (p + i\varepsilon)^2}y), & y > 0, \\ -A(k)\exp(+\sqrt{k^2 - (p + i\varepsilon)^2}y), & y < 0. \end{cases} \tag{7.1.48}$$

If we can determine $A(k)$, we will be done.

Let us recall everything we know. We know that $\psi(x, y)$ is continuous for $x > 0$ when $y = 0$, but is discontinuous for $x < 0$. So, consider the function

$$\psi(x, 0^+) - \psi(x, 0^-) = \begin{cases} 0 & \text{for} \quad x > 0, \\ \text{unknown} & \text{for} \quad x < 0. \end{cases} \tag{7.1.49}$$

But we know the asymptotic form of the latter unknown function to be like $e^{\varepsilon x \cos\theta}$ as $x \to -\infty$ from our earlier discussion. Take the Fourier transform of the above discontinuity (7.1.49) to obtain

$$\hat{\psi}(k, 0^+) - \hat{\psi}(k, 0^-) = \int_{-\infty}^{0} dx e^{-ikx} (\psi(x, 0^+) - \psi(x, 0^-)),$$

the right-hand side of which is a + function, analytic for $k_2 > -\varepsilon \cos\theta$. We define

$$U_+(k) \equiv \hat{\psi}(k, 0^+) - \hat{\psi}(k, 0^-), \quad \text{analytic for} \quad k_2 > -\varepsilon \cos\theta. \tag{7.1.50}$$

Now consider the derivative $\partial\psi(x, y)/\partial y$. This function is continuous at $y = 0$ for all x $(-\infty < x < \infty)$. Furthermore, we know what it is for $x < 0$. Namely,

$$\frac{\partial}{\partial y}\psi(x, 0) = \begin{cases} ip \sin\theta \cdot e^{-ipx\cos\theta} & \text{for} \quad x < 0, \\ \text{unknown} & \text{for} \quad x > 0. \end{cases} \tag{7.1.51}$$

But we know the asymptotic form of the latter unknown function to be like $e^{-\varepsilon x}$ as $x \to \infty$. Taking the Fourier transform of Eq. (7.1.51), we obtain

$$\frac{\partial}{\partial y}\hat{\psi}(k, 0) = \int_{-\infty}^{0} dx e^{-ikx} \cdot ip \sin\theta \cdot^{-ipx\cos\theta} + \int_{0}^{+\infty} dx e^{-ikx} \frac{\partial}{\partial y}\psi(x, 0).$$

Thus we have

$$\frac{\partial}{\partial y}\hat{\psi}(k, 0) = -p \sin\theta/(k + p\cos\theta) + L_-(k), \tag{7.1.52}$$

where in the first term of the right-hand side,

$$p = p_1 + i\varepsilon,$$

and the second term represented as $L_-(k)$ is defined by

$$L_-(k) \equiv \int_{0}^{+\infty} dx e^{-ikx} \frac{\partial}{\partial y}\psi(x, 0).$$

$L_-(k)$ is a $-$ function, analytic for $k_2 < \varepsilon$.

We shall now use Eqs. (7.1.50) and (7.1.52), where $U_+(k)$ and $L_-(k)$ are defined, together with the Wiener–Hopf method to solve for $A(k)$. We know that

$$\hat{\psi}(k, y) = \begin{cases} A(k)\exp(-\sqrt{k^2 - p^2}y) & \text{for} \quad y > 0, \\ -A(k)\exp(\sqrt{k^2 - p^2}y) & \text{for} \quad y < 0. \end{cases}$$

Plugging these equations into Eq. (7.1.50), we have $2A(k) = U_+(k)$. Plugging these equations into Eq. (7.1.52), we have

$$-\sqrt{k^2 - p^2}A(k) = -p\sin\theta/(k + p\cos\theta) + L_-(k).$$

Eliminating $A(k)$ from the last two equations, we obtain the Wiener–Hopf problem,

$$-\sqrt{k^2 - p^2}\,U_+(k)/2 = -p\sin\theta/(k + p\cos\theta) + L_-(k), \tag{7.1.53}$$

$$\begin{cases} L_-(k) & \text{analytic for} \quad k_2 < \varepsilon, \\ U_+(k) & \text{analytic for} \quad k_2 > -\varepsilon\cos\theta. \end{cases} \tag{7.1.54}$$

We divide both sides of Eq. (7.1.53) by $\sqrt{k - p}$ to obtain

$$-\sqrt{k + p}U_+(k)/2 = L_-(k)/\sqrt{k - p} - p\sin\theta/\left[(k + p\cos\theta)\sqrt{k - p}\right].$$

The term involving $U_+(k)$ is a + function, while the term involving $L_-(k)$ is a − function. We decompose the last term on the right-hand side as

$$-\frac{p\sin\theta}{(k + p\cos\theta)\sqrt{k - p}} = -\frac{p\sin\theta}{k + p\cos\theta}\left[\frac{1}{\sqrt{k - p}} - \frac{1}{\sqrt{-p\cos\theta - p}}\right] - \frac{p\sin\theta}{(k + p\cos\theta)\sqrt{-p\cos\theta - p}},$$

where the first term on the right-hand side is a − function and the second term a + function.

Collecting the + functions and the − functions, we obtain by the Wiener–Hopf method

$$U_+(k)/2 = p\sin\theta/[\sqrt{k + p}(k + p\cos\theta)\sqrt{-p\cos\theta - p}], \tag{7.1.55}$$

and hence we finally obtain

$$A(k) = p\sin\theta/\left[(k + p\cos\theta)\sqrt{k + p}\sqrt{-p\cos\theta - p}\right]. \tag{7.1.56}$$

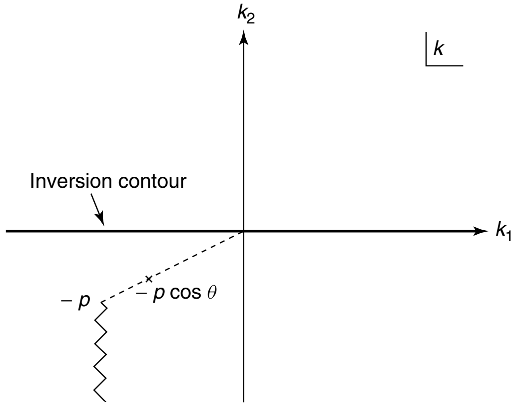

Fig. 7.6 Singularities of $A(k)$ in the complex k plane. $A(k)$ has a simple pole at $k = -p\cos\theta$ and a branch point at $k = -p$, from which the branch cut is extended to $-p - i\infty$.

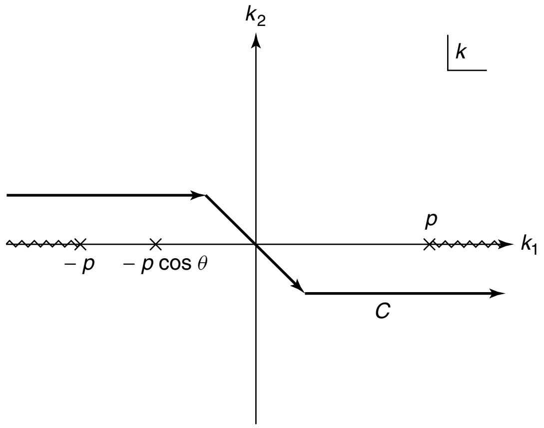

Fig. 7.7 The contour of the complex k integration in Eq. (7.1.57) for $\psi(x, y)$. The integrand has a simple pole at $k = -p\cos\theta$ and the branch points at $k = \pm p$. The branch cuts are extended from $k = \pm p$ to $\pm\infty$ along the real k-axis.

Singularities of $A(k)$ in the complex k plane are drawn in Figure 7.6.

The final solution for the disturbance function $\psi(x, y)$ is given by, in the limit as $\varepsilon \to 0^+$,

$$\psi(x, y) = \frac{\mathrm{sgn}(y)}{2\pi} \int_C dkA(k) \exp(-\sqrt{k^2 - p^2}\,|y| + ikx), \tag{7.1.57}$$

where C is the contour specified in Figure 7.7.

We remark that the choice of

$$p = p_1 + i\varepsilon, \quad \varepsilon > 0,$$

based on the requirement that

$$u_{\text{inc}}(\vec{x}, t) \to 0 \quad \text{as} \quad t \to -\infty,$$

but it grows exponentially in time, i.e.,

$$u_{\text{inc}} = e^{i(\vec{p}\cdot\vec{x} - \omega t)} \quad \operatorname{Im}\omega > 0,$$

is known as *turning on the perturbation adiabatically*.

7.2
Homogeneous Wiener–Hopf Integral Equation of the Second Kind

The Wiener–Hopf integral equations are characterized by *translation kernels*, $K(x, y) = K(x - y)$, and the integral is on the *semi-infinite range*, $0 < x, y < \infty$. We list the Wiener–Hopf integral equations of several types.

Wiener–Hopf integral equation of the first kind:

$$F(x) = \int_0^{+\infty} K(x - y)\phi(y)dy, \quad 0 \le x < \infty.$$

Homogeneous Wiener–Hopf integral equation of the second kind:

$$\phi(x) = \lambda \int_0^{+\infty} K(x - y)\phi(y)dy, \quad 0 \le x < \infty.$$

Inhomogeneous Wiener–Hopf integral equation of the second kind:

$$\phi(x) = f(x) + \lambda \int_0^{+\infty} K(x - y)\phi(y)dy, \quad 0 \le x < \infty.$$

Let us begin with the *homogeneous Wiener–Hopf integral equation of the second kind*:

$$\phi(x) = \lambda \int_0^{+\infty} K(x - y)\phi(y)dy, \quad 0 \le x < \infty. \tag{7.2.1}$$

Here, the translation kernel $K(x - y)$ is defined for its argument both positive and negative. Suppose that

$$K(x) \to \begin{cases} e^{-bx} & \text{as} \quad x \to +\infty, \\ e^{ax} & \text{as} \quad x \to -\infty, \end{cases} \qquad a, b > 0, \tag{7.2.2}$$

so that the Fourier transform of $K(x)$, defined by

$$\hat{K}(k) = \int_{-\infty}^{+\infty} dx e^{-ikx} K(x), \tag{7.2.3}$$

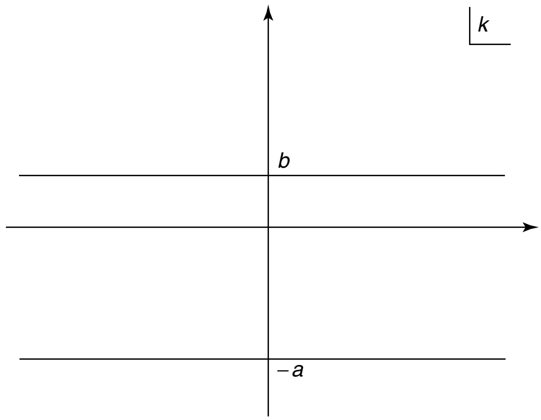

Fig. 7.8 Region of analyticity of $\hat{K}(k)$ in the complex k plane. $\hat{K}(k)$ is defined and analytic inside the strip, $-a < \text{Im}\, k < b$.

is analytic for

$$-a < \text{Im}\, k < b. \tag{7.2.4}$$

The region of analyticity of $\hat{K}(k)$ in the complex k plane is displayed in Figure 7.8. Now define

$$\phi(x) = \begin{cases} \phi(x) & \text{given for} \quad x > 0, \\ 0 & \text{for} \quad x < 0. \end{cases} \tag{7.2.5}$$

But then, although $\phi(x)$ is only known for positive x, since $K(x-y)$ is defined even for negative x, we can certainly define $\psi(x)$ for negative x,

$$\psi(x) \equiv \lambda \int_0^{+\infty} K(x-y)\phi(y)dy \quad \text{for} \quad x < 0. \tag{7.2.6}$$

Take the *Fourier transforms* of Eqs. (7.2.1) and (7.2.6). Adding up the results, and using the *convolution property*, we have

$$\int_0^{+\infty} dxe^{-ikx}\phi(x) + \int_{-\infty}^{0} dxe^{-ikx}\psi(x) = \lambda\hat{K}(k)\hat{\phi}_-(k).$$

Namely, we have

$$\hat{\phi}_-(k) + \hat{\psi}_+(k) = \lambda\hat{K}(k)\hat{\phi}_-(k),$$

or we have

$$\left[1 - \lambda\hat{K}(k)\right]\hat{\phi}_-(k) = -\hat{\psi}_+(k), \tag{7.2.7}$$

where we define

$$\hat{\phi}_-(k) \equiv \int_0^{+\infty} dx e^{-ikx}\phi(x), \quad \hat{\psi}_+(k) \equiv \int_{-\infty}^{0} dx e^{-ikx}\psi(x). \tag{7.2.8,9}$$

Since we have

$$\left|e^{-ikx}\right| = e^{k_2 x}, \quad k = k_1 + ik_2,$$

we know that $\hat{\phi}_-(k)$ is analytic in the lower half plane and $\hat{\psi}_+(k)$ is analytic in the upper half plane. Thus, once again, we have one equation involving two unknown functions, $\hat{\phi}_-(k)$ and $\hat{\psi}_+(k)$, one analytic in the lower half plane and the other analytic in the upper half plane. The precise regions of analyticity for $\hat{\phi}_-(k)$ and $\hat{\psi}_+(k)$ are each determined by the asymptotic behavior of the kernel $K(x)$ as $x \to -\infty$.

In the original equation, Eq. (7.2.1), at the upper limit of the integral, we have $y \to \infty$ so that $x - y \to -\infty$ as $y \to \infty$. By Eq. (7.2.2), we have

$$K(x-y) \sim e^{a(x-y)} \sim e^{-ay} \quad \text{as} \quad y \to \infty.$$

To ensure that the integral in Eq. (7.2.1) converges, we conclude that $\phi(x)$ can grow as fast as

$$\phi(x) \sim e^{(a-\varepsilon)x} \quad \text{with} \quad \varepsilon > 0 \quad \text{as} \quad x \to \infty.$$

By definition of $\hat{\phi}_-(k)$, the region of the analyticity of $\hat{\phi}_-(k)$ is determined by the requirement that

$$\left|e^{-ikx}\phi(x)\right| \sim e^{(k_2+a-\varepsilon)x} \to 0 \quad \text{as} \quad x \to \infty.$$

Thus $\hat{\phi}_-(k)$ is analytic in the lower half plane, $\operatorname{Im} k = k_2 < -a + \varepsilon$, $\varepsilon > 0$, which includes

$$\operatorname{Im} k \leq -a.$$

As for the behavior of $\psi(x)$ as $x \to -\infty$, we observe that $x - y \to -\infty$ as $x \to -\infty$, and

$$K(x-y) \sim e^{a(x-y)} \quad \text{as} \quad x \to -\infty.$$

By definition of $\psi(x)$, we have

$$\psi(x) = \lambda \int_0^{+\infty} K(x-y)\phi(y)dy \sim \lambda e^{ax} \int_0^{+\infty} e^{-ay}\phi(y)dy \quad \text{as} \quad x \to -\infty,$$

where the integral is convergent due to the asymptotic behavior of $\phi(x)$ as $x \to \infty$. The region of analyticity of $\hat{\psi}_+(k)$ is determined by the requirement that

$$\left|e^{-ikx}\psi(x)\right| \sim e^{(k_2+a)x} \to 0 \quad \text{as} \quad x \to -\infty.$$

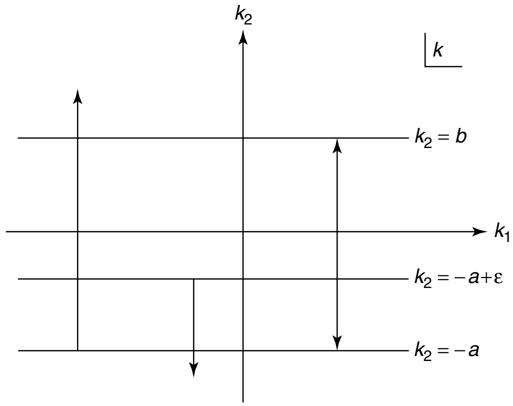

Fig. 7.9 Region of analyticity of $\hat{\phi}_-(k)$, $\hat{\psi}_+(k)$, and $\hat{K}(k)$. $\hat{\phi}_-(k)$ is analytic in the lower half plane, $\operatorname{Im} k < -a + \varepsilon$. $\hat{\psi}_+(k)$ is analytic in the upper half plane, $\operatorname{Im} k > -a$. $\hat{K}(k)$ is analytic inside the strip, $-a < \operatorname{Im} k < b$.

Thus $\hat{\psi}_+(k)$ is analytic in the upper half plane,

$$\operatorname{Im} k = k_2 > -a.$$

To summarize, we know

$$\begin{cases} \phi(x) \to e^{(a-\varepsilon)x} & \text{as} \quad x \to \infty, \\ \psi(x) \to e^{ax} & \text{as} \quad x \to -\infty. \end{cases} \tag{7.2.10}$$

Hence we have

$$\hat{\phi}_-(k) = \int_0^{+\infty} dx e^{-ikx} \phi(x) \quad \text{analytic for} \quad \operatorname{Im} k = k_2 < -a + \varepsilon, \tag{7.2.11a}$$

$$\hat{\psi}_+(k) = \int_{-\infty}^{0} dx e^{-ikx} \psi(x) \quad \text{analytic for} \quad \operatorname{Im} k = k_2 > -a. \tag{7.2.11b}$$

Various regions of the analyticity are drawn in Figure 7.9.

Recalling Eq. (7.2.7), we write $1 - \lambda \hat{K}(k)$ as the ratio of the $-$ function and the $+$ function,

$$1 - \lambda \hat{K}(k) = Y_-(k)/Y_+(k). \tag{7.2.12}$$

From Eqs. (7.2.7) and (7.2.12), we have

$$Y_-(k)\hat{\phi}_-(k) = -Y_+(k)\hat{\psi}_+(k) \equiv F(k), \tag{7.2.13}$$

where $F(k)$ is an entire function. The asymptotic behavior of $F(k)$ as $|k| \to \infty$ will determine $F(k)$ completely.

We know that

$$\begin{aligned} &\hat{\phi}_-(k) \to 0 \quad \text{as} \quad |k| \to \infty, \\ &\hat{\psi}_+(k) \to 0 \quad \text{as} \quad |k| \to \infty, \\ &\hat{K}(k) \to 0 \quad \text{as} \quad |k| \to \infty. \end{aligned} \tag{7.2.14}$$

By Eq. (7.2.12), we know then

$$Y_-(k)/Y_+(k) \to 1, \quad \text{as} \quad |k| \to \infty. \tag{7.2.15}$$

We only need to know the asymptotic behavior of $Y_-(k)$ or $Y_+(k)$ as $|k| \to \infty$ in order to determine the entire function $F(k)$. Once $F(k)$ is determined, we have from Eq. (7.2.13)

$$\hat{\phi}_-(k) = F(k)/Y_-(k) \tag{7.2.16}$$

and we are done.

We remark that it is convenient to choose a function $Y_-(k)$ which is not only analytic in the lower half plane but also has no zeros in the lower half plane, so that $F(k)/Y_-(k)$ is itself a $-$ function for all entire $F(k)$. Otherwise, we need to choose $F(k)$ so as to have zeros exactly at zeros of $Y_-(k)$ to cancel the possible poles in $F(k)/Y_-(k)$ and yield the $-$ function $\hat{\phi}_-(k)$.

The *factorization of* $1 - \lambda\hat{K}(k)$ is essential in solving the Wiener–Hopf integral equation of the second kind. As noted earlier, it can be done either by inspection or by the general method based on the Cauchy integral formula. As a general rule, we assign

$$\begin{aligned} &\textit{Any pole in the lower half plane } (k = p_l) && \text{to} \quad Y_+(k), \\ &\textit{Any zero in the lower half plane } (k = z_l) && \text{to} \quad Y_+(k), \\ &\textit{Any pole in the upper half plane } (k = p_u) && \text{to} \quad Y_-(k), \\ &\textit{Any zero in the upper half plane } (k = z_u) && \text{to} \quad Y_-(k). \end{aligned} \tag{7.2.17}$$

We first solve the following simple example where the factorization is carried out by inspection and illustrate the rational of this general rule for the assignment.

❑ **Example 7.3.** Solve

$$\phi(x) = \lambda \int_0^{+\infty} e^{-|x-y|}\phi(y)dy, \quad x \geq 0. \tag{7.2.18}$$

Solution. Define

$$\psi(x) = \lambda \int_0^{+\infty} e^{-|x-y|}\phi(y)dy, \quad x < 0. \tag{7.2.19}$$

Also, define

$$\phi(x) = \begin{cases} \phi(x) & \text{for} \quad x \geq 0, \\ 0 & \text{for} \quad x < 0. \end{cases}$$

Take the Fourier transform of Eqs. (7.2.18) and (7.2.19) and add the results together to obtain

$$\hat{\phi}_-(k) + \hat{\psi}_+(k) = \lambda \cdot \frac{2}{k^2+1} \cdot \hat{\phi}_-(k). \tag{7.2.20}$$

Now,

$$K(x) = e^{-|x|} \to \begin{cases} e^{-x} & \text{as} \quad x \to \infty, \\ e^{x} & \text{as} \quad x \to -\infty. \end{cases} \tag{7.2.21}$$

Therefore, $\phi(y)$ can be allowed to grow as fast as $e^{(1-\varepsilon)y}$ as $y \to \infty$. Thus $\hat{\phi}_-(k)$ is analytic for $k_2 < -1 + \varepsilon$. We also find that $\psi(x) \to e^x$ as $x \to -\infty$. Thus $\hat{\psi}_+(k)$ is analytic for $k_2 > -1$.

To solve Eq. (7.2.20), we first write

$$\frac{k^2+1-2\lambda}{k^2+1}\hat{\phi}_-(k) = -\hat{\psi}_+(k), \tag{7.2.22}$$

and then decompose

$$\frac{k^2+1-2\lambda}{k^2+1} \tag{7.2.23}$$

into a ratio of a − function to a + function. The common region of analyticity of $\hat{\phi}_-(k)$, $\hat{\psi}_+(k)$, and $\hat{K}(k)$ of this example is drawn in Figure 7.10.

The designations, the lower half plane, and the upper half plane, are to be made relative to a line with

$$-1 < \operatorname{Im} k = k_2 < -1 + \varepsilon. \tag{7.2.24}$$

Referring to Eq. (7.2.23),

$$k^2 + 1 = (k+i)(k-i)$$

so that $k = i$ is a pole of Eq. (7.2.23) in the upper half plane and $k = -i$ is a pole of Eq. (7.2.23) in the lower half plane. Now look at the numerator of Eq. (7.2.23),

$$k^2 + 1 - 2\lambda.$$

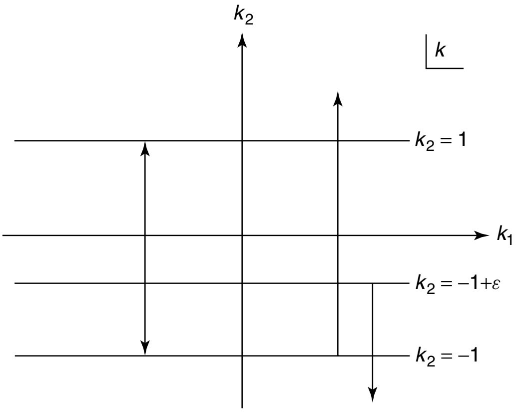

Fig. 7.10 Common region of analyticity of $\hat{\phi}_-(k)$, $\hat{\psi}_+(k)$, and $\hat{K}(k)$ of Example 7.3. $\hat{\phi}_-(k)$ is analytic in the lower half plane, $\mathrm{Im}\, k < -1 + \varepsilon$. $\hat{\psi}_+(k)$ is analytic in the upper half plane, $\mathrm{Im}\, k > -1$. $\hat{K}(k)$ is analytic inside the strip, $-1 < \mathrm{Im}\, k < 1$.

Case 1. $\lambda < 0. \Rightarrow 1 - 2\lambda > 1.$

$$k^2 + 1 - 2\lambda = (k + i\sqrt{1-2\lambda})(k - i\sqrt{1-2\lambda}). \tag{7.2.25}$$

The first factor corresponds to a zero in the lower half plane, while the second factor corresponds to a zero in the upper half plane.

Case 2. $0 < \lambda < 1/2. \Rightarrow 0 < 1 - 2\lambda < 1.$

$$k^2 + 1 - 2\lambda = (k + i\sqrt{1-2\lambda})(k - i\sqrt{1-2\lambda}). \tag{7.2.26}$$

Both factors correspond to a zero in the upper half plane.

Case 3. $\lambda > 1/2. \Rightarrow 1 - 2\lambda < 0.$

$$k^2 + 1 - 2\lambda = (k + \sqrt{2\lambda-1})(k - \sqrt{2\lambda-1}). \tag{7.2.27}$$

Both factors correspond to a zero in the upper half plane.

Now, in general, when we write

$$1 - \lambda\hat{K}(k) = Y_-(k)/Y_+(k),$$

since we will end up with

$$Y_-(k)\hat{\phi}_-(k) = -Y_+(k)\hat{\psi}_+(k) \equiv G(k),$$

which is entire, we wish to have

$$\hat{\phi}_-(k) = G(k)/Y_-(k)$$

analytic in the lower half plane. So, $Y_-(k)$ must not have any zeros in the lower half plane. Hence assign any zeros or poles in the lower half plane to $Y_+(k)$ so that $Y_-(k)$ has neither poles nor zeros in the lower half plane.

Presently we have

$$\begin{aligned} 1-\lambda\hat{K}(k) &= (k^2+1-2\lambda)/(k^2+1) \\ &= (k+i\sqrt{1-2\lambda})(k-i\sqrt{1-2\lambda})/(k+i)(k-i). \end{aligned}$$

Case 1. $\lambda < 0$.

$$\begin{array}{llll} k+i\sqrt{1-2\lambda} & \Rightarrow & \text{zero in the lower half plane} & \Rightarrow \quad Y_+(k) \\ k-i\sqrt{1-2\lambda} & \Rightarrow & \text{zero in the upper half plane} & \Rightarrow \quad Y_-(k) \\ k+i & \Rightarrow & \text{pole in the lower half plane} & \Rightarrow \quad Y_+(k) \\ k-i & \Rightarrow & \text{pole in the upper half plane} & \Rightarrow \quad Y_-(k) \end{array}$$

Thus we obtain

$$\left\{ \begin{array}{lll} Y_-(k) & = & (k-i\sqrt{1-2\lambda})/(k-i) \\ Y_+(k) & = & (k+i)/(k+i\sqrt{1-2\lambda}) \end{array} \right. . \tag{7.2.28}$$

Hence, in the equation

$$Y_-(k)\hat{\phi}_-(k) = -Y_+(k)\hat{\psi}_+(k) = G(k),$$

we know $Y_-(k) \to 1$, $\hat{\phi}_-(k) \to 0$, as $k \to \infty$, so that $G(k) \to 0$ as $k \to \infty$. By Liouville's theorem , we conclude that $G(k) = 0$, from which it follows that

$$\hat{\phi}_-(k) = 0, \quad \hat{\psi}_+(k) = 0. \tag{7.2.29}$$

So, there exists no nontrivial solution, i.e.,

$$\phi(x) = 0 \quad \text{for} \quad \lambda < 0.$$

Case 2. $0 < \lambda < 1/2$.
With a similar analysis as in Case 1, we obtain

$$\left\{ \begin{array}{lll} Y_-(k) & = & (k+i\sqrt{1-2\lambda})(k-i\sqrt{1-2\lambda})/(k-i), \\ Y_+(k) & = & (k+i). \end{array} \right. \tag{7.2.30}$$

Noting that

$$Y_-(k) \to k \quad \text{as} \quad k \to \infty,$$

and

$$Y_-(k)\hat{\phi}_-(k) = G(k),$$

we find that $G(k)$ grows less fast than k as $k \to \infty$. By Liouville's theorem, we find

$$G(k) = A, \quad \text{constant},$$

and thus conclude that

$$\hat{\phi}_-(k) = A(k-i)/(k+i\sqrt{1-2\lambda})(k-i\sqrt{1-2\lambda}). \tag{7.2.31}$$

Case 3. $\lambda > 1/2$.
With a similar analysis as in Case 1, we obtain

$$\begin{cases} Y_-(k) = (k+\sqrt{2\lambda-1})(k-\sqrt{2\lambda-1})/(k-i) & \to k \quad \text{as} \quad k \to \infty, \\ Y_+(k) = (k+i) & \to k \quad \text{as} \quad k \to \infty. \end{cases} \tag{7.2.32}$$

Again, we find that

$$G(k) = A, \quad \text{constant},$$

and thus conclude that

$$\hat{\phi}_-(k) = A(k-i)/(k+\sqrt{2\lambda-1})(k-\sqrt{2\lambda-1}). \tag{7.2.33}$$

To summarize, we find the following:

$$\lambda \le 0 \quad \Rightarrow \quad \phi(x) = 0. \tag{7.2.34}$$

$$\lambda > 0 \quad \Rightarrow \quad \phi(x) = \frac{1}{2\pi}\int_C dk e^{ikx} A(k-i)/(k^2+1-2\lambda), \tag{7.2.35}$$

where the inversion contour C is indicated in Figure 7.11.
For $x > 0$, we shall close the contour in the upper half plane and get the contribution from both poles in the upper half plane in either Case 2 or Case 3. The result of inversions are the following:

Case 2. $0 < \lambda < 1/2$.

$$\phi(x) = C\left(\cosh\sqrt{1-2\lambda}x + \frac{\sinh\sqrt{1-2\lambda}x}{\sqrt{1-2\lambda}}\right), \quad x > 0. \tag{7.2.36a}$$

Case 3. $\lambda > 1/2$.

$$\phi(x) = C\left(\cos\sqrt{2\lambda-1}x + \frac{\sin\sqrt{2\lambda-1}x}{\sqrt{2\lambda-1}}\right), \quad x > 0. \tag{7.2.36b}$$

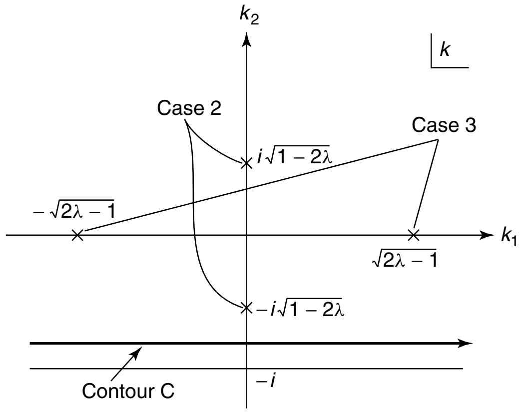

Fig. 7.11 The inversion contour for $\phi(x)$ of Eq. (7.2.35). Simple poles are located at $k = \pm i\sqrt{1-2\lambda}$ for Case 2 and at $k = \pm\sqrt{2\lambda - 1}$ for Case 3.

We shall now consider another example where the factorization also is carried out by inspection after some juggling of the gamma functions.

❑ **Example 7.4.** Solve

$$\phi(x) = \lambda \int_0^{+\infty} \frac{1}{\cosh[\frac{1}{2}(x-y)]}\phi(y)dy, \quad x \geq 0. \tag{7.2.37}$$

Solution. We begin with the Fourier transform of the kernel $K(x)$:

$$K(x) = \frac{1}{\cosh(\frac{1}{2}x)} \to 2e^{-\frac{1}{2}|x|}, \quad \text{as} \quad |x| \to \infty.$$

Then $\hat{K}(k)$ is analytic inside the strip, $-\frac{1}{2} < \operatorname{Im} k = k_2 < \frac{1}{2}$. We calculate $\hat{K}(k)$ as follows:

$$\hat{K}(k) = \int_{-\infty}^{+\infty} dx e^{-ikx} \frac{1}{\cosh(\frac{1}{2}x)} = 2\int_{-\infty}^{+\infty} dx e^{-ikx} e^{\frac{1}{2}x}/(e^x + 1).$$

Setting $e^x = t$, $x = \ln t$, $dx = dt/t$, we have

$$\hat{K}(k) = 2\int_0^{+\infty} dt[t^{-ik-\frac{1}{2}}]/(t+1).$$

Further change of variable $\rho = 1/(t+1)$, $t = (1-\rho)/\rho$, $dt = -d\rho/\rho^2$ results in

$$\hat{K}(k) = 2\int_0^1 d\rho \rho^{ik-\frac{1}{2}}(1-\rho)^{-ik-\frac{1}{2}}.$$

Recalling the definition of the *beta function* $B(n, m)$,

$$B(n,m) = \int_0^1 d\rho \rho^{n-1}(1-\rho)^{m-1} = \Gamma(n)\Gamma(m)/\Gamma(n+m),$$

we have

$$\begin{aligned}\hat{K}(k) &= 2\Gamma\left(ik+\frac{1}{2}\right)\Gamma\left(-ik+\frac{1}{2}\right)/\Gamma\left(ik+\frac{1}{2}-ik+\frac{1}{2}\right)\\ &= 2\Gamma\left(ik+\frac{1}{2}\right)\Gamma\left(-ik+\frac{1}{2}\right).\end{aligned}$$

Recalling the property of the gamma function,

$$\Gamma(z)\Gamma(1-z) = \pi/\sin\pi z, \qquad (7.2.39)$$

we thus obtain the Fourier transform of $K(x)$ as

$$\hat{K}(k) = 2\pi/\cosh\pi k. \qquad (7.2.40)$$

Defining $\psi(x)$ by

$$\psi(x) = \lambda\int_0^{+\infty}\frac{1}{\cosh\left[\frac{1}{2}(x-y)\right]}\phi(y)dy, \quad x<0, \qquad (7.2.41)$$

we obtain the following equation as usual:

$$(1-\lambda\hat{K}(k))\hat{\phi}_-(k) = -\hat{\psi}_+(k), \qquad (7.2.42)$$

where

$$1-\lambda\hat{K}(k) = 1-\frac{2\pi\lambda}{\cosh\pi k} = Y_-(k)/Y_+(k), \qquad (7.2.43)$$

and the regions of the analyticity of $\hat{\phi}_-(k)$ and $\hat{\psi}_+(k)$ are such that

$$\begin{aligned}&(i) \quad \hat{\phi}_-(k) \text{ is analytic in the lower half plane } (\operatorname{Im} k \le -1/2),\\ &(ii) \quad \hat{\psi}_+(k) \text{ is analytic in the upper half plane } (\operatorname{Im} k > -1/2).\end{aligned} \qquad (7.2.44)$$

Rewriting Eq. (7.2.42) in terms of $Y_\pm(k)$'s, we have

$$Y_-(k)\hat{\phi}_-(k) = -Y_+(k)\hat{\psi}_+(k) \equiv G(k), \qquad (7.2.45)$$

where $G(k)$ is entire in k.

Factorization of $1 - \lambda \hat{K}(k)$:

$$Y_-(k)/Y_+(k) = 1 - \frac{2\pi\lambda}{\cosh \pi k} = \frac{\cosh \pi k - 2\pi\lambda}{\cosh \pi k}. \tag{7.2.46}$$

Case 1. $0 < 2\pi\lambda \leq 1$.
Setting

$$\cos \pi\alpha \equiv 2\pi\lambda, \quad 0 \leq \alpha < 1/2, \tag{7.2.47}$$

we have

$$\begin{aligned} Y_-(k)/Y_+(k) &= [\cos(i\pi k) - \cos \pi\alpha]/\sin \pi\left(ik + \frac{1}{2}\right) \\ &= 2\sin[\frac{\pi}{2}(\alpha + ik)]\sin\left[\frac{\pi}{2}(\alpha - ik)\right]/\sin \pi\left(ik + \frac{1}{2}\right). \end{aligned} \tag{7.2.48}$$

Replacing all sine functions in Eq. (7.2.48) with the appropriate product of the gamma functions through the use of the formula

$$\sin \pi z = \pi/\Gamma(z)\Gamma(1-z), \tag{7.2.49}$$

we obtain

$$\begin{aligned} &Y_-(k)/Y_+(k) \\ &= 2\pi\Gamma\left(\frac{1}{2} + ik\right)\Gamma\left(\frac{1}{2} - ik\right)/\Gamma\left(\frac{\alpha + ik}{2}\right)\Gamma\left(\frac{\alpha - ik}{2}\right) \\ &\quad \Gamma\left(1 - \frac{\alpha + ik}{2}\right)\Gamma\left(1 - \frac{\alpha - ik}{2}\right). \end{aligned} \tag{7.2.50}$$

We note that

$$\begin{cases} \Gamma(z) \text{ has simple pole} & \text{at} \quad z = 0, -1, -2, \ldots, \\ \Gamma(1-z) \text{ has simple pole} & \text{at} \quad z = 1, 2, 3, \ldots. \end{cases} \tag{7.2.51}$$

(1) $\Gamma\left(\frac{1}{2} + ik\right)$ has simple poles at $k = i\frac{1}{2}, i\frac{3}{2}, i\frac{5}{2}, i\frac{7}{2}, \ldots$, all of which are assigned to $Y_-(k)$.
(2) $\Gamma\left(\frac{1}{2} - ik\right)$ has simple poles at $k = -i\frac{1}{2}, -i\frac{3}{2}, -i\frac{5}{2}, -i\frac{7}{2}, \ldots$, all of which are assigned to $Y_+(k)$.
(3) $\Gamma\left(\frac{\alpha+ik}{2}\right)$ has simple poles at $k = i\alpha, i(2+\alpha), i(4+\alpha), \ldots$, all of which are assigned to $Y_-(k)$.
(4) $\Gamma\left(\frac{\alpha-ik}{2}\right)$ has simple poles at $k = -i\alpha, -i(2+\alpha), -i(4+\alpha), \ldots$. Since $0 < \alpha < 1/2$, the first pole at $k = -i\alpha$ is assigned to $Y_-(k)$, while the remaining poles are assigned to $Y_+(k)$. Using the property of the gamma function, $\Gamma(z) = \Gamma(z+1)/z$, we rewrite

$$\Gamma\left(\frac{\alpha - ik}{2}\right) = \frac{2}{\alpha - ik}\Gamma\left(1 + \frac{\alpha - ik}{2}\right),$$

(5) where $(\alpha - ik)/2$ is assigned to $Y_-(k)$ while $\Gamma(1 + \frac{\alpha - ik}{2})$ is assigned to $Y_+(k)$.

(6) $\Gamma\left(1 - \frac{\alpha+ik}{2}\right)$ has simple poles at $k = -i(2-\alpha), -i(4-\alpha), -i(6-\alpha), \ldots$, all of which are assigned to $Y_+(k)$.

(7) $\Gamma\left(1 - \frac{\alpha-ik}{2}\right)$ has simple poles at $k = i(2-\alpha), i(4-\alpha), i(6-\alpha), \ldots$, all of which are assigned to $Y_-(k)$.

Then we obtain $Y_\pm(k)$ as follows:

$$Y_-(k) = -2\pi\Gamma\left(\frac{1}{2} + ik\right) \Big/ \Gamma\left(\frac{\alpha + ik}{2}\right)\Gamma\left(-\frac{\alpha - ik}{2}\right), \tag{7.2.52a}$$

$$Y_+(k) = \Gamma\left(1 + \frac{\alpha - ik}{2}\right)\Gamma\left(1 - \frac{\alpha + ik}{2}\right) \Big/ \Gamma\left(\frac{1}{2} - ik\right). \tag{7.2.52b}$$

Now $G(k)$ is determined by the asymptotic behavior of $Y_\pm(k)$ as $k \to \infty$. Making use of the *Duplication formula* and the *Stirling formula*,

$$\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma\left(z + \frac{1}{2}\right)/\sqrt{\pi}, \tag{7.2.53}$$

$$\lim_{|z|\to\infty} \Gamma(z+\beta)/\Gamma(z) \sim z^{\beta}, \tag{7.2.54}$$

we find the asymptotic behavior of $Y_-(k)$ to be given by

$$Y_-(k) \sim -i\sqrt{\frac{\pi}{2}}2^{ik} \cdot k. \tag{7.2.55}$$

Defining

$$Z_\pm(k) \equiv 2^{-ik} \cdot Y_\pm(k), \tag{7.2.56}$$

we find

$$Z_\pm(k) \sim -i\sqrt{\frac{\pi}{2}}k, \tag{7.2.57}$$

since

$$Z_-(k)/Z_+(k) = Y_-(k)/Y_+(k) = 1 - \lambda\hat{K}(k) \to 1 \quad \text{as} \quad k \to \infty.$$

Then Eq. (7.2.45) becomes

$$Z_-(k)\hat{\phi}_-(k) = -Z_+(k)\hat{\psi}_+(k) = 2^{-ik}G(k) \equiv g(k), \tag{7.2.58}$$

where $g(k)$ is now entire. Since

$$\hat{\phi}_-(k), \hat{\psi}_+(k) \to 0 \quad \text{as} \quad k \to \infty,$$

and Eq. (7.2.57) for $Z_\pm(k)$, $g(k)$ cannot grow as fast as k. By Liouville's theorem, we then have

$$g(k) = C', \quad \text{constant.}$$

Thus we obtain

$$\hat{\phi}_-(k) = C'/Z_-(k) = C''2^{ik}\Gamma\left(\frac{\alpha+ik}{2}\right)\Gamma\left(-\frac{\alpha-ik}{2}\right)/\Gamma\left(\frac{1}{2}+ik\right). \quad (7.2.59)$$

We now invert $\hat{\phi}_-(k)$ to obtain $\phi(x)$,

$$\phi(x) = C''\int_{-\infty}^{+\infty}\frac{dk}{2\pi}e^{ikx}2^{ik}\Gamma\left(\frac{\alpha+ik}{2}\right)\Gamma\left(-\frac{\alpha-ik}{2}\right)/\Gamma\left(\frac{1}{2}+ik\right), \quad x \geq 0, \quad (7.2.60)$$

$$\phi(x) = 0, \quad x < 0.$$

For $x > 0$, we close the contour in the upper half plane, picking up the pole contributions from $\Gamma\left(\frac{\alpha+ik}{2}\right)$ and $\Gamma\left(-\frac{\alpha-ik}{2}\right)$. Poles of $\Gamma\left(\frac{\alpha+ik}{2}\right)$ are located at $k = i(2n+\alpha), n = 0, 1, 2, \ldots$. Poles of $\Gamma\left(-\frac{\alpha-ik}{2}\right)$ are located at $k = i(2n-\alpha), \quad n = 0, 1, 2, \ldots$. Since

$$\text{Res.}\Gamma(z)\,|_{z=-n} = (-1)^n\frac{1}{n!}, \quad (7.2.61)$$

we have

$$\phi(x) = C''\sum_{n=0}^{\infty}\left\{e^{-(2n+\alpha)x}2^{-(2n+\alpha)}\frac{(-1)^n}{n!}\frac{\Gamma(-n-\alpha)}{\Gamma\left(\frac{1}{2}-2n-\alpha\right)} + (\alpha \to -\alpha)\right\}. \quad (7.2.62)$$

Since

$$\Gamma(z) = \frac{\pi}{\sin\pi z}\frac{1}{\Gamma(1-z)},$$

we have

$$\Gamma(-n-\alpha) = \frac{(-1)^{n+1}\pi}{\sin\alpha\pi}\frac{1}{\Gamma(n+1+\alpha)},$$

and with the use of the duplication formula,

$$\Gamma\left(\frac{1}{2}-2n-\alpha\right) = \frac{\pi}{\cos\alpha\pi}\cdot\sqrt{2\pi}2^{-(2n+\alpha)}\cdot\frac{1}{\Gamma(\frac{1}{4}+\frac{\alpha}{2}+n)\Gamma\left(\frac{3}{4}+\frac{\alpha}{2}+n\right)},$$

we have

$$\frac{\Gamma(-n-\alpha)}{\Gamma\left(\frac{1}{2}-2n-\alpha\right)} = \frac{(-1)^{n+1}}{\sqrt{2\pi}} \cdot 2^{(2n+\alpha)} \cdot \frac{\cos\alpha\pi}{\sin\alpha\pi} \cdot \frac{\Gamma\left(\frac{1}{4}+\frac{\alpha}{2}+n\right)\Gamma\left(\frac{3}{4}+\frac{\alpha}{2}+n\right)}{\Gamma(1+\alpha+n)}.$$

Our solution $\phi(x)$ is given by

$$\begin{aligned}\phi(x) \\ = C''' \left(\frac{\cos\alpha\pi}{\sin\alpha\pi}\right) \cdot \sum_{n=0}^{\infty} \left\{ e^{-\alpha x} \frac{(e^{-2x})^n}{n!} \frac{\Gamma(\frac{1}{4}+\frac{\alpha}{2}+n)\Gamma\left(\frac{3}{4}+\frac{\alpha}{2}+n\right)}{\Gamma \times (1+\alpha+n)} - (\alpha \to -\alpha) \right\}.\end{aligned} \tag{7.2.63}$$

Recall that the *hypergeometric function* $F(a, b, c; z)$ is given by

$$F(a,b,c;z) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)}{\Gamma(a)} \cdot \frac{\Gamma(b+n)}{\Gamma(b)} \cdot \frac{\Gamma(c)}{\Gamma(c+n)} \cdot \frac{z^n}{n!}. \tag{7.2.64}$$

Setting

$$a = \frac{1}{4}+\frac{\alpha}{2}, \quad b = \frac{3}{4}+\frac{\alpha}{2}, \quad c = 1+\alpha,$$

we have

$$\begin{aligned}\phi(x) = C''' \left(\frac{\cos\alpha\pi}{\sin\alpha\pi}\right) \Bigg[& e^{-\alpha x} \frac{\Gamma(\frac{1}{4}+\frac{\alpha}{2})\Gamma(\frac{3}{4}+\frac{\alpha}{2})}{\Gamma(1+\alpha)} \\ & \times F\left(\frac{1}{4}+\frac{\alpha}{2}, \frac{3}{4}+\frac{\alpha}{2}, 1+\alpha; e^{-2x}\right) \\ & -(\alpha \to -\alpha) \Bigg].\end{aligned} \tag{7.2.65}$$

Recall that the *Legendre function of the second kind*, $Q_{\alpha-\frac{1}{2}}(z)$ is given by

$$\begin{aligned}Q_{\alpha-\frac{1}{2}}(z) &= \frac{\sqrt{\pi}}{2^{\alpha+\frac{1}{2}}} \cdot \frac{\Gamma(\frac{1}{2}+\alpha)}{\Gamma(1+\alpha)} \cdot z^{-(\frac{1}{2}+\alpha)} \cdot F\left(\frac{1}{4}+\frac{\alpha}{2}, \frac{3}{4}+\alpha, 1+\alpha; z^{-2}\right) \\ &= \frac{1}{2} \cdot \frac{\Gamma(\frac{1}{4}+\frac{\alpha}{2})\Gamma(\frac{3}{4}+\frac{\alpha}{2})}{\Gamma(1+\alpha)} \cdot z^{-(\frac{1}{2}+\alpha)} \cdot F\left(\frac{1}{4}+\frac{\alpha}{2}, \frac{3}{4}+\frac{\alpha}{2}, 1+\alpha; z^{-2}\right),\end{aligned} \tag{7.2.66}$$

so our solution given above can be simplified as

$$\phi(x) = C''' \left(\frac{\cos\alpha\pi}{\sin\alpha\pi}\right) e^{\frac{x}{2}} \left\{ Q_{\alpha-\frac{1}{2}}(e^x) - Q_{-\alpha-\frac{1}{2}}(e^x) \right\}. \tag{7.2.67}$$

Recall that the *Legendre function of the first kind*, $P_\beta(z)$, is given by

$$P_\beta(z) = \frac{1}{\pi}\left(\frac{\sin\beta\pi}{\cos\beta\pi}\right)\left\{Q_\beta(z) - Q_{-\beta-1}(z)\right\}, \tag{7.2.68}$$

so

$$P_{\alpha-\frac{1}{2}}(z) = -\frac{1}{\pi}\left(\frac{\cos\alpha\pi}{\sin\alpha\pi}\right)\left\{Q_{\alpha-\frac{1}{2}}(z) - Q_{-\alpha-\frac{1}{2}}(z)\right\}. \tag{7.2.69}$$

So, the final expression for $\phi(x)$ is given by

$$\phi(x) = C\cdot\left(\exp\frac{x}{2}\right)\cdot P_{\alpha-\frac{1}{2}}\left(e^x\right), \quad x \geq 0, \quad 0 \leq \alpha < 1/2, \quad 2\pi\lambda = \cos\alpha\pi. \tag{7.2.70}$$

Similar analysis can be carried out for the cases $2\pi\lambda > 1$ and $\lambda < 0$. We only list the final answers for all cases.

Summary of Example 7.4:

Case 1: $0 < 2\pi\lambda \leq 1, \quad 2\pi\lambda = \cos\alpha\pi, \quad 0 \leq \alpha < 1/2.$

$$\phi(x) = C_1\cdot(\exp\frac{x}{2})\cdot P_{\alpha-\frac{1}{2}}(e^x), \quad x \geq 0.$$

Case 2: $2\pi\lambda > 1, \quad 2\pi\lambda = \cosh\alpha\pi, \quad \alpha > 0.$

$$\phi(x) = C_2\cdot(\exp\frac{x}{2})\cdot P_{i\alpha-\frac{1}{2}}(e^x), \quad x \geq 0.$$

Case 3: $2\pi\lambda \leq 0.$

$$\phi(x) = 0, \quad x \geq 0.$$

7.3 General Decomposition Problem

In the original Wiener–Hopf problem we examined in Section 7.1,

$$\phi_-(k) = \psi_+(k) + F(k), \tag{7.3.1}$$

we need to make the decomposition

$$F(k) = F_+(k) + F_-(k). \tag{7.3.2}$$

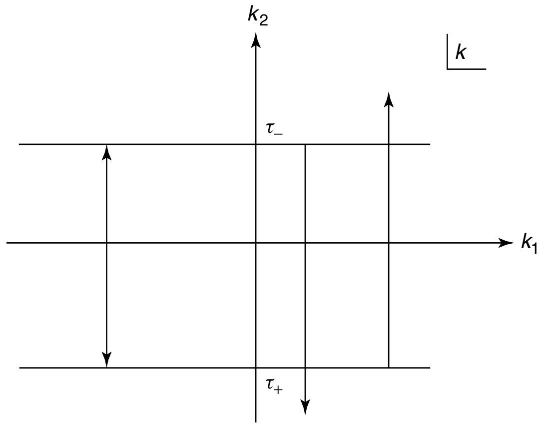

Fig. 7.12 Sum splitting of $F(k)$. $\phi_-(k)$ is analytic in the lower half plane, $\mathrm{Im}\, k < \tau_-$. $\psi_+(k)$ is analytic in the upper half plane, $\mathrm{Im}\, k > \tau_+$. $F(k)$ is analytic inside the strip, $\tau_+ < \mathrm{Im}\, k < \tau_-$.

In the problems we just examined in Section 7.2, i.e., the homogeneous Wiener–Hopf integral equation of the second kind, we need to make the decomposition,

$$1 - \lambda \hat{K}(k) = Y_-(k)/Y_+(k). \tag{7.3.3}$$

Here we discuss how this can be done in general, as opposed to by inspection.

Consider the first problem (*sum splitting*) first (Figure 7.12):

$$\phi_-(k) = \psi_+(k) + F(k).$$

Assume that

$$\phi_-(k), \psi_+(k), F(k) \to 0 \quad \text{as} \quad k \to \infty. \tag{7.3.4}$$

Examine the decomposition of $F(k)$. Since $F(k)$ is analytic inside the strip,

$$\tau_+ < \mathrm{Im}\, k = k_2 < \tau_-, \tag{7.3.5}$$

by the Cauchy integral formula, we have

$$F(k) = \frac{1}{2\pi i} \int_C \frac{F(\zeta)}{\zeta - k} d\zeta \tag{7.3.6}$$

where the complex integration contour C consists of the following path as in Figure 7.13:

$$C = C_1 + C_2 + C_\uparrow + C_\downarrow. \tag{7.3.7}$$

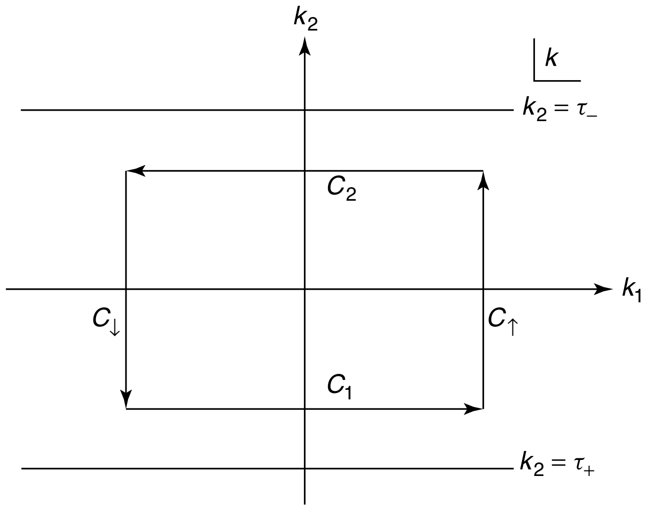

Fig. 7.13 Sum splitting contour C of Eq. (7.3.6) for $F(k)$ inside the strip, $\tau_+ < \mathrm{Im}\, k < \tau_-$.

The contributions from $C_\uparrow$ and $C_\downarrow$ vanish as these contours tend to infinity, since $\left|F(\zeta)\right|$ is bounded (actually→0), and $\left|1/(\zeta - k)\right| \to 0$ as $\zeta \to \infty$.

Thus we have

$$F(k) = \frac{1}{2\pi i}\int_{C_1} \frac{F(\zeta)}{\zeta - k} d\zeta + \frac{1}{2\pi i}\int_{C_2} \frac{F(\zeta)}{\zeta - k} d\zeta, \tag{7.3.8}$$

where the contribution from C_1 is a + function, analytic for $\mathrm{Im}\, k = k_2 > \tau_+$, while the contribution from C_2 is a − function, analytic for $\mathrm{Im}\, k = k_2 < \tau_-$, i.e.,

$$F_+(k) = \frac{1}{2\pi i}\int_{-\infty+i\tau_+}^{+\infty+i\tau_+} \frac{F(\zeta)}{\zeta - k} d\zeta, \tag{7.3.9a}$$

$$F_-(k) = \frac{1}{2\pi i}\int_{-\infty+i\tau_-}^{+\infty+i\tau_-} \frac{F(\zeta)}{\zeta - k} d\zeta. \tag{7.3.9b}$$

Consider now the *factorization of* $1 - \lambda\hat{K}(k)$ *into a ratio ofthe − functiontothe + function*. The function $1 - \lambda\hat{K}(k)$ is analytic inside the strip,

$$-a < \mathrm{Im}\, k = k_2 < b, \tag{7.3.10}$$

and the inversion contour is somewhere inside the strip,

$$-a < \mathrm{Im}\, k = k_2 < -a + \varepsilon. \tag{7.3.11}$$

The analytic function $1 - \lambda\hat{K}(k)$ may have some zeros inside the strip, $-a < \mathrm{Im}\, k = k_2 < b$. Choose a rectangular contour, as indicated in Figure 7.14, below all zeros of $1 - \lambda\hat{K}(k)$ inside the strip, $-a < \mathrm{Im}\, k = k_2 < b$.

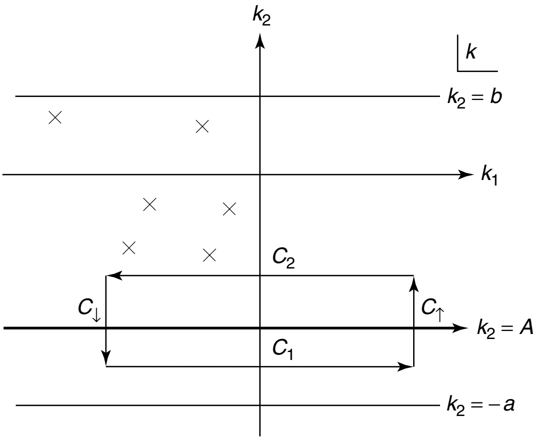

Fig. 7.14 Rectangular contour for the factorization of $1 - \lambda \hat{K}(k)$. This contour is chosen below all the zeros of $1 - \lambda \hat{K}(k)$ inside the strip, $-a < \text{Im}\, k < b$.

Note if $1 - \lambda \hat{K}(k)$ has a zero on $k_2 = -a$, it is all right, since it just remains in the lower half plane. The inversion contour $k_2 = A$ will be chosen inside this rectangle. Now, $1 - \lambda \hat{K}(k)$ is analytic inside the rectangle

$$C_1 + C_\uparrow + C_2 + C_\downarrow$$

and has no zeros inside this rectangle. Also since

$$\hat{K}(k) \to 0 \quad \text{as} \quad k \to \infty,$$

we know

$$1 - \lambda \hat{K}(k) \to 1 \quad \text{as} \quad k \to \infty.$$

In order to express $1 - \lambda \hat{K}(k)$ as the ratio $Y_-(k)/Y_+(k)$, we shall take the logarithm of (7.3.3) to find

$$\ln\left[1 - \lambda \hat{K}(k)\right] = \ln\left[Y_-(k)/Y_+(k)\right] = \ln Y_-(k) - \ln Y_+(k). \qquad (7.3.12)$$

Now, $\ln[1 - \lambda \hat{K}(k)]$ is itself analytic in the rectangle (because it has no branch points since $1 - \lambda \hat{K}(k)$ has no zeros there), so we can apply the Cauchy integral formula

$$\ln\left[1 - \lambda \hat{K}(k)\right] = \frac{1}{2\pi i} \int_C \frac{\ln\left[1 - \lambda \hat{K}(\zeta)\right]}{\zeta - k} d\zeta,$$

with k inside the rectangle and C consisting of $C_1 + C_2 + C_\uparrow + C_\downarrow$. Thus we write

$$\ln\left[1-\lambda\hat{K}(k)\right] = \frac{1}{2\pi i}\int_{C_1}\frac{\ln\left[1-\lambda\hat{K}(\zeta)\right]}{\zeta-k}d\zeta - \frac{1}{2\pi i}\int_{-C_2}\frac{\ln\left[1-\lambda\hat{K}(\zeta)\right]}{\zeta-k}d\zeta + \frac{1}{2\pi i}\int_{C\uparrow+C\downarrow}\frac{\ln\left[1-\lambda\hat{K}(\zeta)\right]}{\zeta-k}d\zeta. \tag{7.3.13}$$

In Eq. (7.3.13), it is tempting to drop the contributions from $C_\uparrow$ and $C_\downarrow$ altogether. It is, however, not always possible to do so. Because of the multivaluedness of the logarithm, we may have, in the limit $|\zeta| \to \infty$,

$$\ln\left[1-\lambda\hat{K}(\zeta)\right] \to \ln e^{2\pi in} = 2\pi in \quad (n = 0, \pm 1, \pm 2, \ldots) \tag{7.3.14}$$

and we have no guarantee that the contributions from $C_\uparrow$ and $C_\downarrow$ cancel each other. In other words, $1-\lambda\hat{K}(\zeta)$ may develop a phase angle as ζ ranges from $-\infty + iA$ to $+\infty + iA$.

Let us define *index* ν of $1-\lambda\hat{K}(\zeta)$ by

$$\nu \equiv \frac{1}{2\pi i}\ln\left[1-\lambda\hat{K}(\zeta)\right]\Big|_{\zeta=-\infty+iA}^{\zeta=+\infty+iA}. \tag{7.3.15}$$

Graphically what we do is the following: plot $z = [1-\lambda\hat{K}(\zeta)]$ as ζ ranges from $-\infty + iA$ to $+\infty + iA$ in the complex z plane, and count the number of counterclockwise revolutions z makes about the origin. The index ν is equal to the number of these revolutions.

We now examine the properties of index ν; in particular, a *relationship between index ν and the zeros and the poles of $1-\lambda\hat{K}(k)$ in the complex k plane*. Suppose $1-\lambda\hat{K}(k)$ has a zero in the upper half plane, say, $1-\lambda\hat{K}(k) = k - z_u$, $\operatorname{Im} z_u > -a$. Then the contribution from this z_u to the index ν is $\nu(z_u) = \frac{1}{2\pi i}\left[0-(-i\pi)\right] = \frac{1}{2}$. Similar analysis yields the following results:

$$\begin{cases} \textit{zero in the upper half plane} & \Rightarrow \quad \nu(z_u) = +\frac{1}{2}, \\ \textit{pole in the upper half plane} & \Rightarrow \quad \nu(p_u) = -\frac{1}{2}, \\ \textit{zero in the lower half plane} & \Rightarrow \quad \nu(z_l) = -\frac{1}{2}, \\ \textit{pole in the lower half plane} & \Rightarrow \quad \nu(p_l) = +\frac{1}{2}. \end{cases} \tag{7.3.16}$$

In many cases, the translation kernel $K(x-y)$ is of the form

$$K(x-y) = K(|x-y|). \tag{7.3.17}$$

Then $\hat{K}(k)$ is even in k,

$$\hat{K}(k) = \hat{K}(-k). \tag{7.3.18}$$

Then $1-\lambda\hat{K}(k)$ (which is even) has an equal number of zeros (poles) in the upper half plane and in the lower half plane, *number of* z_u = *number of* z_l, *number of* p_u = *number of* p_l.

Thus the *index of* $1-\lambda\hat{K}(k)$ *on the real line is equal to zero* ($\nu \equiv 0$) *for* $\hat{K}(k)$ *even.*

Suppose we now lift the path above the real line ($\text{Im}\, k = 0$) into the upper half plane. As the path C ($\text{Im}\, k = A$) passes by a zero of $1-\lambda\hat{K}(k)$ in $\text{Im}\, k > 0$, index ν of $1-\lambda\hat{K}(k)$ with respect to the path C ($\text{Im}\, k = A$) decreases by 1. This is because the point $k = z_0$ is the zero in the upper half plane with respect to the path $C_<$ ($\text{Im}\, k = A^-$) while it is the zero in the lower half plane with respect to the path $C_>$ ($\text{Im}\, k = A^+$), and hence

$$\triangle\nu = \nu(z_l) - \nu(z_u) = -\frac{1}{2} - \frac{1}{2} = -1. \tag{7.3.19z}$$

Likewise, for a pole of $1-\lambda\hat{K}(k)$ in $\text{Im}\, k > 0$, we find

$$\triangle\nu = \nu(p_l) - \nu(p_u) = +\frac{1}{2} + \frac{1}{2} = +1. \tag{7.3.19p}$$

Consider first the case when index ν is equal to *zero*,

$$\nu = 0. \tag{7.3.20}$$

We choose a branch so that $\ln[1-\lambda\hat{K}(\zeta)]$ vanishes on $C_\uparrow$ and $C_\downarrow$. We have

$$\ln Y_-(k) - \ln Y_+(k) = \frac{1}{2\pi i}\int_{C_1} \frac{\ln\left[1-\lambda\hat{K}(\zeta)\right]}{\zeta - k} d\zeta - \frac{1}{2\pi i}\int_{-C_2} \frac{\ln\left[1-\lambda\hat{K}(\zeta)\right]}{\zeta - k} d\zeta. \tag{7.3.21}$$

In the first integral on the right-hand side of Eq. (7.3.21), we let k be anywhere above C_1 where C_1 is arbitrarily close to $\text{Im}\, k = k_2 = -a$ from above. Then

$$\ln Y_+(k) = -\frac{1}{2\pi i}\int_{C_1} \frac{\ln\left[1-\lambda\hat{K}(\zeta)\right]}{\zeta - k} d\zeta, \quad \text{Im}\, k = k_2 > -a, \tag{7.3.22}$$

is analytic in the upper half plane, and hence is identified to be a + function. It also vanishes as $|k| \to \infty$ in the upper half plane ($\text{Im}\, k > -a$).

In the second integral on the right-hand side of Eq. (7.3.21), we let k be anywhere below $-C_2$ where $-C_2$ is arbitrarily close to $\text{Im}\, k = k_2 = -a$ from above. Then

$$\ln Y_-(k) = -\frac{1}{2\pi i}\int_{-C_2} \frac{\ln\left[1-\lambda\hat{K}(\zeta)\right]}{\zeta - k} d\zeta, \quad \text{Im}\, k = k_2 \leq -a, \tag{7.3.23}$$

is analytic in the lower half plane, and hence is identified to be a − function. It also vanishes as $|k| \to \infty$ in the lower half plane ($\text{Im}\, k \leq -a$).

Thus

$$Y_+(k) = \exp[-\frac{1}{2\pi i}\int_{C_1} \frac{\ln\left[1-\lambda\hat{K}(\zeta)\right]}{\zeta - k} d\zeta], \tag{7.3.24}$$

$$Y_-(k) = \exp\left[-\frac{1}{2\pi i}\int_{-C_2} \frac{\ln\left[1-\lambda\hat{K}(\zeta)\right]}{\zeta - k} d\zeta\right]. \tag{7.3.25}$$

We also note that

$$Y_\pm(k) \to 1 \quad \text{as} \quad |k| \to \infty \quad \text{in} \quad \begin{cases} \operatorname{Im} k > -a, \\ \operatorname{Im} k \le -a. \end{cases} \tag{7.3.26}$$

Then the entire function $G(k)$ in the following equation:

$$Y_-(k)\hat{\phi}_-(k) = -Y_+(k)\hat{\psi}_+(k) = G(k),$$

must vanish identically by Liouville's theorem . Hence

$$\hat{\phi}_-(k) = 0 \quad \text{or} \quad \phi(x) = 0 \quad \text{when } \nu = 0. \tag{7.3.27}$$

Consider next the case when index ν is *positive,*

$$\nu > 0. \tag{7.3.28}$$

Instead of dealing with $C_\uparrow$ and $C_\downarrow$ of the integral (7.3.13), we *construct the object whose index is equal to zero,*

$$\prod_{i=1}^{\nu}\left(\frac{k - z_l(i)}{k - p_u(i)}\right)\left[1-\lambda\hat{K}(k)\right] = Z_-(k)/Z_+(k). \tag{7.3.29}$$

Here $z_l(i)$ is a point in the lower half plane ($\operatorname{Im} k \le A$) which contributes $-\nu/2$ in its totality ($i = 1,\ldots, \nu$) to the index and $p_u(i)$ is a point in the upper half plane ($\operatorname{Im} k > A$) which contributes $-\nu/2$ in its totality ($i = 1,\ldots, \nu$) to the index. Then expression (7.3.29) has the index equal to zero with respect to $\operatorname{Im} k = A$,

$$-\frac{\nu}{2}(\text{from } z_l(i)'\text{s}) - \frac{\nu}{2}(\text{from } p_u(i)'\text{s}) + \nu(\text{from } 1-\lambda\hat{K}(k)) = 0. \tag{7.3.30}$$

By factoring of Eq. (7.3.29), using Eqs. (7.3.24) and (7.3.25), we obtain

$$Z_-(k) = \exp\left(-\frac{1}{2\pi i}\int_{-C_2} \frac{\ln\left[(1-\lambda\hat{K}(\zeta))\prod_{i=1}^{\nu}\left(\frac{\zeta - z_l(i)}{\zeta - p_u(i)}\right)\right]}{\zeta - k} d\zeta\right), \tag{7.3.31}$$

$$Z_+(k) = \exp\left(-\frac{1}{2\pi i}\int_{C_1} \frac{\ln\left[(1-\lambda\hat{K}(\zeta))\prod_{i=1}^{\nu}\left(\frac{\zeta - z_l(i)}{\zeta - p_u(i)}\right)\right]}{\zeta - k} d\zeta \right), \tag{7.3.32}$$

with the following properties:

(1) $Z_{\pm}(k) \to 1 \quad \text{as} \quad |k| \to \infty$,
(2) $Z_-(k)$ $(Z_+(k))$ is analytic in the lower (upper) half plane,
(3) $Z_-(k)$ $(Z_+(k))$ has no zero in the lower (upper) half plane.

We write Eq. (7.3.29) as

$$1 - \lambda\hat{K}(k) = \frac{Y_-(k)}{Y_+(k)} = \frac{Z_-(k)}{Z_+(k)} \cdot \frac{\prod_{i=1}^{\nu}(k - p_u(i))}{\prod_{i=1}^{\nu}(k - z_l(i))}. \tag{7.3.33}$$

By the prescription stated in Eq. (7.2.17), we obtain

$$Y_-(k) = Z_-(k) \cdot \prod_{i=1}^{\nu}(k - p_u(i)), \tag{7.3.34}$$

$$Y_+(k) = Z_+(k) \cdot \prod_{i=1}^{\nu}(k - z_l(i)). \tag{7.3.35}$$

We observe that

$$Y_{\pm}(k) \to k^{\nu} \quad \text{as} \quad |k| \to \infty. \tag{7.3.36}$$

Thus the entire function $G(k)$ in the following equation:

$$Y_-(k)\hat{\phi}_-(k) = -Y_+(k)\hat{\psi}_+(k) = G(k),$$

cannot grow as fast as k^{ν} as $k \to \infty$. By Liouville's theorem , we have

$$G(k) = \sum_{j=0}^{\nu-1} C_j k^j, \quad 0 \le j \le \nu - 1, \tag{7.3.37}$$

where C_j's are arbitrary ν constants. Then we obtain

$$\hat{\phi}_-(k) = G(k)/Y_-(k) = \sum_{j=0}^{\nu-1} C_j k^j / Y_-(k), \quad \operatorname{Im} k \le A. \tag{7.3.38}$$

Inverting this expression along $\operatorname{Im} k = A$, we obtain

$$\phi(x) = \frac{1}{2\pi i}\int_{-\infty+iA}^{+\infty+iA} dk e^{ikx}\hat{\phi}_-(k) = \sum_{j=0}^{\nu-1} C_j \phi_{(j)}(x), \quad \text{when} \quad \nu > 0, \tag{7.3.39}$$

where

$$\phi_{(j)}(x) = \frac{1}{2\pi i}\int_{-\infty+iA}^{+\infty+iA} dk e^{ikx} k^j / Y_-(k), \quad j = 0, \ldots, \nu - 1. \tag{7.3.40}$$

We have ν *independent homogeneous solutions*, $\phi_{(j)}(x)$, $j = 0, \ldots, \nu - 1$, which are related to each other by differentiation,

$$\left(-i\frac{d}{dx}\right)\phi_{(j)}(x) = \phi_{(j+1)}(x), \quad 0 \le j \le \nu - 2.$$

Thus it is sufficient to compute $\phi_{(0)}(x)$,

$$\phi_{(j)}(x) = \left(-i\frac{d}{dx}\right)^j \phi_{(0)}(x), \quad j = 0, 1, \ldots, \nu - 1. \tag{7.3.41}$$

Differentiation under the integral, Eq. (7.3.40), is justified by

$$\frac{k^{j+1}}{Y_-(k)} \to \frac{k^{j+1}}{k^{\nu}} \to 0 \quad \text{as} \quad k \to \infty, \quad j \le \nu - 2.$$

Consider thirdly the case when index ν is *negative*,

$$\nu < 0. \tag{7.3.42}$$

As before, we *construct the object whose index is equal to zero*:

$$\prod_{i=1}^{|\nu|}\frac{(k - z_u(i))}{(k - p_l(i))} \cdot \left[1 - \lambda \hat{K}(k)\right] = Z_-(k)/Z_+(k), \tag{7.3.43}$$

which has indeed index zero as shown below.

$$+\frac{|\nu|}{2}(\text{from} z_u(i)'\text{s}) + \frac{|\nu|}{2}(\text{from} p_l(i)'\text{s}) + \nu(\text{from} 1 - \lambda\hat{K}(k)) = 0. \tag{7.3.44}$$

We apply the factorization to the left-hand side of Eq. (7.3.43). Then we write

$$1 - \lambda\hat{K}(k) = \frac{Y_-(k)}{Y_+(k)} = \frac{Z_-(k)}{Z_+(k)} \cdot \frac{\prod_{i=1}^{|\nu|}(k - p_l(i))}{\prod_{i=1}^{|\nu|}(k - z_u(i))}. \tag{7.3.45}$$

By the prescription stated in Eq. (7.2.17), we obtain

$$Y_-(k) = Z_-(k) / \prod_{i=1}^{|\nu|}(k - z_u(i)), \tag{7.3.46}$$

$$Y_+(k) = Z_+(k) / \prod_{i=1}^{|\nu|}(k - p_l(i)). \tag{7.3.47}$$

Then we have

$$Z_{\pm}(k) \to 1 \quad \text{and} \quad Y_{\pm}(k) \to 1/k^{|\nu|}, \quad \text{as} \quad k \to \infty. \tag{7.3.48}$$

Thus the entire function $G(k)$ in the following equation:

$$Y_{-}(k)\hat{\phi}_{-}(k) = -Y_{+}(k)\hat{\psi}_{+}(k) = G(k),$$

must vanish identically by Liouville's theorem. Hence we obtain

$$\phi(x) = 0, \quad \text{when} \quad \nu < 0. \tag{7.3.49}$$

7.4 Inhomogeneous Wiener–Hopf Integral Equation of the Second Kind

Let us consider the *inhomogeneous Wiener–Hopf integral equation of the second kind*,

$$\phi(x) = f(x) + \lambda \int_{0}^{+\infty} K(x-y)\phi(y)dy, \quad x \geq 0, \tag{7.4.1}$$

where we assume as in Section 7.3 that the asymptotic behavior of the kernel $K(x)$ is given by

$$K(x) \sim \begin{cases} O(e^{ax}) & \text{as} \quad x \to -\infty, \\ O(e^{-bx}) & \text{as} \quad x \to +\infty, \end{cases} \quad a, b > 0, \tag{7.4.2}$$

and the asymptotic behavior of the inhomogeneous term $f(x)$ is given by

$$f(x) \to O(e^{cx}) \quad \text{as} \quad x \to +\infty. \tag{7.4.3}$$

We define $\psi(x)$ for $x < 0$ as before

$$\psi(x) = \lambda \int_{0}^{+\infty} K(x-y)\phi(y)dy, \quad x < 0. \tag{7.4.4}$$

We take the Fourier transform of $\phi(x)$ and $\psi(x)$ for $x \geq 0$ and $x < 0$ and add the results together,

$$\hat{\phi}_{-}(k) + \hat{\psi}_{+}(k) = \hat{f}_{-}(k) + \lambda\hat{K}(k)\hat{\phi}_{-}(k), \tag{7.4.5}$$

where $\hat{K}(k)$ is analytic inside the strip,

$$-a < \operatorname{Im} k = k_2 < b. \tag{7.4.6}$$

Difficulty may arise when the inhomogeneous term $f(x)$ grows too fast as $x \to \infty$ so there may not exist a *common region of analyticity* for Eq. (7.4.5) to hold. The Fourier transform $\hat{f}_-(k)$ is defined by

$$\hat{f}_-(k) = \int_0^{+\infty} dx e^{-ikx} f(x), \tag{7.4.7}$$

where

$$\left| e^{-ikx} f(x) \right| \sim e^{(k_2+c)x} \quad \text{as} \quad x \to \infty \quad \text{with} \quad k = k_1 + ik_2.$$

That is, $\hat{f}_-(k)$ is analytic in the lower half plane,

$$\operatorname{Im} k = k_2 < -c. \tag{7.4.8}$$

We require that a and c satisfy

$$a > c. \tag{7.4.9}$$

In other words, $f(x)$ grows at most as fast as

$$f(x) \sim e^{(a-\varepsilon)x}, \quad \varepsilon > 0, \quad \text{as} \quad x \to \infty.$$

We try to solve Eq. (7.4.5) in the narrower strip,

$$-a < \operatorname{Im} k = k_2 < \min(-c, b). \tag{7.4.10}$$

Writing Eq. (7.4.5) as

$$(1 - \lambda \hat{K}(k))\hat{\phi}_-(k) = \hat{f}_-(k) - \hat{\psi}_+(k), \tag{7.4.11}$$

we are content to obtain *one particular solution* of Eq. (7.4.1). We factor $1 - \lambda \hat{K}(k)$ as before,

$$1 - \lambda \hat{K}(k) = Y_-(k)/Y_+(k). \tag{7.4.12}$$

Thus we have from Eqs. (7.4.11) and (7.4.12)

$$Y_-(k)\hat{\phi}_-(k) = Y_+(k)\hat{f}_-(k) - Y_+(k)\hat{\psi}_+(k), \tag{7.4.13}$$

where $Y_-(k)\hat{\phi}_-(k)$ is analytic in the lower half plane and $Y_+(k)\hat{\psi}_+(k)$ is analytic in the upper half plane. We split $Y_+(k)\hat{f}_-(k)$ into a sum of two functions, one analytic in the upper half plane and the other analytic in the lower half plane,

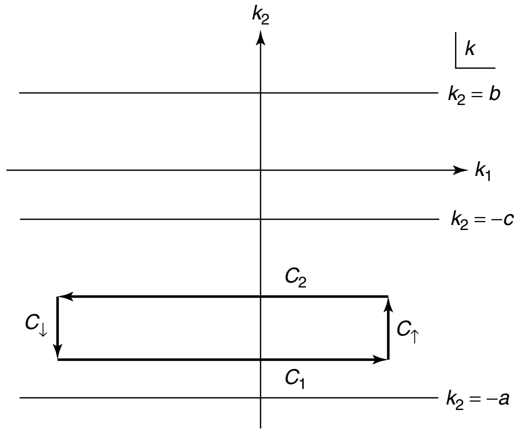

Fig. 7.15 Region of analyticity and the integration contour C for $\hat{F}(k)$ inside the strip, $-a < \operatorname{Im} k < \min(-c, b)$.

$$Y_+(k)\hat{f}_-(k) = (Y_+(k)\hat{f}_-(k))_+ + (Y_+(k)\hat{f}_-(k))_-.$$

In order to do this, we must construct $Y_+(k)$ such that

$$Y_+(k)\hat{f}_-(k) \to 0 \quad \text{as} \quad k \to \infty,$$

or

$$Y_+(k) \to \text{constant} \quad \text{as} \quad k \to \infty. \tag{7.4.14}$$

Suppose $\hat{F}(k)$ is analytic inside the strip,

$$-a < \operatorname{Im} k = k_2 < \min(-c, b). \tag{7.4.15}$$

By choosing the contour C inside the strip as in Figure 7.15, we apply the Cauchy integral formula.

$$\hat{F}(k) = \frac{1}{2\pi i}\int_C \frac{\hat{F}(\zeta)}{\zeta - k}d\zeta = \frac{1}{2\pi i}\int_{C_1} \frac{\hat{F}(\zeta)}{\zeta - k}d\zeta - \frac{1}{2\pi i}\int_{-C_2} \frac{\hat{F}(\zeta)}{\zeta - k}d\zeta. \tag{7.4.16}$$

By the same argument as in the previous section, we identify

$$\hat{F}_-(k) = -\frac{1}{2\pi i}\int_{-C_2} \frac{\hat{F}(\zeta)}{\zeta - k}d\zeta, \tag{7.4.17}$$

$$\hat{F}_+(k) = \frac{1}{2\pi i}\int_{C_1} \frac{\hat{F}(\zeta)}{\zeta - k}d\zeta. \tag{7.4.18}$$

Thus, under the assumption that $Y_+(k)$ satisfy the above-stipulated condition (7.4.14), we obtain

$$(Y_+(k)\hat{f}_-(k))_- = -\frac{1}{2\pi i}\int_{-C_2}[Y_+(\zeta)\hat{f}_-(\zeta)/(\zeta - k)]d\zeta, \tag{7.4.19}$$

$$(Y_+(k)\hat{f}_-(k))_+ = \frac{1}{2\pi i}\int_{C_1}[Y_+(\zeta)\hat{f}_-(\zeta)/(\zeta - k)]d\zeta. \tag{7.4.20}$$

Then we write

$$Y_-(k)\hat{\phi}_-(k) - (Y_+(k)\hat{f}_-(k))_- = (Y_+(k)\hat{f}_-(k))_+ - Y_+(k)\hat{\psi}_+(k) \equiv G(k), \tag{7.4.21}$$

where $G(k)$ is entire in k. If we are looking for the most general homogeneous solutions, we set $\hat{f}_-(k) \equiv 0$ and determine the most general form of the entire function $G(k)$. Now we are just looking for *one particular solution to the inhomogeneous equation*, so that we set $G(k) = 0$. Then we have

$$\hat{\phi}_-(k) = (Y_+(k)\hat{f}_-(k))_-/Y_-(k), \tag{7.4.22}$$

$$\hat{\psi}_+(k) = (Y_+(k)\hat{f}_-(k))_+/Y_+(k). \tag{7.4.23}$$

Choices of $Y_\pm(k)$ for an inhomogeneous solution are different from those for a homogeneous solution. In view of Eqs. (7.4.22) and (7.4.23), we require that

(1) $1/Y_-(k)$ is analytic in the lower half plane,

$$\operatorname{Im} k < c', \quad -a < c' < b, \tag{7.4.24}$$

and $1/Y_+(k)$ is analytic in the upper half plane,

$$\operatorname{Im} k > c'', \quad -a < c'' < b. \tag{7.4.25}$$

(2)

$$\hat{\phi}_-(k) \to 0, \quad \hat{\psi}_+(k) \to 0 \quad \text{as} \quad k \to \infty. \tag{7.4.26}$$

According to this requirement, $Y_-(k)$ for an inhomogeneous solution can have a pole in the lower half plane. This is all right because then $1/Y_-(k)$ has a zero in the lower half plane. Once requirements (1) and (2) are satisfied, $\hat{\phi}_-(k)$ and $\hat{\psi}_+(k)$ given by Eqs. (7.4.22) and (7.4.23) are analytic in the respective half plane. Then we construct the following expression from Eqs. (7.4.22) and (7.4.23):

$$[1 - \lambda\hat{K}(k)]\hat{\phi}_-(k) = \hat{f}_-(k) - \hat{\psi}_+(k). \tag{7.4.27}$$

Inverting for $x \geq 0$, we obtain

$$\phi(x) - \lambda \int_0^{+\infty} K(x-y)\phi(y)dy = f(x), \quad x \geq 0.$$

Thus $\hat{\phi}_-(k)$ and $\hat{\psi}_+(k)$ derived in Eqs. (7.4.22) and (7.4.23) under the requirements (1) and (2) do provide a particular solution to Eq. (7.4.1).

Case 1: Index $\nu = 0$.
When index ν of $1 - \lambda\hat{K}(k)$ with respect to the line $\operatorname{Im} k = A$ is equal to *zero*, no nontrivial homogeneous solution exists. From Eqs. (7.3.22) and (7.3.23), we have

$$1 - \lambda\hat{K}(k) = Y_-(k)/Y_+(k), \tag{7.4.28}$$

where

$$Y_-(k) = \exp\left(-\frac{1}{2\pi i}\int_{-C_2} d\zeta \frac{\ln\left[1-\lambda\hat{K}(\zeta)\right]}{\zeta - k}\right), \quad \operatorname{Im} k \leq A, \tag{7.4.29}$$

$$Y_+(k) = \exp\left(-\frac{1}{2\pi i}\int_{C_1} d\zeta \frac{\ln\left[1-\lambda\hat{K}(\zeta)\right]}{\zeta - k}\right), \quad \operatorname{Im} k > A, \tag{7.4.30}$$

and

$$Y_\pm(k) \to 1 \quad \text{as} \quad |k| \to \infty. \tag{7.4.31}$$

Since $Y_+(k)$ ($Y_-(k)$) has no zeros in the upper half plane (the lower half plane), $1/Y_+(k)$ ($1/Y_-(k)$) is analytic in the upper half plane (the lower half plane). Then we have

$$Y_+(k)\hat{f}_-(k) \to 0 \quad \text{as} \quad k \to \infty,$$

so that $Y_+(k)\hat{f}_-(k)$ can be split into the + part and the − part as in Eqs. (7.4.19) and (7.4.20).

Using $Y_\pm(k)$ given for the $\nu \equiv 0$ case, Eqs. (7.4.29) and (7.4.30), we construct a *resolvent kernel* $H(x, y)$. Since $1/Y_-(k)$ is analytic in the lower plane and approaches to 1 as $k \to \infty$, we define $y_-(x)$ by

$$\frac{1}{Y_-(k)} - 1 \equiv \int_0^{+\infty} dx e^{-ikx} y_-(x), \tag{7.4.32}$$

where the left-hand side is analytic in the lower half plane and vanishes as $k \to \infty$. Inverting Eq. (7.4.32) for $y_-(x)$, we have

$$y_-(x) = \int_{-\infty+iA}^{+\infty+iA} \frac{dk}{2\pi} e^{ikx} \left[\frac{1}{Y_-(k)} - 1 \right] \quad \text{for} \quad x \geq 0, \tag{7.4.33}$$

$$y_-(x) = 0 \quad \text{for} \quad x < 0. \tag{7.4.34}$$

Similarly, we define $y_+(x)$ by

$$Y_+(k) - 1 \equiv \int_{-\infty}^{0} dx e^{-ikx} y_+(x), \tag{7.4.35}$$

where the left-hand side is analytic in the upper half plane and vanishes as $k \to \infty$. Inverting Eq. (7.4.35) for $y_+(x)$, we have

$$y_+(x) = \int_{-\infty+iA}^{+\infty+iA} \frac{dk}{2\pi} e^{ikx} [Y_+(k) - 1] \quad \text{for} \quad x < 0, \tag{7.4.36}$$

$$y_+(x) = 0 \quad \text{for} \quad x \geq 0. \tag{7.4.37}$$

We define $\hat{y}_\pm(k)$ by

$$\frac{1}{Y_-(k)} \equiv 1 + \int_0^{+\infty} dx e^{-ikx} y_-(x) \equiv 1 + \hat{y}_-(k), \tag{7.4.38}$$

$$Y_+(k) \equiv 1 + \int_{-\infty}^{0} dx e^{-ikx} y_+(x) \equiv 1 + \hat{y}_+(k). \tag{7.4.39}$$

Then $\hat{\phi}_-(k)$ given by Eq. (7.4.23) becomes

$$\begin{aligned} \hat{\phi}_-(k) &= \frac{1}{Y_-(k)} (Y_+(k)\hat{f}_-(k))_- = (1 + \hat{y}_-(k))(\hat{f}_-(k) + \hat{y}_+(k)\hat{f}_-(k))_- \\ &= \hat{f}_-(k) + \hat{y}_-(k)\hat{f}_-(k) + (\hat{y}_+(k)\hat{f}_-(k))_- + \hat{y}_-(k)(\hat{y}_+(k)\hat{f}_-(k))_-. \end{aligned} \tag{7.4.40}$$

Inverting Eq. (7.4.40) for $x > 0$,

$$\begin{aligned} \phi(x) &= f(x) + \int_0^{+\infty} y_-(x-y) f(y) dy + \int_0^{+\infty} y_+(x-y) f(y) dy \\ &\quad + \int_0^{+\infty} y_-(x-z) dz \int_0^{+\infty} y_+(z-y) f(y) dy \\ &= f(x) + \int_0^{+\infty} H(x,y) f(y) dy, \quad x \geq 0, \end{aligned} \tag{7.4.41}$$

where

$$H(x, y) \equiv y_-(x-y) + y_+(x-y) + \int_0^{+\infty} y_-(x-z)y_+(z-y)dz. \tag{7.4.42}$$

It is noted that the existence of the resolvent kernel $H(x, y)$ given above is solely due to the analyticity of $1/Y_-(k)$ in the lower half plane and that of $Y_+(k)$ in the upper half plane. Thus, when index $\nu = 0$, we have a *unique solution, solely consisting of a particular solution alone* to Eq. (7.4.11).

Case 2: Index $\nu > 0$.
When index ν is *positive*, we have ν independent homogeneous solutions given by Eq. (7.3.40). We observed in Section 7.3 that

$$Y_\pm(k) \to k^\nu \quad \text{as} \quad k \to \infty, \tag{7.4.43}$$

where $Y_\pm(k)$ are given by Eqs. (7.3.34) and (7.3.35). On the other hand, in solving for a particular solution, we want $Y_\pm(k)$ to be such that (1) $1/Y_-(k)$ ($Y_+(k)$) is analytic in the lower half plane (the upper half plane),

$$Y_\pm(k) \to 1 \quad \text{as} \quad |k| \to \infty. \tag{7.4.44}$$

We construct $W_\pm(k)$ as

$$W_\pm(k) = Y_\pm(k) / \prod_{j=1}^{\nu} (k - p_l(j)), \quad \operatorname{Im} p_l(j) \le A, \quad 1 \le j \le \nu, \tag{7.4.45}$$

where the locations of $p_l(j)$'s are quite arbitrary as long as $\operatorname{Im} p_l(j) \le A$. We notice that $W_\pm(k)$ satisfy requirements (1) and (2):

(1)

$$1/W_-(k) = \prod_{j=1}^{\nu} (k - p_l(j))/Y_-(k)$$

is analytic in the lower half plane, while

$$W_+(k) = Y_+(k) / \prod_{j=1}^{\nu} (k - p_l(j))$$

is analytic in the upper half plane;

(2)

$$W_\pm(k) \to 1 \quad \text{as} \quad |k| \to \infty. \tag{7.4.46}$$

Thus we use $W_\pm(k)$, Eq. (7.4.45), instead of $Y_\pm(k)$, in the construction of the resolvent $H(x, y)$.

Case 3: Index $\nu < 0$.
When index ν is *negative*, we have no nontrivial homogeneous solution. From Eqs. (7.3.46) and (7.3.47), we have

$$Y_{\pm}(k) \to 1/k^{|\nu|} \quad \text{as} \quad |k| \to \infty. \tag{7.4.47}$$

Then we have

$$1/Y_{-}(k) \to k^{|\nu|} \quad \text{as} \quad |k| \to \infty, \tag{7.4.48}$$

while

$$Y_{+}(k)\hat{f}_{-}(k) \to 0 \quad \text{as} \quad |k| \to \infty. \tag{7.4.49}$$

By Liouville's theorem, $\hat{\phi}_{-}(k)$ can grow at most as fast as $k^{|\nu|-1}$ as $|k| \to \infty$,

$$\hat{\phi}_{-}(k) = (Y_{+}(k)\hat{f}_{-}(k))_{-}/Y_{-}(k) \sim k^{|\nu|-1} \quad \text{as} \quad |k| \to \infty. \tag{7.4.50}$$

In general, we have

$$\hat{\phi}_{-}(k) \nrightarrow 0 \quad \text{as} \quad |k| \to \infty,$$

so that a particular solution to the inhomogeneous problem may not exist. There are some *exceptions* to this. We analyze $(Y_{+}(k)\hat{f}_{-}(k))_{-}$ more carefully. We know

$$(Y_{+}(k)\hat{f}_{-}(k))_{-} = -\frac{1}{2\pi i}\int_{-C_2} \frac{Y_{+}(\zeta)\hat{f}_{-}(\zeta)}{\zeta - k}d\zeta. \tag{7.4.51}$$

Expanding $1/(\zeta - k)$ in power series of ζ/k,

$$1/(\zeta - k) = -(1/k)\left(1 + \frac{\zeta}{k} + \frac{\zeta^2}{k^2} + \cdots + \frac{\zeta^{|\nu|-1}}{k^{|\nu|-1}} + \cdots\right), \quad \left|\frac{\zeta}{k}\right| < 1,$$

we write

$$\begin{aligned}(Y_{+}(k)\hat{f}_{-}(k))_{-} &= \frac{1}{2\pi i}\int_{-C_2} \frac{1}{k}\sum_{j=0}^{\infty}\left(\frac{\zeta}{k}\right)^j Y_{+}(\zeta)\hat{f}_{-}(\zeta)d\zeta \\ &= \frac{1}{k}\sum_{j=0}^{\infty}\frac{1}{k^j}\frac{1}{2\pi i}\int_{-C_2} \zeta^j Y_{+}(\zeta)\hat{f}_{-}(\zeta)d\zeta.\end{aligned} \tag{7.4.52}$$

In view of Eqs. (7.4.22) and (7.4.52), we realize that

$$\hat{\phi}_{-}(k) = (Y_{+}(k)\hat{f}_{-}(k))_{-}/Y_{-}(k) \to 0 \quad \text{as} \quad |k| \to \infty, \tag{7.4.53}$$

if and only if

$$\frac{1}{2\pi i}\int_{-C_2}\zeta^j Y_+(\zeta)\hat{f}_-(\zeta)d\zeta = 0, \quad j = 0, \ldots, |\nu| - 1. \tag{7.4.54}$$

If this condition is satisfied, we get from the $j = |\nu|$ term onward,

$$\hat{\phi}_-(k) \to C/k^{1+|\nu|} \quad \text{as} \quad |k| \to \infty, \tag{7.4.55}$$

so that $\hat{\phi}_-(k)$ can be inverted for $\phi(x)$, which is the *unique solution* to the inhomogeneous problem. To understand this solvability condition (7.4.54), we first recall the *Parseval identity*,

$$\int_{-\infty}^{+\infty} dk\hat{h}(k)\hat{g}(-k) = 2\pi\int_{-\infty}^{+\infty} h(y)g(y)dy, \tag{7.4.56}$$

where $\hat{h}(k)$ and $\hat{g}(k)$ are the Fourier transforms of $h(y)$ and $g(y)$, respectively. Then we consider the *homogeneous adjoint problem*. Recall that for a real kernel,

$$K^{\text{adj}}(x, y) = K(y, x).$$

Thus, corresponding to the original homogeneous problem,

$$\phi(x) = \lambda\int_0^{+\infty} K(x - y)\phi(y)dy,$$

there exists the homogeneous adjoint problem,

$$\phi^{\text{adj}}(x) = \lambda\int_0^{+\infty} K(y - x)\phi^{\text{adj}}(y)dy, \tag{7.4.57}$$

whose translation kernel is related to the original one by

$$K^{\text{adj}}(\xi) = K(-\xi). \tag{7.4.58}$$

Now, when we take the Fourier transform of the homogeneous adjoint problem, we find

$$\left[1 - \lambda\hat{K}(-k)\right]\hat{\phi}_-^{\text{adj}}(k) = -\hat{\psi}_+^{\text{adj}}(k), \tag{7.4.59}$$

where the only difference from the original equation is the sign of k inside $\hat{K}(-k)$. However, since $1 - \lambda\hat{K}(-k)$ is just the reflection of $1 - \lambda\hat{K}(k)$ through the origin, a zero of $1 - \lambda\hat{K}(k)$ in the upper half plane corresponds to a zero of $1 - \lambda\hat{K}(-k)$ in the lower half plane, etc. Thus, when the original $1 - \lambda\hat{K}(k)$ has a negative index $\nu < 0$ with respect to a line, $\operatorname{Im} k = k_2 = A$, the homogeneous adjoint problem $1 - \lambda\hat{K}(-k)$ has a positive index $|\nu|$ relative to the line $\operatorname{Im} k = k_2 = -A$. So, in that case, although the original problem may have no solutions, the homogeneous adjoint problem has $|\nu|$ independent solutions. Now presently, $1 - \lambda\hat{K}(k)$ is found to have the decomposition

$$1 - \lambda\hat{K}(k) = Y_-(k)/Y_+(k),$$

with

$$Y_\pm(k) \to 1/k^{|\nu|} \quad \text{as} \quad k \to \infty.$$

We conclude that

$$1 - \lambda\hat{K}(-k) = Y_-(-k)/Y_+(-k) = Y_-^{\text{adj}}(k)/Y_+^{\text{adj}}(k), \tag{7.4.60}$$

from which we recognize

$$\begin{cases} Y_-^{\text{adj}}(k) = 1/Y_+(-k) & \text{analytic in the lower half plane,} \quad \to k^{|\nu|}, \\ Y_+^{\text{adj}}(k) = 1/Y_-(-k) & \text{analytic in the upper half plane,} \quad \to k^{|\nu|}. \end{cases}$$

Thus the homogeneous adjoint problem reads

$$Y_-^{\text{adj}}(k)\hat{\phi}_-^{\text{adj}}(k) = -Y_+^{\text{adj}}(k)\psi_+^{\text{adj}}(k) \equiv G(k), \tag{7.4.61}$$

with

$$G(k) = C_0 + C_1 k + \cdots + C_{|\nu|-1}k^{|\nu|-1}. \tag{7.4.62}$$

We then know that the $|\nu|$ independent solutions to the homogeneous adjoint problem are of the form

$$\hat{\phi}_-^{\text{adj}}(k) = \{1/Y_-^{\text{adj}}(k), k/Y_-^{\text{adj}}(k), \ldots, k^{|\nu|-1}/Y_-^{\text{adj}}(k)\},$$

which is equivalent to

$$\hat{\phi}_-^{\text{adj}}(k) = \{Y_+(-k), kY_+(-k), \ldots, k^{|\nu|-1}Y_+(-k)\}. \tag{7.4.63}$$

As such, to within a constant factor, which is irrelevant, we can write the solvability condition (7.4.54) as

$$\int_{-C_2} \zeta^j Y_+(\zeta)\hat{f}_-(\zeta)d\zeta = 0, \quad j = 0, 1, \ldots, |\nu| - 1, \tag{7.4.64}$$

which is equivalent to

$$\int_{-C_2} \hat{\phi}_{-(j)}^{\text{adj}}(-\zeta)\hat{f}_-(\zeta)d\zeta = 0, \quad j = 0, 1, \ldots, |\nu| - 1, \tag{7.4.65}$$

where

$$\hat{\phi}^{\text{adj}}_{-(j)}(\zeta) \equiv \zeta^j Y_+(-\zeta), \quad j = 0, 1, \ldots, |\nu| - 1. \tag{7.4.66}$$

By the Parseval identity (7.4.56), the *solvability condition* (7.4.65) can now be written as

$$\int_0^{+\infty} \phi^{\text{adj}}_{(j)}(x) f(x) dx = 0, \quad j = 0, 1, \ldots, |\nu| - 1, \tag{7.4.67}$$

where

$$\phi^{\text{adj}}_{(j)}(x) = \frac{1}{2\pi} \int_{-\infty - iA}^{+\infty - iA} e^{i\zeta x} \hat{\phi}^{\text{adj}}_{-(j)}(\zeta) d\zeta, \quad j = 0, 1, \ldots, |\nu| - 1, \quad x \geq 0. \tag{7.4.68}$$

Namely, *if and only if the inhomogeneous term $f(x)$ is orthogonal to all of the homogeneous solutions $\phi^{\text{adj}}(x)$ of the homogeneous adjoint problem, the inhomogeneous equation (7.4.1) has a unique solution, when index ν is negative.*

Summary of the Wiener–Hopf integral equation:

$$\phi(x) = \lambda \int_0^{\infty} K(x - y)\phi(y) dy, \quad x \geq 0,$$

$$\phi^{\text{adj}}(x) = \lambda \int_0^{\infty} K(y - x)\phi^{\text{adj}}(y) dy, \quad x \geq 0,$$

$$\phi(x) = f(x) + \lambda \int_0^{\infty} K(x - y)\phi(y) dy, \quad x \geq 0.$$

(1) Index $\nu = 0$:
Homogeneous problem and its homogeneous adjoint problem have no solutions.
Inhomogeneous problem has a unique solution.

(2) Index $\nu > 0$:
Homogeneous problem has ν independent solutions and its homogeneous adjoint problem has no solutions.
Inhomogeneous problem has a nonunique solutions (but there are no solvability conditions).

(3) Index $\nu < 0$:
Homogeneous problem has no solutions, and its homogeneous adjoint problem has $|\nu|$ independent solutions.
Inhomogeneous problem has a unique solution, if and only if the inhomogeneous term is orthogonal to all $|\nu|$ independent solutions to the homogeneous adjoint problem.

7.5
Toeplitz Matrix and Wiener–Hopf Sum Equation

In this section, we consider the application of the Wiener–Hopf method to the *infinite system of the inhomogeneous linear algebraic equation,*

$$M\vec{X} = \vec{f}, \quad \text{or} \quad \sum_m M_{nm} X_m = f_n, \tag{7.5.1,2}$$

where the coefficient matrix M is *real* and has the *Toeplitz structure,*

$$M_{nm} = M_{n-m}. \tag{7.5.3}$$

Case A: Infinite Toeplitz matrix.
The system of the infinite inhomogeneous linear algebraic equations (7.5.1) becomes

$$\sum_{m=-\infty}^{\infty} M_{n-m} X_m = f_n, \quad -\infty < n < \infty. \tag{7.5.4}$$

We look for the solution $\{X_m\}_{m=-\infty}^{+\infty}$ under the assumption of the *uniform convergence* of $\{X_m\}_{m=-\infty}^{+\infty}$ and $\{M_m\}_{m=-\infty}^{+\infty}$,

$$\sum_{m=-\infty}^{\infty} |X_m| < \infty \quad \text{and} \quad \sum_{m=-\infty}^{\infty} |M_m| < \infty. \tag{7.5.5}$$

Multiplying Eq. (7.5.4) by ξ^n and summing over n, we have

$$\sum_{n,m} \xi^n M_{n-m} X_m = \sum_n \xi^n f_n. \tag{7.5.6}$$

The left-hand side of Eq. (7.5.6) can be expressed as

$$\sum_{n,m} \xi^n M_{n-m} X_m = \sum_m \xi^m X_m \sum_n \xi^{n-m} M_{n-m} = M(\xi) X(\xi),$$

with the interchange of the order of the summations, where $X(\xi)$ and $M(\xi)$ are defined by

$$X(\xi) \equiv \sum_{n=-\infty}^{\infty} X_n \xi^n, \quad M(\xi) \equiv \sum_{n=-\infty}^{\infty} M_n \xi^n. \tag{7.5.7,8}$$

We also define

$$f(\xi) \equiv \sum_{n=-\infty}^{\infty} f_n \xi^n. \tag{7.5.9}$$

Thus Eq. (7.5.6) takes the following form:

$$M(\xi)X(\xi) = f(\xi). \tag{7.5.10}$$

We assume the following bounds on M_n and f_n:

$$|M_n| = \begin{cases} O(a^{-|n|}) & \text{as} \quad n \to -\infty, \\ O(b^{-n}) & \text{as} \quad n \to +\infty, \end{cases} \qquad a, b > 0, \tag{7.5.11a}$$

$$|f_n| = \begin{cases} O(c^{-|n|}) & \text{as} \quad n \to -\infty, \\ O(d^{-n}) & \text{as} \quad n \to +\infty, \end{cases} \qquad c, d > 0. \tag{7.5.11b}$$

Then $M(\xi)$ is analytic in the annulus in the complex ξ plane,

$$1/a < |\xi| < b, \quad 1 < ab, \tag{7.5.12a}$$

and $f(\xi)$ is analytic in the annuls in the complex ξ plane,

$$1/c < |\xi| < d, \quad 1 < cd. \tag{7.5.12b}$$

Hence we obtain

$$X(\xi) = \frac{f(\xi)}{M(\xi)} = \sum_{n=-\infty}^{\infty} X_n \xi^n \quad \text{provided} \quad M(\xi) \neq 0. \tag{7.5.13}$$

$X(\xi)$ is analytic in the annulus

$$\max(1/a, 1/c) < |\xi| < \min(b, d). \tag{7.5.12c}$$

By the *Fourier series inversion formula* on the *unit circle*, we obtain

$$X_n = \frac{1}{2\pi} \int_0^{2\pi} d\theta \exp[-in\theta] X(\exp[i\theta]) = \frac{1}{2\pi i} \oint_{|\xi|=1} d\xi \xi^{-n-1} X(\xi), \tag{7.5.14}$$

with

$$M(\xi) \neq 0 \quad \text{for} \quad |\xi| = 1. \tag{7.5.15}$$

Next, consider the *eigenvalue problem* of the following form:

$$\sum_{m=-\infty}^{\infty} M_{n-m}X_m = \mu X_n. \tag{7.5.16}$$

We try

$$X_m = \xi^m. \tag{7.5.17}$$

Then we obtain

$$M(\xi) = \mu. \tag{7.5.18}$$

The roots of Eq. (7.5.18) provide the solutions to Eq. (7.5.16). For $\mu = 0$, we obtain

$$X_m = (\xi_0)^m, \tag{7.5.19}$$

where ξ_0 is a zero of $M(\xi)$.

Case B: Semi-infinite Toeplitz matrix.

The system of the semi-infinite inhomogeneous linear algebraic equations (7.5.1) becomes

$$\sum_{m=0}^{\infty} M_{n-m}X_m = f_n, \quad n \geq 0, \tag{7.5.20a}$$

which is the *inhomogeneous Wiener–Hopf sum equation*.

We let

$$M(\xi) \equiv \sum_{n=-\infty}^{\infty} M_n \xi^n, \quad 0 \leq \arg\xi \leq 2\pi, \tag{7.5.21}$$

be such that $M(\exp[i\theta]) \neq 0$ for $0 \leq \theta \leq 2\pi$. We assume that

$$\sum_{n=0}^{\infty} |f_n| < \infty, \tag{7.5.22}$$

and

$$|X_m| < (1+\varepsilon)^{-m}, \quad m \geq 0, \quad \varepsilon > 0. \tag{7.5.23}$$

We define

$$y_n \equiv \sum_{m=0}^{\infty} M_{n-m}X_m \quad \text{for} \quad n \leq -1 \quad \text{and} \quad y_n \equiv 0 \quad \text{for} \quad n \geq 0, \tag{7.5.24a}$$

$$f_n \equiv 0 \quad \text{for} \quad n \leq -1, \tag{7.5.24b}$$

$$\overline{X}(\xi) \equiv \sum_{n=0}^{\infty} X_n \xi^n, \tag{7.5.25}$$

$$\overline{f}(\xi) \equiv \sum_{n=0}^{\infty} f_n \xi^n, \tag{7.5.26}$$

$$Y(\xi) \equiv \sum_{n=-\infty}^{-1} y_n \xi^n. \tag{7.5.27}$$

We look for the solution which satisfies

$$\sum_{n=0}^{\infty} |X_n| < \infty. \tag{7.5.28}$$

Then Eq. (7.5.20a) is rewritten as

$$\sum_{m=-\infty}^{\infty} M_{n-m} X_m = f_n + y_n \quad \text{for} -\infty < n < \infty. \tag{7.5.20b}$$

We note that

$$\sum_{n=-\infty}^{\infty} |y_n| = \sum_{n=-\infty}^{\infty} \left| \sum_{m=0}^{\infty} M_{n-m} X_m \right| < \sum_{n=-\infty}^{\infty} |M_n| \sum_{m=0}^{\infty} |X_m| < \infty,$$

where the interchange of the order of the summation is justified since the final expression is finite. Multiplying by $\exp[in\theta]$ on both sides of Eq. (7.5.20b) and summing over n, we obtain

$$M(\xi)\overline{X}(\xi) = \overline{f}(\xi) + Y(\xi). \tag{7.5.29}$$

Homogeneous problem: We set

$$f_n = 0 \quad \text{or} \quad \overline{f}(\xi) = 0.$$

The problem we will solve first is

$$M(\xi)\overline{X}(\xi) = Y(\xi), \tag{7.5.30}$$

where

$$\begin{array}{lll} \overline{X}(\xi) & \text{analytic for} & |\xi| < 1+\varepsilon, \\ Y(\xi) & \text{analytic for} & |\xi| > 1. \end{array} \tag{7.5.31}$$

We define the *index* ν of $M(\xi)$ in the counter-clockwise direction by

$$\nu \equiv \frac{1}{2\pi i} \ln[M(\exp[i\theta])]\Big|_{\theta=0}^{\theta=2\pi} . \tag{7.5.32}$$

Suppose that $M(\xi)$ has been factored into the following form:

$$M(\xi) = N_{\text{in}}(\xi)/N_{\text{out}}(\xi), \tag{7.5.33a}$$

where

$$\left\{ \begin{array}{lllll} N_{\text{in}}(\xi) & \text{analytic for} & |\xi| < 1+\varepsilon, & \text{and continuous for} & |\xi| \le 1, \\ N_{\text{out}}(\xi) & \text{analytic for} & |\xi| > 1, & \text{and continuous for} & |\xi| \ge 1. \end{array} \right. \tag{7.5.33b}$$

Then Eq. (7.5.30) is rewritten as

$$N_{\text{in}}(\xi)\overline{X}(\xi) = N_{\text{out}}(\xi)Y(\xi) \equiv G(\xi), \tag{7.5.34}$$

where $G(\xi)$ is entire in the complex ξ plane. The form of $G(\xi)$ is examined.

Case 1: the *index* $\nu = 0$.
By the now familiar formula, Eq. (7.3.25), we have

$$\oint \frac{\ln[M(\xi')]}{\xi' - \xi} \frac{d\xi'}{2\pi i} = \left(\oint_{C_1} - \oint_{C_2} \right) \left[\frac{\ln[M(\xi')]}{\xi' - \xi} \frac{d\xi'}{2\pi i} \right], \tag{7.5.35}$$

where the integration contours, C_1 and C_2, are displayed in Figure 7.16.
Thus we have

$$N_{\text{in}}(\xi) = \exp\left[\oint_{C_1} \frac{\ln[M(\xi')]}{\xi' - \xi} \frac{d\xi'}{2\pi i} \right] \to 1 \quad \text{as} \quad |\xi| \to \infty, \tag{7.5.36a}$$

$$N_{\text{out}}(\xi) = \exp\left[\oint_{C_2} \frac{\ln[M(\xi')]}{\xi' - \xi} \frac{d\xi'}{2\pi i} \right] \to 1 \quad \text{as} \quad |\xi| \to \infty. \tag{7.5.36b}$$

We find by Liouville's theorem,

$$G(\xi) = 0, \tag{7.5.37}$$

and hence we have *no nontrivial homogeneous solution* when $\nu = 0$.

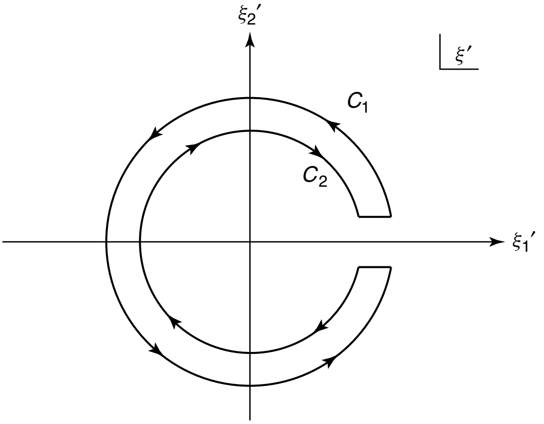

Fig. 7.16 Integration contour of $\ln[M(\xi')]$, when the index $\nu = 0$.

Case 2: the *index* $\nu > 0$ (*positive integer*).
We construct the *object with the index zero,* $M(\xi)/\xi^{\nu}$ and obtain

$$M(\xi) = \xi^{\nu} \exp\left(\oint_{C_1} - \oint_{C_2}\right)\left[\frac{\ln[M(\xi')/(\xi')^{\nu}]}{\xi' - \xi}\frac{d\xi'}{2\pi i}\right], \tag{7.5.38}$$

from which, we obtain

$$N_{\text{in}}(\xi) = \xi^{\nu} \exp\left[\oint_{C_1} \frac{\ln[M(\xi')/(\xi')^{\nu}]}{\xi' - \xi}\frac{d\xi'}{2\pi i}\right] \to \xi^{\nu} \quad \text{as} \quad |\xi| \to \infty, \tag{7.5.39a}$$

$$N_{\text{out}}(\xi) = \exp\left[\oint_{C_2} \frac{\ln[M(\xi')/(\xi')^{\nu}]}{\xi' - \xi}\frac{d\xi'}{2\pi i}\right] \to 1 \quad \text{as} \quad |\xi| \to \infty. \tag{7.5.39b}$$

By Liouville's theorem, $G(\xi)$ cannot grow as fast as ξ^{ν}. Hence we have

$$G(\xi) = \sum_{m=0}^{\nu-1} G_m \xi^m, \tag{7.5.40}$$

where G_m's are ν arbitrary constants. Thus $\overline{X}(\xi)$ is given by

$$\overline{X}(\xi) = \sum_{m=0}^{\nu-1} G_m \xi^m / N_{\text{in}}(\xi), \tag{7.5.41a}$$

from which, we obtain ν *independent homogeneous solutions* X_n's by the *Fourier series inversion formula* on the *unit circle,*

$$X_n = \frac{1}{2\pi}\int_0^{2\pi} d\theta \exp[-in\theta] X(\exp[i\theta]) = \frac{1}{2\pi i}\oint_{|\xi|=1} d\xi\, \xi^{-n-1} X(\xi). \tag{7.5.41b}$$

Case 3: the *index* $\nu < 0$ (*negative integer*).
We construct the *object with the index zero,* $M(\xi)\xi^{|\nu|}$, and obtain

$$M(\xi) = \frac{1}{\xi^{|\nu|}} \exp\left(\oint_{C_1} - \oint_{C_2}\right)\left[\frac{\ln[M(\xi')(\xi')^{|\nu|}]}{\xi' - \xi}\frac{d\xi'}{2\pi i}\right], \tag{7.5.42}$$

from which, we obtain

$$N_{\text{in}}(\xi) = \frac{1}{\xi^{|\nu|}} \exp\left[\oint_{C_1} \frac{\ln[M(\xi')(\xi')^{|\nu|}]}{\xi' - \xi}\frac{d\xi'}{2\pi i}\right] \to \frac{1}{\xi^{|\nu|}} \quad \text{as} \quad |\xi| \to \infty, \tag{7.5.43a}$$

$$N_{\text{out}}(\xi) = \exp\left[\oint_{C_2} \frac{\ln[M(\xi')(\xi')^{|\nu|}]}{\xi' - \xi}\frac{d\xi'}{2\pi i}\right] \to 1 \quad \text{as} \quad |\xi| \to \infty. \tag{7.5.43b}$$

By Liouville's theorem, we have

$$G(\xi) = 0, \tag{7.5.44}$$

and hence we have *no nontrivial solution.*

Inhomogeneous problem: We restate the inhomogeneous problem below,

$$M(\xi)\overline{X}(\xi) = \overline{f}(\xi) + Y(\xi), \tag{7.5.45}$$

where we assume that $\overline{f}(\xi)$ is analytic for $|\xi| < 1 + \varepsilon$. Factoring $M(\xi)$ as before,

$$M(\xi) = N_{\text{in}}(\xi)/N_{\text{out}}(\xi), \tag{7.5.46}$$

and multiplying by $N_{\text{out}}(\xi)$ on both sides of Eq. (7.5.45), we have

$$\overline{X}(\xi)N_{\text{in}}(\xi) = \overline{f}(\xi)N_{\text{out}}(\xi) + Y(\xi)N_{\text{out}}(\xi), \tag{7.5.47a}$$

or, splitting $\overline{f}(\xi)N_{\text{out}}(\xi)$ into a sum of the *in* function and the *out* function,

$$\overline{X}(\xi)N_{\text{in}}(\xi) - [\overline{f}(\xi)N_{\text{out}}(\xi)]_{\text{in}} = [\overline{f}(\xi)N_{\text{out}}(\xi)]_{\text{out}} + Y(\xi)N_{\text{out}}(\xi) \equiv F(\xi), \tag{7.5.47b}$$

where $F(\xi)$ is entire in the complex ξ plane. Since we are intent to obtain *one particular solution* to Eq. (7.5.45), we set in Eq. (7.5.47b)

$$F(\xi) = 0, \tag{7.5.48}$$

resulting in the particular solution,

$$\overline{X}_{\text{part}}(\xi) = [\overline{f}(\xi)N_{\text{out}}(\xi)]_{\text{in}}/N_{\text{in}}(\xi), \tag{7.5.49a}$$

$$Y(\xi) = -[\overline{f}(\xi)N_{\text{out}}(\xi)]_{\text{out}}/N_{\text{out}}(\xi). \tag{7.5.49b}$$

The fact that Eqs. (7.5.49a) and (7.5.49b) satisfy Eq. (7.5.47a) can be easily demonstrated. From Eq. (7.5.49a), the particular solution, $X_{n,\text{part}}$'s can be obtained by the *Fourier series inversion formula* on the *unit circle*.

We note that in writing Eq. (7.5.47b), the following property of $N_{\text{out}}(\xi)$ is essential:

$$N_{\text{out}}(\xi) \to 1 \quad \text{as} \quad |\xi| \to \infty. \tag{7.5.50}$$

Case 1: the *index* $\nu = 0$.
Since the homogeneous problem has no nontrivial solution, the *unique particular solution*, $X_{n,\text{part}}$'s, is obtained for the inhomogeneous problem.

Case 2: the *index* $\nu > 0$ (*positive integer*).
In this case, since the homogeneous problem has ν independent solutions, the solution to the inhomogeneous problem is *not unique*.

Case 3: the *index* $\nu < 0$ (*negative integer*).
In this case, consider the *homogeneous adjoint problem*,

$$\sum_{m=0}^{\infty} M_{m-n} X_m^{\text{adj}} = 0. \tag{7.5.51}$$

Its M function, $M^{\text{adj}}(\xi)$, is defined by

$$M^{\text{adj}}(\xi) \equiv \sum_{n=-\infty}^{+\infty} M_{-n}\xi^n = \sum_{n=-\infty}^{+\infty} M_n \xi^{-n} = M(1/\xi). \tag{7.5.52}$$

The index ν^{adj} of $M^{\text{adj}}(\xi)$ is defined by

$$\begin{aligned} \nu^{\text{adj}} &\equiv \frac{1}{2\pi i} \ln[M^{\text{adj}}(\exp[i\theta])]\Big|_{\theta=0}^{\theta=2\pi} = \frac{1}{2\pi i} \ln[M(\exp[-i\theta])]\Big|_{\theta=0}^{\theta=2\pi} \\ &= \frac{1}{2\pi i} \ln[\{M(\exp[i\theta])\}^*]\Big|_{\theta=0}^{\theta=2\pi} = -\frac{1}{2\pi i} \ln[M(\exp[i\theta])]\Big|_{\theta=0}^{\theta=2\pi} = -\nu. \end{aligned} \tag{7.5.53}$$

The factorization of $M^{\text{adj}}(\xi)$ is carried out as in the case of $M(\xi)$, with the result

$$M^{\text{adj}}(\xi) = N_{\text{in}}^{\text{adj}}(\xi)/N_{\text{out}}^{\text{adj}}(\xi) = M(1/\xi) = N_{\text{in}}(1/\xi)/N_{\text{out}}(1/\xi). \tag{7.5.54}$$

From this, we recognize that

$$\begin{cases} N_{\text{in}}^{\text{adj}}(\xi) = N_{\text{out}}^{-1}(1/\xi) & \text{analytic in} \quad |\xi| < 1, \quad \text{and continuous for} \quad |\xi| \le 1, \\ N_{\text{out}}^{\text{adj}}(\xi) = N_{\text{in}}^{-1}(1/\xi) & \text{analytic in} \quad |\xi| > 1, \quad \text{and continuous for} \quad |\xi| \ge 1. \end{cases} \tag{7.5.55}$$

Then, in this case, the homogeneous adjoint problem has $|\nu|$ independent solutions,

$$X_m^{\mathrm{adj}(j)} \quad j = 1, \ldots, |\nu|, \quad m \geq 0. \tag{7.5.56}$$

By the argument similar to the derivation of the solvability condition for the inhomogeneous Wiener–Hopf integral equation of the second kind discussed in Section 7.4, noting Eq. (7.5.50), we obtain the *solvability condition for the inhomogeneous Wiener–Hopf sum equation* as follows:

$$\sum_{m=0}^{\infty} f_m X_m^{\mathrm{adj}(j)} = 0, \quad j = 1, \ldots, |\nu|. \tag{7.5.57}$$

Thus, *if and only if the solvability condition (7.5.57) is satisfied, i.e., the inhomogeneous term f_m is orthogonal to all the $|\nu|$ independent solutions $X_m^{\mathrm{adj}(j)}$ to the homogeneous adjoint problem (7.5.51)*, the inhomogeneous Wiener–Hopf sum equation has the *unique solution*, $X_{n,\mathrm{part}}$.

From this analysis of the *inhomogeneous Wiener–Hopf sum equation*, we find that the problem at hand is the *discrete analog of the inhomogeneous Wiener–Hopf integral equation of the second kind, not of the first kind*, despite its formal appearance.

For an interesting application of the Wiener–Hopf sum equation to the phase transition of the two-dimensional Ising model, the reader is referred to the article by *T.T. Wu*, cited in the bibliography.

For another interesting application of the Wiener–Hopf sum equation to the Yagi–Uda semi-infinite arrays, the reader is referred to the articles by *W. Wasylkiwskyj* and *A.L. VanKoughnett*, cited in the bibliography.

The Cauchy integral formula used in this section should actually be Pollard's theorem which is the generalization of the Cauchy integral formula. We avoided the mathematical technicalities in the presentation of the Wiener–Hopf sum equation.

As for the mathematical details related to the Wiener–Hopf sum equation, Liouville's theorem, the Wiener–Lévy theorem, and Pollard's theorem, we refer the reader to Chapter IX of the book by *B. McCoy* and *T.T. Wu*, cited in the bibliography.

Summary of the Wiener–Hopf sum equation:

$$\sum_{m=0}^{\infty} M_{n-m} X_m = f_n, \quad n \geq 0,$$

$$\sum_{m=0}^{\infty} M_{m-n} X_m^{\mathrm{adj}} = 0, \quad n \geq 0.$$

(1) Index $\nu = 0$.
Homogeneous problem has no nontrivial solution.
Homogeneous adjoint problem has no nontrivial solution.
Inhomogeneous problem has a unique solution.
(2) Index $\nu > 0$.
Homogeneous problem has ν independent nontrivial solutions.
Homogeneous adjoint problem has no nontrivial solution.
Inhomogeneous problem has nonunique solutions.
(3) Index $\nu < 0$.
Homogeneous problem has no nontrivial solution.
Homogeneous adjoint problem has $|\nu|$ independent nontrivial solutions.
Inhomogeneous problem has a unique solution, if and only if the inhomogeneous term is orthogonal to all $|\nu|$ independent solutions to the homogeneous adjoint problem.

7.6 Wiener–Hopf Integral Equation of the First Kind and Dual Integral Equations

In this section, we shall reexamine the *mixed boundary value problem* considered in Section 7.1 with some generality and show its equivalence to the *Wiener–Hopf integral equation of the first kind* and to the *dual integral equations*. This equivalence does not constitute a solution to the original problem; rather it provides a hint for solving the Wiener–Hopf integral equation of the first kind and the dual integral equations .

❑ **Example 7.5.** Solve the mixed boundary value problem of two-dimensional Laplace equation in half plane:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi(x, y) = 0, \quad y \geq 0, \tag{7.6.1}$$

with the boundary conditions specified on the x-axis,

$$\phi(x, 0) = f(x), x \geq 0; \phi_y(x, 0) = g(x), x < 0; \phi(x, y) \to 0 \text{ as } x^2 + y^2 \to \infty. \tag{7.6.2}$$

Solution. We write

$$\phi(x, y) = \lim_{\varepsilon \to 0^+} \int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{ikx - \sqrt{k^2+\varepsilon^2}y} \hat{\phi}(k), \quad y \geq 0. \tag{7.6.3}$$

Setting $y = 0$ in Eq. (7.6.3),

$$\int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{ikx} \hat{\phi}(k) = \begin{cases} f(x), & x \geq 0, \\ \phi(x, 0), & x < 0. \end{cases} \tag{7.6.4}$$

Thus we have

$$\hat{\phi}(k) = \int_{-\infty}^{+} \infty dx e^{-ikx}\phi(x,0) = \hat{\phi}_+(k) + \hat{f}_-(k), \tag{7.6.5}$$

where

$$\hat{\phi}_+(k) = \int_{-\infty}^{0} dx e^{-ikx}\phi(x,0), \tag{7.6.6}$$

$$\hat{f}_-(k) = \int_{0}^{+\infty} dx e^{-ikx} f(x). \tag{7.6.7}$$

We know that $\hat{\phi}_+(k)$ ($\hat{f}_-(k)$) is analytic in the upper half plane (the lower half plane). Differentiating Eq. (7.6.3) with respect to y, and setting $y = 0$, we have

$$\int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{ikx} \left(-\sqrt{k^2+\varepsilon^2}\right) \hat{\phi}(k) = \begin{cases} \phi_y(x,0), & x \geq 0, \\ g(x), & x < 0. \end{cases} \tag{7.6.8}$$

Then, by inversion, we obtain

$$\left(-\sqrt{k^2+\varepsilon^2}\right) \hat{\phi}(k) = \hat{g}_+(k) + \hat{\psi}_-(k), \tag{7.6.9}$$

where

$$\hat{g}_+(k) = \int_{-\infty}^{0} dx e^{-ikx} g(x), \tag{7.6.10}$$

$$\hat{\psi}_-(k) = \int_{0}^{+\infty} dx e^{-ikx} \phi_y(x,0). \tag{7.6.11}$$

As before, we know that $\hat{g}_+(k)$ ($\hat{\psi}_-(k)$) is analytic in the upper half plane (the lower half plane).

Eliminating $\hat{\phi}(k)$ from Eqs. (7.6.5) and (7.6.9), we obtain

$$\hat{\phi}_+(k) + \hat{f}_-(k) = \left(-\frac{1}{\sqrt{k^2+\varepsilon^2}}\right) \hat{g}_+(k) + \left(-\frac{1}{\sqrt{k^2+\varepsilon^2}}\right) \hat{\psi}_-(k). \tag{7.6.12}$$

Inverting Eq. (7.6.12) for $x > 0$, we obtain

$$\begin{aligned} &\int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{ikx} (\hat{\phi}_+(k) + \hat{f}_-(k) + \frac{1}{\sqrt{k^2+\varepsilon^2}} \hat{g}_+(k)) \\ &= \int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{ikx} \left(-\frac{1}{\sqrt{k^2+\varepsilon^2}}\right) \hat{\psi}_-(k), \quad x > 0, \end{aligned} \tag{7.6.13}$$

where

$$\int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{ikx} \hat{\phi}_+(k) = 0, \quad \text{for} \quad x > 0, \tag{7.6.14}$$

because $\hat{\phi}_+(k)$ is analytic in the upper half plane and the contour of the integration is closed in the upper half plane for $x > 0$.

The remaining terms on the left-hand side of Eq. (7.6.13) are identified as

$$\int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{ikx} \hat{f}_-(k) = f(x), \quad x > 0, \tag{7.6.15}$$

$$\int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{ikx} \frac{1}{\sqrt{k^2 + \varepsilon^2}} \hat{g}_+(k) \equiv G(x), \quad x > 0. \tag{7.6.16}$$

The right-hand side of Eq. (7.6.13) is identified as

$$\int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{ikx} \left(-\frac{1}{\sqrt{k^2 + \varepsilon^2}} \right) \hat{\psi}_-(k) = \int_0^{+\infty} \psi(y) K(x - y) dy, \quad x > 0, \tag{7.6.17}$$

where $\psi(x)$ and $K(x)$ are defined by

$$\psi(x) \equiv \phi_y(x, 0), \quad x > 0, \tag{7.6.18}$$

$$K(x) \equiv \int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{ikx} \left(-\frac{1}{\sqrt{k^2 + \varepsilon^2}} \right). \tag{7.6.19}$$

Thus we obtain the integral equation for $\psi(x)$ from Eq. (7.6.13),

$$\int_0^{+\infty} K(x - y) \psi(y) dy = f(x) + G(x), \quad x > 0. \tag{7.6.20}$$

This is the integral equation with a translational kernel of semi-infinite range and is called the *Wiener–Hopf integral equation of the first kind*. As noted earlier, this reduction of the mixed boundary value problem, Eqs. (7.6.1)–(7.6.2c), to the Wiener–Hopf integral equation of the first kind by no means constitutes a solution to the original mixed boundary value problem.

In order to solve the Wiener–Hopf integral equation of the first kind

$$\int_0^{+\infty} K(x - y) \psi(y) dy = F(x), \quad x \geq 0, \tag{7.6.21}$$

we rather argue backward. Equation (7.6.21) is reduced to the form of Eq. (7.6.12). We define the left-hand side of Eq. (7.6.21) for $x < 0$ by

$$\int_0^{+\infty} K(x - y) \psi(y) dy = H(x), \quad x < 0. \tag{7.6.22}$$

We consider the Fourier transforms of Eqs. (7.6.21) and (7.6.22),

$$\int_0^{+\infty} dx e^{-ikx} \int_0^{+\infty} dy K(x-y)\psi(y) = \int_0^{+\infty} dx e^{-ikx} F(x) \equiv \hat{F}_-(k), \tag{7.6.23a}$$

$$\int_{-\infty}^{0} dx e^{-ikx} \int_0^{+\infty} dy K(x-y)\psi(y) = \int_{-\infty}^{0} dx e^{-ikx} H(x) \equiv \hat{H}_+(k), \tag{7.6.23b}$$

where $\hat{F}_-(k)$ $(\hat{H}_+(k))$ is analytic in the lower half plane (the upper half plane). Adding Eqs. (7.6.23a) and (7.6.23b) together, we obtain

$$\int_0^{+\infty} dy e^{-iky}\psi(y) \int_{-\infty}^{+\infty} dx e^{-ik(x-y)} K(x-y) = \hat{F}_-(k) + \hat{H}_+(k).$$

Hence we have

$$\hat{\psi}_-(k)\hat{K}(k) = \hat{F}_-(k) + \hat{H}_+(k), \tag{7.6.24}$$

where $\hat{\psi}_-(k)$ and $\hat{K}(k)$, respectively, are defined by

$$\hat{\psi}_-(k) \equiv \int_0^{+\infty} e^{-ikx}\psi(x)dx, \tag{7.6.25}$$

$$\hat{K}(k) \equiv \int_{-\infty}^{+\infty} e^{-ikx} K(x)dx. \tag{7.6.26}$$

From Eq. (7.6.24), we have

$$\hat{\psi}_-(k) = (1/\hat{K}(k))(\hat{F}_-(k) + \hat{H}_+(k)). \tag{7.6.27}$$

Carrying out the sum splitting on the right-hand side of Eq. (7.6.27) either by inspection or by the general method discussed in Section 7.3, we can obtain $\hat{\psi}_-(k)$ as in Section 7.1.

Returning to Example 7.5, we note that the mixed boundary value problem we examined belongs to the general class of the equation,

$$\hat{\phi}_+(k) + \hat{f}_-(k) = \hat{K}(k)(\hat{g}_+(k) + \hat{\psi}_-(k)). \tag{7.6.28}$$

If we directly invert for $\hat{\psi}_-(k)$ for $x > 0$ in Eq. (7.6.28), we obtain the Wiener–Hopf integral equation of the first kind (7.6.20). Instead, we may write Eq. (7.6.28) as a pair of equations,

$$\Phi(k) = \hat{g}_+(k) + \hat{\psi}_-(k), \tag{7.6.29a}$$

$$\hat{K}(k)\Phi(k) = \hat{\phi}_+(k) + \hat{f}_-(k). \tag{7.6.29b}$$

Inverting Eqs. (7.6.29a) and (7.6.29b) for $x < 0$ and $x \geq 0$, respectively, we find a pair of integral equations for $\Phi(k)$ of the following form:

$$\int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{ikx} \Phi(k) = g(x), \quad x < 0, \tag{7.6.30a}$$

$$\int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{ikx} \hat{K}(k)\Phi(k) = f(x), \quad x \geq 0. \tag{7.6.30b}$$

A pair of integral equations, one holding in some range of the independent variable and the other holding in the complementary range, are called the *dual integral equations*. This pair is equivalent to the mixed boundary value problem, Eqs. (7.6.1)–(7.6.2c). A solution to the dual integral equations is again provided by the methods we developed in Sections 7.1 and 7.3.

7.7
Problems for Chapter 7

7.1. (due to H. C.). Solve the Sommerfeld diffraction problem in two spatial dimensions with the boundary condition

$$\phi_x(x, 0) = 0 \quad \text{for} \quad x < 0.$$

7.2. (due to H. C.). Solve the half-line problem

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - p^2\right)\phi(x, y) = 0 \quad \text{with} \quad \phi(x, 0) = e^x \quad \text{for} \quad x \leq 0,$$

and

$$\phi(x, y) \to 0 \quad \text{as} \quad x^2 + y^2 \to \infty.$$

It is assumed that $\phi(x, y)$ and $\phi_y(x, y)$ are continuous except on the half-line $y = 0$ with $x \leq 0$.

7.3. (Due to D. M.) Solve the boundary value problem,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + p^2\right)\phi(x, y) = 0 \quad \text{with} \quad \phi_y(x, 0) = e^{i\alpha x} \quad \text{for} \quad x \geq 0,$$

and

$$\sqrt{x^2 + y^2}(\phi_y(x, y) + ip\phi(x, y)) \to 0 \quad \text{as} \quad \sqrt{x^2 + y^2} \to \infty,$$

by using the Wiener–Hopf method. In the above, the point (x, y) lies in the entire (x, y)-plane with the exception of the half-line,

$$\{(x, y) : y = 0, x \geq 0\},$$

p is real and positive, and

$$0 < \alpha < p.$$

We note that the given condition for $\sqrt{x^2 + y^2} \to \infty$ is called the Sommerfeld radiation condition.

7.4. (Due to D. M.) Solve the boundary value problem,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - p^2\right)\phi(x, y) = 0 \quad \text{with} \quad \phi_y(x, 0) = e^{i\alpha x} \quad \text{for} \quad x \geq 0,$$

and

$$\phi(x, y) \to 0 \quad \text{as} \quad \sqrt{x^2 + y^2} \to \infty,$$

by using the Wiener–Hopf method. In this problem, the point (x, y) lies in the region stated in the previous problem. Note that the Sommerfeld radiation condition is now replaced by the usual condition of zero limit. Compare your answer with the previous one.

7.5. (due to H. C.). Solve

$$\nabla^2\phi(x, y) = 0,$$

with a cut on the positive x-axis, subject to the boundary conditions

$$\phi(x, 0) = e^{-ax} \quad \text{for} \quad x \geq 0,$$

$$\phi(x, y) \to 0 \quad \text{as} \quad x^2 + y^2 \to \infty.$$

7.6. (due to H. C.). Solve

$$\nabla^2\phi(x, y) = 0, \quad 0 < y < 1,$$

subject to the boundary conditions

$$\phi(x, 0) = 0 \quad \text{for} \quad x \geq 0,$$

$$\phi(x,1) = e^{-x} \quad \text{for} \quad x \geq 0,$$

and

$$\phi_y(x,1) = 0 \quad \text{for} \quad x < 0.$$

7.7. (due to H. C.). Solve

$$\nabla^2\phi(x,y) = 0, \quad 0 < y < 1,$$

subject to the boundary conditions

$$\phi_y(x,0) = 0 \quad \text{for} \quad -\infty < x < \infty, \quad \phi_y(x,1) = 0 \quad \text{for} \quad x < 0,$$

and

$$\phi(x,1) = e^{-x} \quad \text{for} \quad x \geq 0.$$

7.8. (due to H. C.). Solve

$$\left(\frac{\partial^2}{\partial x^2} + 2\frac{\partial}{\partial x} + \frac{\partial^2}{\partial y^2}\right)\phi(x,y) = \phi(x,y), \quad y > 0,$$

with the boundary conditions

$$\phi(x,0) = e^{-x} \quad \text{for} \quad x > 0, \quad \phi_y(x,0) = 0 \quad \text{for} \quad x < 0,$$

and

$$\phi(x,y) \to 0 \quad \text{as} \quad x^2 + y^2 \to \infty.$$

7.9. (due to H. C.). Solve

$$\phi(x) = \lambda \int_0^{+\infty} K(x-y)\phi(y)dy, \quad x \geq 0,$$

with

$$K(x) \equiv \int_{-\infty}^{+\infty} e^{-ikx} \frac{1}{\sqrt{k^2+1}} \frac{dk}{2\pi} \quad \text{and} \quad \lambda > 1.$$

Find also the resolvent $H(x,y)$ of this kernel.

7.10. (due to H. C.). Solve

$$\phi(x) = \lambda \int_0^{+\infty} e^{-(x-y)^2} \phi(y) dy, \quad 0 \le x < \infty.$$

7.11. Solve

$$\phi(x) = \frac{\lambda}{2} \int_0^{+\infty} E_1(|x-y|)\phi(y) dy, \quad x \ge 0, \quad 0 < \lambda \le 1,$$

with

$$E_1(x) \equiv \int_x^{+\infty} (e^{-\zeta}/\zeta) d\zeta.$$

7.12. (due to H. C.). Consider the eigenvalue equation,

$$\phi(x) = \lambda \int_0^{+\infty} K(x-y)\phi(y) dy, \quad \text{where} \quad K(x) = x^2 e^{-x^2}.$$

(a) What is the behavior of $\phi(x)$ so that the integral above is convergent?

(b) What is the behavior of $\psi(x)$ as $x \to -\infty$ (where $\psi(x)$ is the integral above for $x < 0$)? What is the region of analyticity for $\hat{\psi}(k)$?

(c) Find $\hat{K}(k)$. What is the region of analyticity for $\hat{K}(k)$?

(d) It is required that $\phi(x)$ does not blow up faster than a polynomial of x as $x \to \infty$. Find the spectrum of λ and the number of independent eigenfunctions for each eigenvalue λ.

7.13. Solve

$$\phi(x) = e^{-|x|} + \lambda \int_0^{+\infty} e^{-|x-y|} \phi(y) dy, \quad x \ge 0.$$

Hint:

$$\int_{-\infty}^{\infty} dx e^{ikx} e^{-|x|} = \frac{2}{k^{2.} + 1}.$$

7.14. (due to H. C.). Solve

$$\phi(x) = \cosh \frac{x}{2} + \lambda \int_0^{+\infty} e^{-|x-y|} \phi(y) dy, \quad x \ge 0.$$

Hint:

$$\int_{-\infty}^{+\infty} \frac{e^{ikx}}{\cosh x} dx = \frac{\pi}{\cosh(\pi k/2)}.$$

7.15. (due to H. C.). Solve

$$\phi(x) = 1 + \lambda \int_0^1 \frac{1}{x + x'} \phi(x')dx', \quad 0 \le x \le 1.$$

Hint: Perform the change of the variables from x and x' to t and t',

$$x = \exp(-t) \quad \text{and} \quad x' = \exp(-t') \quad \text{with} \quad t, t' \in [0, +\infty).$$

7.16. Solve

$$\phi(x) = \lambda \int_0^{+\infty} \frac{1}{\alpha^2 + (x-y)^2} \phi(y)dy + f(x), \quad x \ge 0, \quad \alpha > 0.$$

7.17. Solve

$$T_{n+1}(z) + T_{n-1}(z) = 2zT_n(z), \quad n \ge 1, \quad -1 \le z \le 1,$$

with

$$T_0(z) = 1 \quad \text{and} \quad T_1(z) = z.$$

Hint: Factor the $M(\xi)$ function by inspection.

7.18. Solve

$$U_{n+1}(z) + U_{n-1}(z) = 2zU_n(z), \quad n \ge 1, \quad -1 \le z \le 1,$$

with

$$U_0(z) = 0 \quad \text{and} \quad U_1(z) = \sqrt{1 - z^2}.$$

7.19. Solve

$$\sum_{k=0}^{\infty} \exp[i\rho \left|j - k\right|]\xi_k - \lambda\xi_j = q^j, \quad j = 0, 1, 2, \ldots,$$

with

$$\operatorname{Im} \rho > 0 \quad \text{and} \quad \left|q\right| < 1.$$

7.20. (due to H. C.). Solve the inhomogeneous Wiener–Hopf sum equation that originates from the two-dimensional Ising model

$$\sum_{m=0}^{\infty} M_{n-m} X_m = f_n, \quad n \geq 0,$$

with

$$M(\xi) = \sum_{n=-\infty}^{\infty} M_n \xi^n \equiv \sqrt{\frac{(1-\alpha_1\xi)(1-\alpha_2\xi^{-1})}{(1-\alpha_1\xi^{-1})(1-\alpha_2\xi)}} \quad \text{and} \quad f_n = \delta_{n0}.$$

Consider the following five cases:

(a) $\alpha_1 < 1 < \alpha_2$,

(b) $\alpha_1 < \alpha_2 < 1$,

(c) $\alpha_1 < \alpha_2 = 1$,

(d) $1 < \alpha_1 < \alpha_2$,

(e) $\alpha_1 = \alpha_2$.

Hint: Factorize the $M(\xi)$ function by inspection for the above five cases and determine the functions $N_{\text{in}}(\xi)$ and $N_{\text{out}}(\xi)$.

7.21. Solve the Wiener–Hopf integral equation of the first kind,

$$\int_0^{+\infty} \frac{1}{2\pi} K_0(\alpha\,|x-y|)\phi(y)dy = 1, \quad x \geq 0,$$

where the kernel is given by

$$K_0(x) \equiv \int_0^{+\infty} \frac{\cos kx}{\sqrt{k^2+1}} dk = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{e^{ikx}}{\sqrt{k^2+1}} dk.$$

7.22. (due to D. M.) Solve the Wiener–Hopf integral equation of the first kind,

$$\int_0^{\infty} K(x-y)\phi(y)dy = 1, \quad x \geq 0,$$

where the kernel is given by

$$K(x) = |x| \exp\left[-|x|\right].$$

7.23. Solve the Wiener–Hopf integral equation of the first kind,

$$\int_0^{+\infty} K(z-\varsigma)\phi(\varsigma)d\varsigma = 0, \quad z \geq 0,$$

$$K(z) \equiv \frac{1}{2}[H_0^{(1)}(k\,|z|) + H_0^{(1)}(k\sqrt{d^2+z^2})],$$

where $H_0^{(1)}(k\,|z|)$ is the 0th-order *Hankel function of the first kind.*

7.24. Solve the Wiener–Hopf integral equation of the first kind,

$$\int_0^{+\infty} K(z-\varsigma)\phi(\varsigma)d\varsigma = 0, \quad z \geq 0,$$

$$K(z) \equiv \frac{1}{2}[H_0^{(1)}(k\,|z|) - H_0^{(1)}(k\sqrt{d^2+z^2})],$$

where $H_0^{(1)}(k\,|z|)$ is the 0th-order *Hankel function of the first kind.*

7.25. Solve the integro-differential equation

$$\left(\frac{\partial^2}{\partial z^2} + k^2\right)\int_0^{+\infty} K(z-\varsigma)\phi(\varsigma)d\varsigma = 0, \quad z \geq 0,$$

$$K(z) \equiv \frac{1}{2}\left[H_0^{(1)}(k\,|z|) + H_0^{(1)}\left(k\sqrt{d^2+z^2}\right)\right],$$

where $H_0^{(1)}(k\,|z|)$ is the 0th-order *Hankel function of the first kind.*

7.26. Solve the integro-differential equation,

$$\left(\frac{\partial^2}{\partial z^2} + k^2\right)\int_0^{+\infty} K(z-\varsigma)\phi(\varsigma)d\varsigma = 0, \quad z \geq 0,$$

$$K(z) \equiv \frac{1}{2}[H_0^{(1)}(k\,|z|) - H_0^{(1)}\left(k\sqrt{d^2+z^2}\right)],$$

where $H_0^{(1)}(k\,|z|)$ is the 0th-order *Hankel function of the first kind.*

Hint for Problems 7.23 through 7.26:

The 0th-order *Hankel function of the first kind* $H_0^{(1)}(kD)$ is given by

$$H_0^{(1)}(kD) = \frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{\exp[ik\sqrt{D^2+\xi^2}]}{\sqrt{D^2+\xi^2}}d\xi, \quad D > 0,$$

and hence its Fourier transform is given by

$$\frac{1}{2}\int_{-\infty}^{\infty} H_0^{(1)}(k\sqrt{D^2+z^2})\exp[i\omega z]dz = \frac{\exp[iv(\omega)D]}{v(\omega)}, \quad v(\omega) = \sqrt{k^2-\omega^2},$$

$\operatorname{Im} v(\omega) > 0.$

The problems are thus reduced to factoring the following functions:

$$\psi(\omega) \equiv 1 + \exp[iv(\omega)d] = \psi_+(\omega)\psi_-(\omega),$$
$$\varphi(\omega) \equiv 1 - \exp[iv(\omega)d] = \varphi_+(\omega)\varphi_-(\omega),$$

where $\psi_+(\omega)$ ($\varphi_+(\omega)$) is analytic and has no zeros in the upper half plane, $\operatorname{Im}\omega \geq 0$, and $\psi_-(\omega)$ ($\varphi_-(\omega)$) is analytic and has no zeros in the lower half plane, $\operatorname{Im}\omega \leq 0$.

As for the integro-differential equations, the differential operator

$$\frac{\partial^2}{\partial z^2} + k^2$$

can be brought inside the integral symbol and we obtain the extra factor

$$v^2(\omega) = k^2 - \omega^2,$$

for the Fourier transforms, multiplying onto the functions to be factored. The functions to be factored are given by

$$\tilde{K}(\omega)_{\text{Prob. 7.23}} = \frac{\psi(\omega)}{v(\omega)}, \quad \tilde{K}(\omega)_{\text{Prob. 7.24}} = \frac{\varphi(\omega)}{v(\omega)},$$

$$v(\omega)\tilde{K}(\omega)_{\text{Prob. 7.25}} = v(\omega)\psi(\omega), \quad v(\omega)\tilde{K}(\omega)_{\text{Prob. 7.26}} = v(\omega)\varphi(\omega).$$

7.27. Solve the Wiener–Hopf integral equation of the first kind,

$$\int_0^{+\infty} K(z - \varsigma)\phi(\varsigma)d\varsigma = 0, \quad z \geq 0,$$

$$K(z) \equiv \frac{a}{2}\int_{-\infty}^{\infty} J_1(v(\omega)a)H_1^{(1)}(v(\omega)a)\exp[i\omega z]d\omega,$$

with

$$v(\omega) = \sqrt{k^2 - \omega^2},$$

where $J_1(va)$ is the first-order *Bessel function of the first kind* and $H_1^{(1)}(va)$ is the first-order *Hankel function of the first kind.*

7.28. Solve the integro-differential equation,

$$\left(\frac{\partial^2}{\partial z^2} + k^2\right)\int_0^{+\infty} K(z - \varsigma)\phi(\varsigma)d\varsigma = 0, \quad z \geq 0,$$
$$K(z) \equiv \frac{a}{2}\int_{-\infty}^{\infty} J_0(v(\omega)a)H_0^{(1)}(v(\omega)a)\exp[i\omega z]d\omega,$$

with

$$v(\omega) = \sqrt{k^2 - \omega^2},$$

where $J_0(va)$ is the 0th-order *Bessel function of the first kind* and $H_0^{(1)}(va)$ is the 0th-order *Hankel function of the first kind.*

Hint for Problems 7.27 and 7.28:

The functions to be factored are

$$\tilde{K}(\omega)_{\text{Prob. 7.27}} = \pi a J_1(va) H_1^{(1)}(va),$$
$$v^2 \tilde{K}(\omega)_{\text{Prob. 7.28}} = \pi a v^2 J_0(va) H_0^{(1)}(va).$$

The factorization procedures are identical to the previous problems.

As for the details of the factorizations for Problems 7.23 through 7.28, we refer the reader to the following monograph.

Weinstein, L.A.: *The Theory of Diffraction and the Factorization Method,* Golem Press, (1969). Chapters 1 and 2.

7.29. Solve the dual integral equations of the following form, which shows up in electrostatics,

$$\int_0^\infty y f(y) J_n(yx) dy = x^n \quad \text{for} \quad 0 \le x < 1$$

and

$$\int_0^\infty f(y) J_n(yx) dy = 0 \quad \text{for} \quad 1 \le x < \infty,$$

where n is the nonnegative integer and $J_n(yx)$ is the nth-order *Bessel function of the first kind.*

Hint: Jackson, J.D. : *Classical Electrodynamics*, 3rd edition, John Wiley & Sons, New York (1999). Section 3.13.

7.30. Solve the dual integral equations of the following form, which shows up in magnetstatics,

$$\int_0^\infty f(y) J_n(yx) dy = x^n \quad \text{for} \quad 0 \le x < 1$$

and

$$\int_0^\infty y f(y) J_n(yx) dy = 0 \quad \text{for} \quad 1 \le x < \infty,$$

where n is the nonnegative integer and $J_n(yx)$ is the nth order *Bessel function of the first kind*.

Hint: Jackson, J.D. : *Classical Electrodynamics*, 3rd edition, John Wiley & Sons, New York (1999). Section 5.13.

7.31. Solve the dual integral equations of the following form:

$$\int_0^\infty y^\alpha f(y) J_\mu(yx)dy = g(x) \quad \text{for} \quad 0 \leq x < 1$$

and

$$\int_0^\infty f(y) J_\mu(yx)dy = 0 \quad \text{for} \quad 1 \leq x < \infty.$$

Hint: Kondo, J.: *Integral Equations*, Kodansha Ltd., Tokyo (1991), p. 412.

7.32. Consider the integral equation

$$\phi(x) = \int_{-1}^{+1} H_0^{(1)}(\alpha\,|x-y|)\phi(y)dy \quad \text{with} \quad -1 \leq x \leq +1,$$

where $H_0^{(1)}(x)$ is the 0th order *Hankel function of the first kind*. We can assume that $\phi(x)$ is even. Then the above integral equation becomes

$$\phi(x) = 2\int_0^{+1} H_0^{(1)}(\alpha\,|x-y|)\phi(y)dy \quad \text{with} \quad 0 \leq x \leq +1.$$

(a) By assuming the exponential damping of $\phi(x)$ as $x \to \infty$, we obtain the approximate integral equation

$$\psi(x) = 2\int_0^\infty H_0^{(1)}(\alpha\,|x-y|)\psi(y)dy \quad \text{with} \quad 0 \leq x < \infty.$$

The above approximate integral equation is the homogeneous Wiener–Hopf integral equation of the second kind.

(b) Show that the exact solution to the approximate integral equation is given by

$$\psi(x) = 1 + \chi(\alpha\,|1+x|) + \chi(\alpha\,|1-x|) \quad \text{with} \quad 0 \leq x < \infty,$$

where

$$\chi(x) = \frac{1}{\sqrt{\pi x}}\exp[-x] - \text{erf}[\sqrt{x}].$$

(c) Ascertain that the function $\chi(x)$ exhibits the exponential damping as $x \to \infty$, thus verifying that $\phi(x) \to 0$ exponentially as $x \to \infty$.

8
Nonlinear Integral Equations

8.1
Nonlinear Integral Equation of the Volterra Type

In Chapter 3, the *linear* integral equations of the Volterra type are examined. We applied the Laplace transform technique for a translation kernel. As an application of the *Laplace transform technique*, we can solve a *nonlinear Volterra integral equation of convolution type*:

$$\phi(x) = f(x) + \lambda \int_0^x \phi(y)\phi(x-y)dy. \tag{8.1.1}$$

Taking the Laplace transform of Eq. (8.1.1), we obtain

$$\overline{\phi}(s) = \overline{f}(s) + \lambda[\overline{\phi}(s)]^2. \tag{8.1.2}$$

Hence we have

$$\overline{\phi}(s) = [1 \pm (1 - 4\lambda\overline{f}(s))^{1/2}]/2\lambda.$$

We assume that $\overline{f}(s) \to 0$ as $\mathrm{Re}\, s \to \infty$, and we require that $\overline{\phi}(s) \to 0$ as $\mathrm{Re}\, s \to \infty$. Then only one of the two solutions survives. Specifically it is

$$\overline{\phi}(s) = [1 - (1 - 4\lambda\overline{f}(s))^{1/2}]/2\lambda. \tag{8.1.3a}$$

Inverse Laplace transform provides us the solution

$$\phi(x) = L^{-1}\left([1 - (1 - 4\lambda\overline{f}(s))^{1/2}]/2\lambda\right)$$

$$= \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{ds}{2\pi i} e^{sx}\left[1 - (1 - 4\lambda\overline{f}(s))^{1/2}\right]/2\lambda, \tag{8.1.3b}$$

where the inversion path is to the right of all singularities of the integrand. We examine two specific cases.

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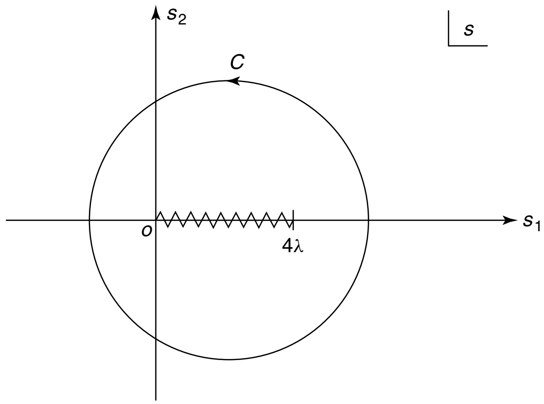

Fig. 8.1 Branch cut of the integrand of Eq. (8.1.5a) from $s = 0$ to $s = 4\lambda$.

❑ **Example 8.1.** $f(x) = 0$.

Solution. In this case, Eq. (8.1.3b) gives

$$\phi(x) = 0.$$

Thus there is no nontrivial solution.

❑ **Example 8.2.** $f(x) = 1$.

Solution. In this case,

$$\bar{f}(s) = 1/s, \tag{8.1.4}$$

and Eq. (8.1.3b) gives

$$\phi(x) = \frac{1}{2\lambda} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{ds}{2\pi i} e^{sx} \left[1 - \sqrt{\frac{s - 4\lambda}{s}} \right]. \tag{8.1.5a}$$

The integrand has a branch cut from $s = 0$ to $s = 4\lambda$ as in Figure 8.1.
Let $\lambda > 0$, then the branch cut is as illustrated in Figure 8.2.
By deforming the contour, we get

$$\phi(x) = \frac{1}{2\lambda} \oint_C \frac{ds}{2\pi i} e^{sx} \left(1 - \sqrt{\frac{s - 4\lambda}{s}} \right) = -\frac{1}{2\lambda} \oint_C \frac{ds}{2\pi i} e^{sx} \sqrt{\frac{s - 4\lambda}{s}}, \tag{8.1.5b}$$

where C is the contour wrapped around the branch cut , as shown in Figure 8.2. By evaluating the values of the integrand on the two sides of the branch cut , we get

$$\phi(x) = \frac{1}{2\pi\lambda} \int_0^{4\lambda} ds e^{sx} \sqrt{\frac{4\lambda - s}{s}} = \frac{2}{\pi} \int_0^1 dt e^{4\lambda t x} \sqrt{\frac{1 - t}{t}}. \tag{8.1.6}$$

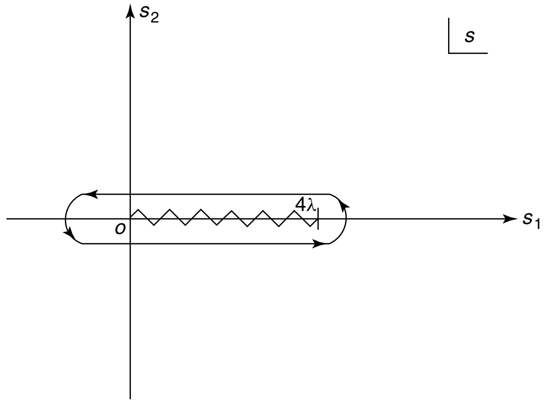

Fig. 8.2 The contour of integration C wrapping around the branch cut of Figure 8.1 for $\lambda > 0$.

Here we have changed the variable,

$$s = 4\lambda t, \quad 0 \leq t \leq 1.$$

The integral in Eq. (8.1.6) can be explicitly evaluated:

$$\phi(x) = \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(4\lambda x)^n}{n!}\int_0^1 dtt^n\sqrt{\frac{1-t}{t}} = \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(4\lambda x)^n}{n!}\frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(n+2)}, \tag{8.1.7}$$

where we have used the formula

$$\int_0^1 dtt^{n-1}(1-t)^{m-1} = \frac{\Gamma(n)\Gamma(m)}{\Gamma(n++m)}.$$

Now, the *confluent hypergeometric function* is given by

$$F(a;c;z) = 1 + \frac{a}{c}\frac{z}{1!} + \frac{a}{c}\frac{a+1}{c+1}\frac{z^2}{2!} + \cdots = \sum_{n=0}^{\infty}\frac{\Gamma(c)}{\Gamma(a)}\frac{\Gamma(a+n)}{\Gamma(c+n)}\frac{z^n}{n!}. \tag{8.1.8}$$

From Eqs. (8.1.7) and (8.1.8), we find that

$$\phi(x) = F\left(\frac{1}{2};2;4\lambda x\right) \tag{8.1.9}$$

satisfies the nonlinear integral equation

$$\phi(x) = 1 + \lambda\int_0^x \phi(x-y)\phi(y)dy. \tag{8.1.10}$$

Although the above is proved only for $\lambda > 0$, we may verify that it is also true for $\lambda < 0$ by repeating the same argument. Alternatively we may prove this in the following way. Let us substitute Eq. (8.1.9) into Eq. (8.1.10). Since $F(\frac{1}{2}; 2; 4\lambda x)$ is an entire function of λ, each side of the resulting equation is the entire function of λ. Since this equation is satisfied for $\lambda > 0$, it must be satisfied for all λ by analytic continuation. Thus the integral equation (8.1.10) has the unique solution given by Eq. (8.1.9), for all values of λ.

At the end of this section, we classify the nonlinear integral equations of Volterra type in the following manner:

(1) Kernel part is nonlinear,

$$\phi(x) - \int_a^x H(x, y, \phi(y))dy = f(x). \tag{VN.1}$$

(2) Particular part is nonlinear,

$$G(\phi(x)) - \int_a^x K(x, y)\phi(y)dy = f(x). \tag{VN.2}$$

(3) Both parts are nonlinear,

$$G(\phi(x)) - \int_a^x H(x, y, \phi(y))dy = f(x). \tag{VN.3}$$

(4) Nonlinear Volterra integral equation of the first kind,

$$\int_a^x H(x, y, \phi(y))dy = f(x). \tag{VN.4}$$

(5) Homogeneous nonlinear Volterra integral equation of the first kind where the kernel part is nonlinear,

$$\phi(x) = \int_a^x H(x, y, \phi(y))dy. \tag{VN.5}$$

(6) Homogeneous nonlinear Volterra integral equation of the first kind where the particular part is nonlinear,

$$G(\phi(x)) = \int_a^x K(x, y)\phi(y)dy. \tag{VN.6}$$

(7) Homogeneous nonlinear Volterra integral equation of the first kind where both parts are nonlinear,

$$G(\phi(x)) = \int_a^x H(x, y, \phi(y))dy. \tag{VN.7}$$

8.2 Nonlinear Integral Equation of the Fredholm Type

In this section, we shall give a brief discussion of the *nonlinear integral equation of Fredholm type*. Recall that the linear algebraic equation

$$\phi = f + \lambda K \phi \tag{8.2.1}$$

has the solution

$$\phi = (1 - \lambda K)^{-1} f. \tag{8.2.2}$$

In particular, the solution of Eq. (8.2.1) exists and is unique as long as the corresponding homogeneous equation

$$\phi = \lambda K \phi \tag{8.2.3}$$

has no nontrivial solutions.

Nonlinear equations behave quite differently. Consider, for example, the nonlinear algebraic equation obtained from Eq. (8.2.1) by replacing K with ϕ,

$$\phi = f + \lambda \phi^2. \tag{8.2.4a}$$

The solutions of Eq. (8.2.4a) are

$$\phi = \frac{1 \pm \sqrt{1 - 4\lambda f}}{2\lambda}. \tag{8.2.4b}$$

We first observe that the solution is not unique. Indeed, if we require the solutions to be real, then Eq. (8.2.4a) has two solutions if

$$1 - 4\lambda f > 0, \tag{8.2.5}$$

and no solution if

$$1 - 4\lambda f < 0. \tag{8.2.6}$$

Thus the number of solutions changes from 2 to 0 as the value of λ passes $1/4f$. The point $\lambda = 1/4f$ is called a *bifurcation point* of Eq. (8.2.4a). Note that at the bifurcation point, Eq. (8.2.4a) has only one solution.

We also observe from Eq. (8.2.4b) that another special point for Eq. (8.2.4a) is $\lambda = 0$. At this point, one of the two solutions is infinite. Since the number of solutions remains to be two as the value of λ passes $\lambda = 0$, the point $\lambda = 0$ is not a bifurcation point. We shall call it a *singular point*.

Consider now the equation

$$\phi = \lambda\phi^2, \tag{8.2.7}$$

obtained from Eq. (8.2.4a) by setting $f = 0$. This equation always has the nontrivial solution

$$\phi = 1/\lambda \tag{8.2.8}$$

provided that

$$\lambda \neq 0. \tag{8.2.9}$$

There is no connection between the existence of the solutions for Eq. (8.2.4a) and the absence of the solutions for Eq. (8.2.4a) with $f = 0$, quite unlike the case of linear algebraic equations.

Nonlinear integral equations share these properties. This is evident in the following examples.

❑ **Example 8.3.** Solve

$$\phi(x) = 1 + \lambda \int_0^1 \phi^2(y)dy. \tag{8.2.10}$$

Solution. The right-hand side of Eq. (8.2.10) is independent of x. Thus $\phi(x)$ is constant. Let

$$\phi(x) = a.$$

Then Eq. (8.2.10) becomes

$$a = 1 + \lambda a^2. \tag{8.2.11}$$

Equation (8.2.11) is just Eq. (8.2.4a) with $f = 1$. Thus

$$\phi(x) = \frac{1 \pm \sqrt{1 - 4\lambda}}{2\lambda}. \tag{8.2.12}$$

There are two real solutions for $\lambda < 1/4$, and no real solutions for $\lambda > 1/4$. Thus $\lambda = 1/4$ is a bifurcation point.

❑ **Example 8.4.** Solve

$$\phi(x) = 1 + \lambda \int_0^1 \phi^3(y)dy. \tag{8.2.13}$$

Solution. The right-hand side of Eq. (8.2.13) is independent of x. Thus $\phi(x)$ is constant. Letting

$$\phi(x) = a,$$

we get

$$a = 1 + \lambda a^3. \tag{8.2.14a}$$

Equation (8.2.14a) is cubic and hence has three solutions. Not all of these solutions are real. Let us rewrite Eq. (8.2.14a) as

$$\frac{a-1}{\lambda} = a^3, \tag{8.2.14b}$$

and plot $(a-1)/\lambda$ as well as a^3 in the figure. The points of intersection between these two curves are the solutions of Eq. (8.2.14a). For λ negative, there is obviously only one real root, while for λ positive and very small, there are three real roots (two positive roots and one negative root). Thus $\lambda = 0$ is a bifurcation point . For λ large and positive, there is again only one real root. The change of numbers of roots can be shown to occur at $\lambda = 4/27$, which is another bifurcation point.

We may generalize the above considerations to

$$\phi(x) = c + \lambda \int_0^1 K(\phi(y))dy. \tag{8.2.15a}$$

The right-hand side of Eq. (8.2.15a) is independent of x. Thus $\phi(x)$ is constant. Letting

$$\phi(x) = a,$$

we get

$$(a - c)/\lambda = K(a). \tag{8.2.15b}$$

The roots of the above equation can be graphically obtained by plotting $K(a)$ and $(a - c)/\lambda$.

Obviously, with a proper choice of $K(a)$, the number of solutions as well as the number of bifurcation points may take any value. For example, if

$$K(\phi) = \phi \sin \pi\phi, \tag{8.2.16}$$

there are infinitely many solutions as long as $|\lambda| < 1$. As another example, for

$$K(\phi) = \sin\phi/(\phi^2 + 1), \tag{8.2.17}$$

there are infinitely many bifurcation points .

In summary, we have found the following conclusions for nonlinear integral equations:

(1) *There may be more than one solution.*
(2) *There may be one or more bifurcation points.*
(3) *There is no significant relationship between the integral equation with* $f \neq 0$ *and the one obtained from it by setting* $f = 0$.

The above considerations for simple examples may be extended to more general cases. Consider the integral equation

$$\phi(x) = f(x) + \int_0^1 K(x, y, \phi(x), \phi(y))dy. \tag{8.2.18}$$

If K is separable, i.e.,

$$K(x, y, \phi(x), \phi(y)) = g(x, \phi(x))h(y, \phi(y)), \tag{8.2.19}$$

then the integral equation (8.2.18) is solved by

$$\phi(x) = f(x) + ag(x, \phi(x)), \tag{8.2.20}$$

with

$$a = \int_0^1 h(x, \phi(x))dx. \tag{8.2.21}$$

We may solve Eq. (8.2.20) for $\phi(x)$ and express $\phi(x)$ as a function of x and a. There may be more than one solution. Substituting any one of these solutions into Eq. (8.2.21), we may obtain an equation for a. Thus the nonlinear integral equation (8.2.18) is equivalent to one or more nonlinear algebraic equations for a.

Similarly, if K is a sum of the separable terms, the integral equation is equivalent to systems of N coupled nonlinear algebraic equations.

In closing this section, we classify the nonlinear integral equations of the Fredholm type in the following manner:

(1) Kernel part is nonlinear,

$$\phi(x) - \int_a^b H(x, y, \phi(y))dy = f(x). \tag{FN.1}$$

(2) Particular part is nonlinear,

$$G(\phi(x)) - \int_a^b K(x, y)\phi(y)dy = f(x). \tag{FN.2}$$

(3) Both parts are nonlinear,

$$G(\phi(x)) - \int_a^b H(x, y, \phi(y))dy = f(x). \tag{FN.3}$$

(4) Nonlinear Fredholm integral equation of the first kind,

$$\int_a^b H(x, y, \phi(y))dy = f(x). \tag{FN.4}$$

(5) Homogeneous nonlinear Fredholm integral equation of the first kind, where the kernel part is nonlinear,

$$\phi(x) = \int_a^b H(x, y, \phi(y))dy. \tag{FN.5}$$

(6) Homogeneous nonlinear Fredholm integral equation of the first kind, where the particular part is nonlinear,

$$G(\phi(x)) = \int_a^b K(x, y)\phi(y)dy. \tag{FN.6}$$

(7) Homogeneous nonlinear Fredholm integral equation of the first kind, where both parts are nonlinear,

$$G(\phi(x)) = \int_a^b H(x, y, \phi(y))dy. \tag{FN.7}$$

8.3 Nonlinear Integral Equation of the Hammerstein Type

The inhomogeneous term $f(x)$ in Eq. (8.2.18) is actually not particularly meaningful. This is because we may define

$$\psi(x) \equiv \phi(x) - f(x), \tag{8.3.1}$$

then Eq. (8.2.18) is of the form

$$\psi(x) = \int_0^1 K(x, y, \psi(x) + f(x), \psi(y) + f(y))dy. \tag{8.3.2}$$

The *nonlinear integral equation of the Hammerstein type* is a special case of Eq. (8.3.2),

$$\psi(x) = \int_0^1 K(x, y)f(y, \psi(y))dy. \tag{8.3.3}$$

For the remainder of this section, we discuss this latter equation, (8.3.3). We shall show that if f satisfies *uniformly* a *Lipschitz condition* of the form

$$\left|f(y, u_1) - f(y, u_2)\right| < C(y)\left|u_1 - u_2\right|, \tag{8.3.4}$$

and if

$$\int_0^1 A(y)C^2(y)dy = M^2 < 1, \tag{8.3.5}$$

then the solution of Eq. (8.3.3) is unique and can be obtained by iteration. The function $A(x)$ in Eq. (8.3.5) is given by

$$A(x) = \int_0^1 K^2(x, y)dy. \tag{8.3.6}$$

We begin by setting

$$\psi_0(x) = 0 \tag{8.3.7}$$

and

$$\psi_n(x) = \int_0^1 K(x, y)f(y, \psi_{n-1}(y))dy. \tag{8.3.8}$$

If $f(y, 0) = 0$, then Eq. (8.3.3) is solved by $\psi(x) = 0$. If

$$f(y, 0) \neq 0, \tag{8.3.9}$$

we have

$$\psi_1^2(x) \leq A(x) \int_0^1 f^2(y, 0)dy = A(x) \|f\|^2, \tag{8.3.10}$$

where

$$\|f\|^2 \equiv \int_0^1 f^2(y, 0)dy. \tag{8.3.11}$$

Also, as a consequence of the Lipschitz condition (8.3.4),

$$|\psi_n(x) - \psi_{n-1}(x)| < \int_0^1 |K(x, y)| C(y) |\psi_{n-1}(y) - \psi_{n-2}(y)| dy.$$

Thus

$$[\psi_n(x) - \psi_{n-1}(x)]^2 \leq A(x) \int_0^1 C^2(y)[\psi_{n-1}(y) - \psi_{n-2}(y)]^2 dy. \tag{8.3.12}$$

From Eqs. (8.3.4) and (8.3.5), we get

$$[\psi_2(x) - \psi_1(x)]^2 \leq A(x) \|f\|^2 M^2.$$

By induction,

$$[\psi_n(x) - \psi_{n-1}(x)]^2 \leq A(x) \left\| f \right\|^2 (M^2)^{n-1}. \tag{8.3.13}$$

Thus the series

$$\psi_1(x) + [\psi_2(x) - \psi_1(x)] + [\psi_3(x) - \psi_2(x)] + \cdots$$

is convergent, due to Eq. (8.3.5).

The proof of uniqueness will be left to the reader.

8.4 Problems for Chapter 8

8.1. (due to H. C.). Prove the uniqueness of the solution to the nonlinear integral equation of the Hammerstein type

$$\psi(x) = \int_0^1 K(x, y) f(y, \psi(y)) dy.$$

8.2. (due to H. C.). Consider

$$\frac{d^2}{dt^2} x(t) + x(t) = \frac{1}{\pi^2} x^2(t), \quad t > 0,$$

with the initial conditions

$$x(0) = 0 \quad \text{and} \quad \left. \frac{dx}{dt} \right|_{t=0} = 1.$$

(a) Transform this nonlinear ordinary differential equation to an integral equation.

(b) Obtain an approximate solution accurate to a few percent.

8.3. (due to H. C.). Consider

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \phi(x, y) = \frac{1}{\pi^2} \phi^2(x, y), \quad x^2 + y^2 < 1,$$

with

$$\phi(x, y) = 1 \quad \text{on} \quad x^2 + y^2 = 1.$$

(a) Construct a Green's function satisfying

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) G(x, y; x', y') = \delta(x - x')\delta(y - y'),$$

with the boundary condition

$$G(x, y; x', y') = 0 \quad \text{on} \quad x^2 + y^2 = 1.$$

Prove that

$$G(x, y; x', y') = G(x', y'; x, y).$$

Hint: To construct Green's function $G(x, y; x', y')$, which vanishes on the unit circle, use the method of images.

(b) Transform the above nonlinear partial differential equation to an integral equation, and obtain an approximate solution accurate to a few percent.

8.4. (due to H. C.).

(a) Discuss a phase transition of $\rho(\theta)$ which is given by the nonlinear integral equation,

$$\begin{aligned} \rho(\theta) &= Z^{-1} \exp[\beta \int_0^{2\pi} \cos(\theta - \phi)\rho(\phi)d\phi], \\ \beta &= J/k_B T, \quad 0 \le \theta \le 2\pi, \end{aligned}$$

where Z is the normalization constant such that $\rho(\theta)$ is normalized to unity,

$$\int_0^{2\pi} \rho(\theta)d\theta = 1.$$

(b) Determine the nature of the phase transition.

Hint: You may need the following special functions:

$$I_0(z) = \sum_{m=0}^{\infty} \frac{1}{(m!)^2} \left(\frac{z}{2}\right)^{2m},$$

$$I_1(z) = \sum_{m=0}^{\infty} \frac{1}{m!(m+1)!} \left(\frac{z}{2}\right)^{2m+1},$$

and generally

$$I_n(z) = \frac{1}{\pi}\int_0^{\pi} e^{z\cos\theta}(\cos n\theta)d\theta.$$

For $|z| \to 0$, we have

$$I_0(z) \to 1,$$

$$I_1(z) \to z/2.$$

8.5. Solve the nonlinear integral equation of the Fredholm type

$$\phi(x) - \int_0^{0.1} xy[1 + \phi^2(y)]dy = x + 1.$$

8.6. Solve the nonlinear integral equation of the Fredholm type

$$\phi(x) - 60\int_0^1 xy\phi^2(y)dy = 1 + 20x - x^2.$$

8.7. Solve the nonlinear integral equation of the Fredholm type

$$\phi(x) - \lambda\int_0^1 xy[1 + \phi^2(y)]dy = x + 1.$$

8.8. Solve the nonlinear integral equation of the Fredholm type

$$\phi^2(x) - \int_0^1 xy\phi(y)dy = 4 + 10x + 9x^2.$$

8.9. Solve the nonlinear integral equation of the Fredholm type

$$\phi^2(x) - \lambda\int_0^1 xy\phi(y)dy = 1 + x^2.$$

8.10. Solve the nonlinear integral equation of the Fredholm type

$$\phi^2(x) + \int_0^2 \phi^3(y)dy = 1 - 2x + x^2.$$

8.11. Solve the nonlinear integral equation of the Fredholm type

$$\phi^2(x) = \frac{5}{2}\int_0^1 xy\phi(y)dy.$$

Hint for Problems 8.5 through 8.11: The integrals are at most linear in x.

8.12. Solve the nonlinear integral equation of the Volterra type

$$\phi^2(x) - \int_0^x (x-y)\phi(y)dy = 1 + 3x + \frac{1}{2}x^2 - \frac{1}{2}x^3.$$

8.13. Solve the nonlinear integral equation of the Volterra type

$$\phi^2(x) + \int_0^x \sin(x-y)\phi(y)dy = \exp[x].$$

8.14. Solve the nonlinear integral equation of the Volterra type

$$\phi(x) - \int_0^x (x-y)^2\phi^2(y)dy = x.$$

8.15. Solve the nonlinear integral equation of the Volterra type

$$\phi^2(x) - \int_0^x (x-y)^3\phi^3(y)dy = 1 + x^2.$$

8.16. Solve the nonlinear integral equation of the Volterra type

$$2\phi(x) - \int_0^x \phi(x-y)\phi(y)dy = \sin x.$$

Hint for Problems 8.12 through 8.16: Take the Laplace transform of the given nonlinear integral equations of the Volterra type.

8.17. Solve the nonlinear integral equation

$$\phi(x) - \lambda \int_0^1 \phi^2(y)dy = 1.$$

In particular, identify the bifurcation points of this equation. What are the nontrivial solutions of the corresponding homogeneous equations?

9
Calculus of Variations: Fundamentals

9.1
Historical Background

The calculus of variations was first found in the late 17th century soon after calculus was invented. The main figures involved are *Newton, the two Bernoulli brothers, Euler, Lagrange, Legendre,* and *Jacobi.*

Isaac Newton (1642–1727) formulated the fundamental laws of motion. The fundamental quantities of motion were established as the momentum and the force. Newton's laws of motion state:

(1) In the inertial frame, every body remains at rest or in uniform motion unless acted on by a force $\vec{F}$. The condition $\vec{F} = \vec{0}$ implies a constant velocity $\vec{v}$ and a constant momentum $\vec{p} = m\vec{v}$.

(2) In the inertial frame, application of force $\vec{F}$ alters the momentum $\vec{p}$ by an amount specified by

$$\vec{F} = \frac{d}{dt}\vec{p}. \tag{9.1.1}$$

(3) To each action of a force, there is an equal and opposite action of a force. Thus if $\vec{F}_{21}$ is the force exerted on particle 1 by particle 2, then

$$\vec{F}_{21} = -\vec{F}_{12}, \tag{9.1.2}$$

and these forces act along the line separating the particles.

Contrary to the common belief that he discovered the gravitational force by observing that the apple dropped from the tree at Trinity College, he actually deduced Newton's laws of motion from the careful analysis of Kepler's laws. He also invented the calculus, named methodus fluxionum in 1666, about 10 year ahead of Leibniz. In 1687, Newton published *Philosophiae naturalis principia mathematica*, often called *Principia*. It consists of three parts: Newton's laws of motion, laws of the gravitational force, and laws of motion of the planets.

The *Bernoulli brothers, Jacques* (1654–1705) and *Jean* (1667–1748), came from the family of mathematicians in Switzerland. They solved the problem of *Brachistochrone*. They established the principle of virtual work as a general principle of

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statics with which all problems of equilibrium could be solved. Remarkably, they also compared the motion of a particle in a given field of force with that of light in an optically heterogeneous medium and tried to give a mechanical theory of the refractive index. The Bernoulli brothers were the forerunners of the theory of Hamilton which has shown that the principle of least action in classical mechanics and Fermat's principle of shortest time in geometrical optics are strikingly analogous to each other. They used the notation, g, for the gravitational acceleration for the first time.

Leonhard Euler (1707–1783) grew up under the influence of Bernoulli family in Switzerland. He made an extensive contribution to the development of calculus after Leibniz, and initiated calculus of variations. He started the systematic study of the isoperimetric problems. He also contributed in an essential way to the variational treatment of classical mechanics, providing the Euler equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0, \tag{9.1.3}$$

for the extremization problem

$$\delta I = \delta \int_{x_1}^{x_2} f(x, y, y')dx = 0, \tag{9.1.4}$$

with

$$\delta y(x_1) = \delta y(x_2) = 0. \tag{9.1.5}$$

Joseph Louis Lagrange (1736–1813) provided the solution to the isoperimetric problems by the method presently known as the method of Lagrange multipliers; quite independent of Euler. He started the whole field of the calculus of variations . He also introduced the notion of generalized coordinates, $\{q_r(t)\}_{r=1}^{f}$, into classical mechanics and completely reduced the mechanical problem to that of the differential equations presently known as Lagrange equations of motion,

$$\frac{d}{dt}\left(\frac{\partial L(q_s, \dot{q}_s, t)}{\partial \dot{q}_r}\right) - \frac{\partial L(q_s, \dot{q}_s, t)}{\partial q_r} = 0, \quad r = 1, \ldots, f, \quad \text{with} \quad \dot{q}_r \equiv \frac{d}{dt} q_r, \tag{9.1.6}$$

with the Lagrangian $L(q_r(t), \dot{q}_r(t), t)$ appropriately chosen in terms of the kinetic energy and the potential energy. He successfully converted classical mechanics into analytical mechanics with the variational principle. He also carried out the research on Fermat problem, the general treatment of the theory of ordinary differential equations, and the theory of elliptic functions.

Adrien Marie Legendre (1752–1833) announced the research on the form of the planet in 1784. In his article, the Legendre polynomials was used for the first time. He provided the Legendre test in the maximization–minimization problem of the calculus of variations, among his numerous and diverse contributions to mathematics. As one of his major accomplishments, his classification of elliptic integrals into three types stated in *Exercices de calcul intégral*, published in 1811, should be mentioned. In 1794, he published *Éléments de géométrie, avec notes.* He further developed the methods of transformations for thermodynamics which are presently known as the Legendre transformations and are used even in quantum field theory today.

William Rowan Hamilton (1805–1865) started research on optics around 1823 and introduced the notion of the characteristic function. His results formed the basis of the later development of the notion of eikonal in optics. He also succeeded in transforming the Lagrange equations of motion which is of the second order into a set of differential equations of the first order with twice as many variables, with the introduction of the momenta $\{p_r(t)\}_{r=1}^{f}$ canonically conjugate to the generalized coordinates $\{q_r(t)\}_{r=1}^{f}$ by

$$p_r(t) = \frac{\partial L(q_s, \dot{q}_s, t)}{\partial \dot{q}_r}, \quad r = 1, \dots, f. \tag{9.1.7}$$

His equations are known as Hamilton's canonical equations of motion:

$$\frac{d}{dt}q_r(t) = \frac{\partial H(q_s(t), p_s(t), t)}{\partial p_r(t)}, \quad \frac{d}{dt}p_r(t) = -\frac{\partial H(q_s(t), p_s(t), t)}{\partial q_r(t)}, \quad r = 1, \dots, f. \tag{9.1.8}$$

He formulated classical mechanics in terms of the principle of least action. The variational principles formulated by Euler and Lagrange apply only to the conservative system. He also recognized that the principle of least action in classical mechanics and Fermat's principle of shortest time in geometrical optics are strikingly analogous, permitting the interpretation of the optical phenomena in terms of mechanical terms and vice versa. He was one step short of discovering wave mechanics in analogy to wave optics as early as 1834, although he did not have any experimentally compelling reason to take such step. On the other hand, by 1924, L. de Broglie and E. Schrödinger had sufficient experimentally compelling reasons to take such step.

Carl Gustav Jacob Jacobi (1804–1851), in 1824, quickly recognized the importance of the work of Hamilton. He realized that Hamilton was using just one particular choice of a set of the variables $\{q_r(t)\}_{r=1}^{f}$ and $\{p_r(t)\}_{r=1}^{f}$ to describe the mechanical system and carried out the research on the canonical transformation theory with the Legendre transformation. He duly arrived at what is presently known as the Hamilton–Jacobi equation . His research on the canonical transformation theory is summarized in *Vorlesungen über Dynamik*, published in 1866. He formulated his version of the principle of least action for the time-independent case. He provided the Jacobi test in the maximization–minimization problem of the calculus of variations . In 1827, he introduced the elliptic functions as the inverse functions of the elliptic integrals.

From what we discussed, we may be led to the conclusion that the calculus of variations is the finished subject of the 19th century. We shall note that, from the 1940s to 1950s, we encountered the resurgence of the action principle for the systemization of quantum field theory. Feynman's action principle and Schwinger's action principle are the subject matter. In contemporary particle physics, if we start out with the Lagrangian density of the system under consideration, the extremization of the action functional is still employed as the starting point of the discussion (see Chapter 10). Furthermore, the Legendre transformation is used in the computation of the effective potential in quantum field theory.

We define the problem of the calculus of variations. Suppose that we have an unknown function $y(x)$ of the independent variable x which satisfies some condition C. We construct the functional $I[y]$ which involves the unknown function $y(x)$ and its derivatives. We now want to determine the unknown function $y(x)$ which extremizes the functional $I[y]$ under the infinitesimal variation $\delta y(x)$ of $y(x)$ subject to the condition C. Simple example is the extremization of the following functional:

$$I[y] = \int_{x_1}^{x_2} L(x, y, y')dx, \quad C : y(x_1) = y(x_2) = 0. \tag{9.1.9}$$

The problem is reduced to solving Euler equation, which we will discuss later. This problem and its solution are the problem of the calculus of variations.

Many basic principles of physics can be cast in the form of the calculus of variations. Most of the problems in classical mechanics and classical field theory are of this form, with a certain generalization, under the following replacements: for classical mechanics, we replace

$$\begin{cases} x & \text{with} & t, \\ y & \text{with} & q(t), \\ y' & \text{with} & \dot{q}(t) \equiv dq(t)/dt, \end{cases} \tag{9.1.10}$$

and for classical field theory, we replace

$$\begin{cases} x & \text{with} & (t, \vec{r}), \\ y & \text{with} & \psi(t, \vec{r}), \\ y' & \text{with} & (\partial\psi(t, \vec{r})/\partial t, \vec{\nabla}\psi(t, \vec{r})). \end{cases} \tag{9.1.11}$$

On the other hand, when we want to solve some differential equation subject to the condition C, we may be able to reduce the problem of solving the original differential equation to that of the extremization of the functional $I[y]$ subject to the condition C, provided that Euler equation of the extremization problem coincides with the original differential equation we want to solve. With this reduction, we can obtain an approximate solution of the original differential equation.

The problem of *Brachistochrone* to be defined later, the isoperimetric problem to be defined later, and the problem of finding the shape of the soap membrane with the minimum surface area are the classic problems of the calculus of variations.

9.2 Examples

We list examples of the problems to be solved.

❑ **Example 9.1.** The shortest distance between two points is a straight line: minimize

$$I = \int_{x_1}^{x_2} \sqrt{1 + (y')^2} dx. \tag{9.2.1}$$

❑ **Example 9.2.** The largest area enclosed by an arc of fixed length is a circle: minimize

$$I = \int y dx, \tag{9.2.2}$$

subject to the condition that

$$\int \sqrt{1 + (y')^2} dx \quad \text{fixed.} \tag{9.2.3}$$

❑ **Example 9.3.** Catenary. Surface formed by two circular wires dipped in a soap solution: minimize

$$\int_{x_1}^{x_2} y\sqrt{1 + (y')^2} dx. \tag{9.2.4}$$

❑ **Example 9.4.** Brachistochrone. Determine a path down which a particle falls under gravity in the shortest time: minimize

$$\int \sqrt{\frac{1 + (y')^2}{y}} dx. \tag{9.2.5}$$

❑ **Example 9.5.** Hamilton's action principle in classical mechanics: minimize the action integral I defined by

$$I \equiv \int_{t_1}^{t_2} L(q, \dot{q}) dt \quad \text{with} \quad q(t_1) \text{ and } q(t_2) \quad \text{fixed,} \tag{9.2.6}$$

where $L(q(t), \dot{q}(t))$ is the *Lagrangian* of the mechanical system.

The last example is responsible for starting the whole field of the calculus of variations.

9.3
Euler Equation

We shall derive the fundamental equation for the calculus of variations, Euler equation.

We shall extremize

$$I = \int_{x_1}^{x_2} f(x, y, y')dx, \tag{9.3.1}$$

with the end points

$$y(x_1), \quad y(x_2) \quad \text{fixed.} \tag{9.3.2}$$

We consider a small variation of $y(x)$ of the following form:

$$y(x) \rightarrow y(x) + \varepsilon\nu(x), \tag{9.3.3a}$$

$$y'(x) \rightarrow y'(x) + \varepsilon\nu'(x), \tag{9.3.3b}$$

with

$$\nu(x_1) = \nu(x_2) = 0, \quad \varepsilon = \text{positive infinitesimal.} \tag{9.3.3c}$$

Then the variation of I is given by

$$\delta I = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\varepsilon\nu(x) + \frac{\partial f}{\partial y'}\varepsilon\nu'(x) \right) dx = \frac{\partial f}{\partial y'}\varepsilon\nu(x)\bigg|_{x=x_1}^{x=x_2}$$

$$+ \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}(\frac{\partial f}{\partial y'}) \right) \varepsilon\nu(x)dx$$

$$= \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}(\frac{\partial f}{\partial y'}) \right) \varepsilon\nu(x)dx = 0, \tag{9.3.4}$$

for $\nu(x)$ arbitrary other than the condition (9.3.3c).

We set

$$J(x) = \frac{\partial f}{\partial y} - \frac{d}{dx}(\frac{\partial f}{\partial y'}). \tag{9.3.5}$$

We suppose that $J(x)$ is positive in the interval $[x_1, x_2]$,

$$J(x) > 0 \quad \text{for} \quad x \in [x_1, x_2]. \tag{9.3.6}$$

Then, by choosing

$$\nu(x) > 0 \quad \text{for} \quad x \in [x_1, x_2], \tag{9.3.7}$$

we can make δI positive,

$$\delta I > 0. \tag{9.3.8}$$

We now suppose that $J(x)$ is negative in the interval $[x_1, x_2]$,

$$J(x) < 0 \quad \text{for} \quad x \in [x_1, x_2]. \tag{9.3.9}$$

Then, by choosing

$$\nu(x) < 0 \quad \text{for} \quad x \in [x_1, x_2], \tag{9.3.10}$$

we can make δI positive,

$$\delta I > 0. \tag{9.3.11}$$

We lastly suppose that $J(x)$ alternates its sign in the interval $[x_1, x_2]$. Then by choosing

$$\nu(x) \gtrless 0 \quad \text{wherever} \quad J(x) \gtrless 0, \tag{9.3.12}$$

we can make δI positive,

$$\delta I > 0. \tag{9.3.13}$$

Thus, in order to have Eq. (9.3.4) for $\nu(x)$ arbitrary together with the condition (9.3.3c), we must have $J(x)$ identically equal to zero,

$$\frac{\partial f}{\partial y} - \frac{d}{dx}(\frac{\partial f}{\partial y'}) = 0, \tag{9.3.14}$$

which is known as the *Euler equation*.

We now illustrate the Euler equation by solving some of the examples listed above.

❑ **Example 9.1.** The shortest distance between two points. In this case, f is given by

$$f = \sqrt{1 + (y')^2}. \tag{9.3.15a}$$

The Euler equation simply gives

$$y' \big/ \sqrt{1+(y')^2} = \text{constant}, \Rightarrow y' = c. \tag{9.3.15b}$$

In general, if $f = f(y')$, independent of x and y, then the Euler equation always gives

$$y' = \text{constant}. \tag{9.3.16}$$

❑ **Example 9.4.** Brachistochrone problem. In this case, f is given by

$$f = \sqrt{\frac{1+(y')^2}{y}}. \tag{9.3.17}$$

The Euler equation gives

$$-\frac{1}{2}\sqrt{\frac{1+(y')^2}{y^3}} - \frac{d}{dx}\frac{y'}{\sqrt{(1+(y')^2)y}} = 0.$$

This appears somewhat difficult to solve. However, there is a simple way which is applicable to many cases.

Suppose

$$f = f(y, y'), \quad \text{independent of } x, \tag{9.3.18a}$$

then

$$\frac{d}{dx} = y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + \frac{\partial}{\partial x},$$

where the last term is absent when acting on $f = f(y, y')$. Thus

$$\frac{d}{dx}f = y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'}.$$

Making use of the Euler equation on the first term of the right-hand side, we have

$$\frac{d}{dx}f = y'\frac{d}{dx}(\frac{\partial f}{\partial y'}) + (\frac{d}{dx}y')\frac{\partial f}{\partial y'} = \frac{d}{dx}(y'\frac{\partial f}{\partial y'}),$$

i.e.,

$$\frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) = 0.$$

Hence we obtain

$$f - y' \frac{\partial f}{\partial y'} = \text{constant}. \tag{9.3.18b}$$

Returning to the *Brachistochrone problem*, we have

$$\sqrt{\frac{1 + (y')^2}{y}} - y' \frac{y'}{\sqrt{(1 + (y')^2)y}} = \text{constant},$$

or,

$$y(1 + (y')^2) = 2R.$$

Solving for y', we obtain

$$\frac{dy}{dx} = y' = \sqrt{\frac{2R - y}{y}}. \tag{9.3.19}$$

Hence we have

$$\int dy \sqrt{\frac{y}{2R - y}} = x.$$

We set

$$y = 2R \sin^2(\frac{\theta}{2}) = R(1 - \cos\theta). \tag{9.3.20a}$$

Then we easily get

$$x = 2R \int \sin^2(\frac{\theta}{2}) d\theta = R(\theta - \sin\theta). \tag{9.3.20b}$$

Equations (9.3.20a) and (9.3.20b) are the *parametric equations for a cycloid*, the curve traced by a point on the rim of a wheel rolling on the x-axis. The shape of a cycloid is displayed in Figure 9.1.

We state several remarks on Example 9.4:

(1) The solution of the fastest fall is not a straight line. It is a cycloid with infinite initial slope.

(2) There exists a unique solution. In our parametric representation of a cycloid, the range of θ is implicitly assumed to be $0 \le \theta \le 2\pi$. Setting $\theta = 0$, we find that the starting point is chosen to be at the origin,

$$(x_1, y_1) = (0, 0). \tag{9.3.21}$$

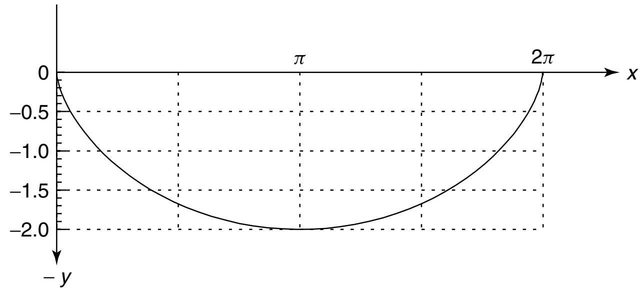

Fig. 9.1 The curve traced by a point on the rim of a wheel rolling on the x-axis.

The question is that given the end point (x_2, y_2), can we uniquely determine a radius of the wheel R. We put $y_2 = R(1 - \cos\theta_0)$, and $x_2 = R(\theta_0 - \sin\theta_0)$, or,

$$\frac{1-\cos\theta_0}{\theta_0 - \sin\theta_0} = \frac{y_2}{x_2}, \quad 2R = \frac{2y_2}{1-\cos\theta_0} = \frac{y_2}{\sin^2(\theta_0/2)},$$

which has a unique solution in the range $0 < \theta_0 < \pi$.

(3) The shortest time of descent is

$$T = \int_0^{x_2} dx \sqrt{\frac{1+(y')^2}{y}} = 2\sqrt{2R}\int_0^{\theta_0/2} d\theta = \sqrt{y_2}\frac{\theta_0}{\sin(\theta_0/2)}. \tag{9.3.22}$$

❑ **Example 9.5.** Hamilton's action principle in classical mechanics. Consider the infinitesimal variation $\delta q(t)$ of $q(t)$, vanishing at $t = t_1$ and $t = t_2$,

$$\delta q(t_1) = \delta q(t_2) = 0. \tag{9.3.23}$$

Then the Hamilton's action principle demands that

$$\delta I = \delta\int_{t_1}^{t_2} L(q(t), \dot{q}(t), t)dt = \int_{t_1}^{t_2}\left(\delta q(t)\frac{\partial L}{\partial q(t)} + \delta\dot{q}(t)\frac{\partial L}{\partial \dot{q}(t)}\right)dt$$

$$= \int_{t_1}^{t_2} dt\left(\delta q(t)\frac{\partial L}{\partial q(t)} + (\frac{d}{dt}\delta q(t))\frac{\partial L}{\partial \dot{q}(t)}\right)$$

$$= \left[\delta q(t)\frac{\partial L}{\partial \dot{q}(t)}\right]_{t=t_1}^{t=t_2} + \int_{t_1}^{t_2} dt\delta q(t)\left(\frac{\partial L}{\partial q(t)} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}(t)})\right)$$

$$= \int_{t_1}^{t_2} dt\delta q(t)\left(\frac{\partial L}{\partial q(t)} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}(t)})\right) = 0, \tag{9.3.24}$$

where $\delta q(t)$ is arbitrary other than the condition (9.3.23). From this, we obtain the *Lagrange equation of motion,*

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}(t)}) - \frac{\partial L}{\partial q(t)} = 0, \tag{9.3.25}$$

which is nothing but the Euler equation (9.3.14), with the identification

$$t \Rightarrow x, \quad q(t) \Rightarrow y(x), \quad L(q(t), \dot{q}(t), t) \Rightarrow f(x, y, y').$$

When the Lagrangian $L(q(t), \dot{q}(t), t)$ does not depend on t explicitly, the following quantity:

$$\dot{q}(t)\frac{\partial L}{\partial \dot{q}(t)} - L(q(t), \dot{q}(t)) \equiv E, \tag{9.3.26}$$

is a constant of motion and is called the *energy integral,* which is nothing but Eq. (9.3.18b). Solving the energy integral for $\dot{q}(t)$, we can obtain the *differential equation* for $q(t)$.

9.4
Generalization of the Basic Problems

❑ **Example 9.6.** Free end point: y_2 arbitrary.

An example is to consider, in the *Brachistochrone problem,* the dependence of the shortest time of fall as a function of y_2. The question is: What is the height of fall y_2 which, for a given x_2, minimizes this time of fall? We may, of course, start by taking the expression for the shortest time of fall:

$$T = \sqrt{y_2}\frac{\theta_0}{\sin(\theta_0/2)} = \sqrt{2x_2}\frac{\theta_0}{\sqrt{\theta_0 - \sin\theta_0}},$$

which, for a given x_2, has a minimum at $\theta_0 = \pi$, where $T = \sqrt{2\pi x_2}$. We shall, however, give a treatment for the general problem of free end point.

To extremize

$$I = \int_{x_1}^{x_2} f(x, y, y')dx, \tag{9.4.1}$$

with

$$y(x_1) = y_1 \quad \text{and} \quad y_2 \text{ arbitrary}, \tag{9.4.2}$$

we require

$$\delta I = \frac{\partial f}{\partial y'} \varepsilon \nu(x)\bigg|_{x=x_2} + \varepsilon \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \right] \nu(x) dx = 0. \tag{9.4.3}$$

By choosing $\nu(x_2) = 0$, we get the Euler equation. Next we choose $\nu(x_2) \neq 0$ and obtain in addition,

$$\frac{\partial f}{\partial y'}\bigg|_{x=x_2} = 0. \tag{9.4.4}$$

Note that y_2 is determined by these equations.

For the *Brachistochrone problem* of arbitrary y_2, we get

$$\frac{\partial f}{\partial y'}\bigg|_{x=x_2} = \frac{y'}{\sqrt{(1 + (y')^2)y}} = 0 \quad \Rightarrow \quad y' = 0.$$

Thus $\theta_0 = \pi$, and $y_2 = x_2(\frac{2}{\pi})$, as obtained previously.

❑ **Example 9.7.** Endpoint on the curve $y = g(x)$: an example is to find the shortest time of descent to a curve. Note that in this problem, neither x_2 nor y_2 are given. They are to be determined and related by $y_2 = g(x_2)$.

Suppose that $y = y(x)$ is the solution. This means that if we make a variation

$$y(x) \to y(x) + \varepsilon \nu(x), \tag{9.4.5}$$

which intersects the curve at $(x_2 + \Delta x_2,\ y_2 + \Delta y_2)$, then $y(x_2 + \Delta x_2) + \varepsilon \nu(x_2 + \Delta x_2) = g(x_2 + \Delta x_2)$, or

$$\varepsilon \nu(x_2) = (g'(x_2) - y'(x_2))\Delta x_2, \tag{9.4.6}$$

and that the variation δI vanishes:

$$\delta I = \int_{x_1}^{x_2+\Delta x_2} f(x, y + \varepsilon \nu, y' + \varepsilon \nu')dx - \int_{x_1}^{x_2} f(x, y, y')dx$$

$$\simeq \left[f(x, y, y')\Delta x_2 + \frac{\partial f}{\partial y'} \varepsilon \nu(x) \right]_{x=x_2} + \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} \right) \varepsilon \nu(x) dx = 0.$$

Thus, in addition to the Euler equation, we have

$$\left[f(x, y, y')\Delta x_2 + \frac{\partial f}{\partial y'} \varepsilon \nu(x) \right]_{x=x_2} = 0,$$

or, using Eq. (9.4.6), we obtain

$$\left[f(x, y, y') + \frac{\partial f}{\partial y'}(g' - y')\right]_{x=x_2} = 0. \tag{9.4.7}$$

Applying the above equation to the *Brachistochrone problem*, we get

$$y'g' = -1.$$

This means that the path of fastest descent intersects the curve $y = g(x)$ at a right angle.

❑ **Example 9.8.** The isoperimetric problem: find $y(x)$ which extremizes

$$I = \int_{x_1}^{x_2} f(x, y, y')dx \tag{9.4.8}$$

while keeping

$$J = \int_{x_1}^{x_2} F(x, y, y')dx \quad \text{fixed.} \tag{9.4.9}$$

An example is the classic problem of finding the maximum area enclosed by a curve of fixed length.

Let $y(x)$ be the solution. This means that if we make a variation of y which does not change the value of J , the variation of I must vanish. Since J cannot change, this variation is not of the form $\varepsilon\nu(x)$, with $\nu(x)$ arbitrary. Instead, we must put

$$y(x) \rightarrow y(x) + \varepsilon_1\nu_1(x) + \varepsilon_2\nu_2(x), \tag{9.4.10}$$

where ε_1 and ε_2 are so chosen that

$$\delta J = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'}\right)\left(\varepsilon_1\nu_1(x) + \varepsilon_2\nu_2(x)\right) dx = 0. \tag{9.4.11}$$

For these kinds of variations , $y(x)$ extremizes I:

$$\delta I = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right)\left(\varepsilon_1\nu_1(x) + \varepsilon_2\nu_2(x)\right) dx = 0. \tag{9.4.12}$$

Eliminating ε_2, we get

$$\varepsilon_1 \int_{x_1}^{x_2} \left[\frac{\partial}{\partial y}(f + \lambda F) - \frac{d}{dx}\frac{\partial}{\partial y'}(f + \lambda F)\right] \nu_1(x)dx = 0, \tag{9.4.13a}$$

where

$$\lambda = \left[\int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right) \nu_2(x)dx\right] \Big/ \left[\int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'}\right) \nu_2(x)dx\right]. \tag{9.4.13b}$$

Thus $(f + \lambda F)$ satisfies the Euler equation. The λ is determined by solving the Euler equation, substituting y into the integral for J, and requiring that J takes the prescribed value.

❑ **Example 9.9.** Integral involves more than one function:
Extremize

$$I = \int_{x_1}^{x_2} f(x, y, y', z, z')dx, \tag{9.4.14}$$

with y and z taking prescribed values at the end points. By varying y and z successively, we get

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0, \tag{9.4.15a}$$

and

$$\frac{\partial f}{\partial z} - \frac{d}{dx}\frac{\partial f}{\partial z'} = 0. \tag{9.4.15b}$$

❑ **Example 9.10.** Integral involves y'':

$$I = \int_{x_1}^{x_2} f(x, y, y', y'')dx, \tag{9.4.16}$$

with y and y' taking prescribed values at the end points. The Euler equation is

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial f}{\partial y''} = 0. \tag{9.4.17}$$

❑ **Example 9.11.** Integral is multidimensional:

$$I = \int dxdtf(x, t, y, y_x, y_t), \tag{9.4.18}$$

with y taking the prescribed values at the boundary. The Euler equation is

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y_x} - \frac{d}{dt}\frac{\partial f}{\partial y_t} = 0. \tag{9.4.19}$$

9.5
More Examples

❑ **Example 9.12.** Catenary:

(a) *Shape of a chain hanging on two pegs*: the gravitational potential of a chain of uniform density is proportional to

$$I = \int_{x_1}^{x_2} y\sqrt{1 + (y')^2}dx. \tag{9.5.1a}$$

The equilibrium position of the chain minimizes I, subject to the condition that the length of the chain is fixed,

$$J = \int_{x_1}^{x_2} \sqrt{1 + (y')^2}dx \quad \text{fixed.} \tag{9.5.1b}$$

Thus we extremize $I + \lambda J$ and obtain

$$\frac{(y + \lambda)}{\sqrt{1 + (y')^2}} = \alpha \quad \text{constant,}$$

or,

$$\int \frac{dy}{\sqrt{(y + \lambda)^2 / \alpha^2 - 1}} = \int dx. \tag{9.5.2}$$

We put

$$(y + \lambda)/\alpha = \cosh\theta, \tag{9.5.3}$$

then

$$x - \beta = \alpha\theta, \tag{9.5.4}$$

and the shape of the chain is given by

$$y = \alpha\cosh(\frac{x - \beta}{\alpha}) - \lambda. \tag{9.5.5}$$

The constants, α β, and λ, are determined by the two boundary conditions and the requirement that J is equal to the length of the chain.

Let us consider the case

$$y(-L) = y(L) = 0. \tag{9.5.6}$$

Then the boundary conditions give

$$\beta = 0, \quad \lambda = \alpha \cosh \frac{L}{\alpha}. \tag{9.5.7}$$

The condition that J is constant gives

$$\frac{\alpha}{L} \sinh \frac{L}{\alpha} = \frac{l}{L}, \tag{9.5.8}$$

where $2l$ is the length of the chain. It is easily shown that, for $l \geq L$, a unique solution is obtained.

(b) *Soap film formed by two circular wires*: surface of soap film takes minimum area as a result of surface tension. Thus we minimize

$$I = \int_{x_1}^{x_2} y\sqrt{1 + (y')^2}dx, \tag{9.5.9}$$

obtaining as in (a),

$$y = \alpha \cosh(\frac{x - \beta}{\alpha}), \tag{9.5.10}$$

where α and β are constants of integration, to be determined from the boundary conditions at $x = \pm L$.

Let us consider the special case in which the two circular wires are of equal radius R. Then

$$y(-L) = y(L) = R. \tag{9.5.11}$$

We easily find that $\beta = 0$, and that α is determined by the equation

$$\frac{R}{L} = \frac{\alpha}{L} \cosh \frac{L}{\alpha}, \tag{9.5.12}$$

which has zero, one, or two solutions depending on the ratio R/L. In order to decide if any of these solutions actually minimizes I, we must study the *second variation*, which we shall discuss in Section 9.7.

❑ **Example 9.5.** Hamilton 's action principle in classical mechanics:
Let the Lagrangian $L(q(t), \dot{q}(t))$ be defined by

$$L(q(t), \dot{q}(t)) \equiv T(q(t), \dot{q}(t)) - V(q(t), \dot{q}(t)), \tag{9.5.13}$$

where T and V are the *kinetic energy* and the *potential energy* of the mechanical system, respectively. In general, T and V can depend on both $q(t)$ and $\dot{q}(t)$. When T and V are given, respectively, by

$$T = \frac{1}{2}m\dot{q}(t)^2, \quad V = V(q(t)), \tag{9.5.14}$$

the Lagrange equation of motion (9.3.25) provides us *Newton's equation of motion*,

$$m\ddot{q}(t) = -\frac{d}{dq(t)}V(q(t)) \quad \text{with} \quad \ddot{q}(t) = \frac{d^2}{dt^2}q(t). \tag{9.5.15}$$

In other words, the extremization of the action integral I given by

$$I = \int_{t_1}^{t_2} [\frac{1}{2}m\dot{q}(t)^2 - V(q(t))]dt$$

with $\delta q(t_1) = \delta q(t_2) = 0$, leads us to Newton's equation of motion (9.5.15). With T and V given by Eq. (9.5.14), the energy integral E given by Eq. (9.3.26) assumes the following form:

$$E = \frac{1}{2}m\dot{q}(t)^2 + V(q(t)), \tag{9.5.16}$$

which represents the total mechanical energy of the system, quite appropriate for the terminology, the *energy integral*.

❑ **Example 9.13.** Fermat's principle in geometrical optics:
The path of a light ray between two given points in a medium is the one which minimizes the time of travel. Thus the path is determined from minimizing

$$T = \frac{1}{c}\int_{x_1}^{x_2} dx\sqrt{1 + (\frac{dy}{dx})^2 + (\frac{dz}{dx})^2}n(x, y, z), \tag{9.5.17}$$

where $n(x, y, z)$ is the index of refraction. If n is independent of x, we get

$$n(y, z)/\sqrt{1 + (\frac{dy}{dx})^2 + (\frac{dz}{dx})^2} = \text{constant}. \tag{9.5.18}$$

From Eq. (9.5.18), we easily derive the *law of reflection* and the *law of refraction* (*Snell's law*).

9.6 Differential Equations, Integral Equations, and Extremization of Integrals

We now consider the inverse problem: If we are to solve a differential or an integral equation, can we formulate the problem in terms of problem of extremizing an integral? This will have practical advantages when we try to obtain approximate solutions of differential equations and approximate eigenvalues.

❑ **Example 9.14.** Solve

$$\frac{d}{dx}\left[p(x)\frac{d}{dx}y(x)\right] - q(x)y(x) = 0, \quad x_1 < x < x_2, \tag{9.6.1}$$

with

$$y(x_1), \quad y(x_2) \quad \text{specified.} \tag{9.6.2}$$

This problem is equivalent to extremizing the integral

$$I = \frac{1}{2}\int_{x_1}^{x_2} \left[p(x)(y'(x))^2 + q(x)(y(x))^2\right] dx. \tag{9.6.3}$$

❑ **Example 9.15.** Solve the *Sturm–Liouville eigenvalue problem*

$$\frac{d}{dx}\left[p(x)\frac{d}{dx}y(x)\right] - q(x)y(x) = \lambda r(x)y(x), \quad x_1 < x < x_2, \tag{9.6.4}$$

with

$$y(x_1) = y(x_2) = 0. \tag{9.6.5}$$

This problem is equivalent to extremizing the integral

$$I = \frac{1}{2}\int_{x_1}^{x_2} \left[p(x)(y'(x))^2 + q(x)(y(x))^2\right] dx, \tag{9.6.6}$$

while keeping

$$J = \frac{1}{2}\int_{x_1}^{x_2} r(x)(y(x))^2 dx \quad \text{fixed.} \tag{9.6.7}$$

In practice, we find the approximation to the lowest eigenvalue and the corresponding eigenfunction of the Sturm–Liouville eigenvalue problem by minimizing I/J. Note that an eigenfunction good to the first order yields an eigenvalue good to the second order.

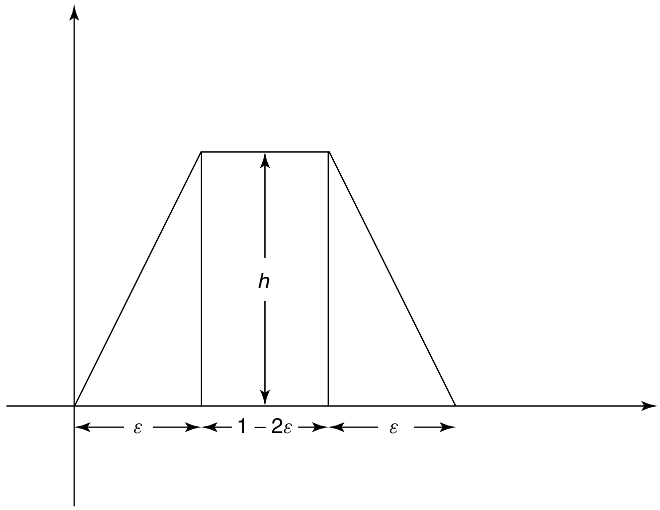

Fig. 9.2 The shape of the trial function for Example 9.16.

❑ **Example 9.16.** Solve

$$\frac{d^2}{dx^2}y(x) = -\lambda y(x), \quad 0 < x < 1, \tag{9.6.8}$$

with

$$y(0) = y(1) = 0. \tag{9.6.9}$$

Solution. We choose the trial function to be the one in Figure 9.2. Then $I = 2 \cdot h^2 / \varepsilon$, $J = h^2(1 - \frac{4}{3}\varepsilon)$, and $I/J = 2/\left[\varepsilon(1 - \frac{4}{3}\varepsilon)\right]$, which has a minimum value of $\frac{32}{3}$ at $\varepsilon = \frac{3}{8}$. This is compared with the exact value, $\lambda = \pi^2$. Note that $\lambda = \pi^2$ is a lower bound for I/J. If we choose the trial function to be

$$y(x) = x(1 - x),$$

we get $I/J = 10$. This is accurate to almost one percent.

In order to obtain an accurate estimate of the eigenvalue, it is important to choose a trial function that satisfies the boundary condition and looks qualitatively like the expected solution. For instance, if we are calculating the lowest eigenvalue, it would be unwise to use a trial function that has a zero inside the interval.

Let us now calculate the next eigenvalue. This is done by choosing the trial function $y(x)$ which is orthogonal to the exact lowest eigenfunction $u_0(x)$, i.e.,

$$\int_0^1 y(x)u_0(x)r(x)dx = 0, \tag{9.6.10}$$

and find the minimum value of I/J with respect to some parameters in the trial function. Let us choose the trial function to be

$$y(x) = x(1-x)(1-ax),$$

and then the requirement that it is orthogonal to $x(1-x)$, instead of $u_0(x)$ which is unknown, gives $a = 2$. Note that this trial function has one zero inside the interval $[0, 1]$. This looks qualitatively like the expected solution. For this trial function, we have

$$I/J = 42,$$

and this compares well with the exact value, $4\pi^2$.

❑ **Example 9.17.** Solve the *Laplace equation*

$$\nabla^2\phi = 0, \tag{9.6.11}$$

with ϕ given at the boundary.

This problem is equivalent to extremizing

$$I = \int (\vec{\nabla}\phi)^2 dV. \tag{9.6.12}$$

❑ **Example 9.18.** Solve the *wave equation*

$$\nabla^2\phi = k^2\phi, \tag{9.6.13a}$$

with

$$\phi = 0 \tag{9.6.13b}$$

at the boundary.

This problem is equivalent to extremizing

$$\int (\vec{\nabla}\phi)^2 dV \Big/ \left[\int \phi^2 dV\right]. \tag{9.6.14}$$

❑ **Example 9.19.** Estimate the lowest frequency of a circular drum of radius R.

Solution.

$$k^2 \leq \int (\vec{\nabla}\phi)^2 dV \Big/ \int \phi^2 dV. \tag{9.6.15}$$

Try a rotationally symmetric trial function,

$$\phi(r) = 1 - \frac{r}{R}, \quad 0 \leq r \leq R. \tag{9.6.16}$$

Then we get

$$k^2 \leq \frac{\int_0^R (\phi_r(r))^2 2\pi r dr}{\int_0^R (\phi(r))^2 2\pi r dr} = \frac{6}{R^2}. \tag{9.6.17}$$

This is compared with the exact value,

$$k^2 = \frac{5.7832}{R^2}. \tag{9.6.18}$$

Note that the numerator of the right-hand side of Eq. (9.6.18) is the square of the smallest zero in magnitude of the 0th order *Bessel function of the first kind*, $J_0(kR)$.

The homogeneous Fredholm integral equations of the second kind for the localized, monochromatic, and highly directive classical current distributions in two and three dimensions can be derived by maximizing the directivity D in the far field while constraining $C = N/T$, where N is the integral of the square of the magnitude of the current density and T is proportional to the total radiated power. The homogeneous Fredholm integral equations of the second kind and the inhomogeneous Fredholm integral equations of the second kind are now derived from the calculus of variations in general term.

❑ **Example 9.20.** Solve the *homogeneous Fredholm integral equation of the second kind*,

$$\phi(x) = \lambda \int_0^h K(x, x')\phi(x')dx', \tag{9.6.19}$$

with the square-integrable kernel $K(x, x')$, and the projection unity on $\psi(x)$,

$$\int_0^h \psi(x)\phi(x)dx = 1. \tag{9.6.20}$$

This problem is equivalent to extremizing the integral

$$I = \int_0^h \int_0^h \psi(x)K(x, x')\phi(x')dxdx', \tag{9.6.21}$$

with respect to $\psi(x)$, while keeping

$$J = \int_0^h \psi(x)\phi(x)dx = 1 \quad \text{fixed}. \tag{9.6.22}$$

The extremization of Eq. (9.6.21) with respect to $\phi(x)$, while keeping J fixed, results in the *homogeneous adjoint integral equation* for $\psi(x)$,

$$\psi(x) = \lambda \int_0^h \psi(x')K(x', x)dx'. \tag{9.6.23}$$

With the real and symmetric kernel , $K(x, x') = K(x', x)$, the homogeneous integral equations for $\phi(x)$ and $\psi(x)$, Eqs. (9.6.19) and (9.6.23), are identical and Eq. (9.6.20) provides the normalization of $\phi(x)$ and $\psi(x)$ to the unity, respectively.

❑ **Example 9.21.** Solve the *inhomogeneous Fredholm integral equation of the second kind*

$$\phi(x) - \lambda \int_0^h K(x, x')\phi(x')dx' = f(x), \tag{9.6.24}$$

with the square-integrable kernel $K(x, x')$, $0 < \|K\|^2 < \infty$.

This problem is equivalent to extremizing the integral

$$I = \int_0^h [\{\frac{1}{2}\phi(x) - \lambda \int_0^h K(x, x')\phi(x')dx'\}\phi(x) + F(x)\frac{d}{dx}\phi(x)]dx, \tag{9.6.25}$$

where $F(x)$ is defined by

$$F(x) = \int^x f(x')dx'. \tag{9.6.26}$$

9.7
The Second Variation

The *Euler equation* is *necessary* for extremizing the integral, but it is *not sufficient*. In order to find out whether the solution of the Euler equation actually extremizes the integral, we must study the *second variation* . This is similar to the case of finding an extremum of a function: To confirm that the point at which the first derivative of a function vanishes is an extremum of the function, we must study the second derivatives.

Consider the extremization of

$$I = \int_{x_1}^{x_2} f(x, y, y')dx, \tag{9.7.1}$$

with

$$y(x_1) \quad \text{and} \quad y(x_2) \text{ specified.} \tag{9.7.2}$$

Suppose $y(x)$ is such a solution.

Let us consider a *weak variation*,

$$\begin{cases} y(x) & \rightarrow & y(x) + \varepsilon\nu(x), \\ y'(x) & \rightarrow & y'(x) + \varepsilon\nu'(x), \end{cases} \tag{9.7.3}$$

as opposed to a *strong variation* in which $y'(x)$ is also varied, independent of $\varepsilon\nu'(x)$. Then

$$I \rightarrow I + \varepsilon I_1 + \frac{1}{2}\varepsilon^2 I_2 + \cdots, \tag{9.7.4}$$

where

$$I_1 = \int_{x_1}^{x_2} [\frac{\partial f}{\partial y}\nu(x) + \frac{\partial f}{\partial y'}\nu'(x)]dx = \int_{x_1}^{x_2} [\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}]\nu(x)dx, \tag{9.7.5}$$

and

$$I_2 = \int_{x_1}^{x_2} [\frac{\partial^2 f}{\partial y^2}\nu^2(x) + 2\frac{\partial^2 f}{\partial y \partial y'}\nu(x)\nu'(x) + \frac{\partial^2 f}{\partial y'^2}\nu'^2(x)]dx. \tag{9.7.6a}$$

If $y = y(x)$ indeed minimizes or maximizes I, then I_2 must be positive or negative for all variations $\nu(x)$ vanishing at the end points, when the solution of the Euler equation, $y = y(x)$, is substituted into the integrand of I_2.

The integrand of I_2 is a quadratic form of $\nu(x)$ and $\nu'(x)$. Therefore, this integral is always positive if

$$(\frac{\partial^2 f}{\partial y \partial y'})^2 - (\frac{\partial^2 f}{\partial y^2})(\frac{\partial^2 f}{\partial y'^2}) < 0, \tag{9.7.7a}$$

and

$$\frac{\partial^2 f}{\partial y'^2} > 0. \tag{9.7.7b}$$

Thus, if the above conditions hold throughout

$$x_1 < x < x_2,$$

I_2 is always positive and $y(x)$ minimizes I. Similar considerations hold, of course, for maximizations. The above conditions are, however, too crude, i.e., stronger than necessary. This is because $\nu(x)$ and $\nu'(x)$ are not independent.

Let us first state the necessary and sufficient conditions for the solution to the Euler equation to give the *weak minimum*:

Let

$$P(x) \equiv f_{yy}, \quad Q(x) \equiv f_{yy'}, \quad R(x) \equiv f_{y'y'}, \tag{9.7.8}$$

where $P(x)$, $Q(x)$, and $R(x)$ are evaluated at the point which extremizes the integral I defined by

$$I \equiv \int_{x_1}^{x_2} f(x, y, y')dx.$$

We express I_2 in terms of $P(x)$, $Q(x)$, and $R(x)$ as

$$I_2 = \int_{x_1}^{x_2} [P(x)\nu^2(x) + 2Q(x)\nu(x)\nu'(x) + R(x)\nu'^2(x)]dx. \tag{9.7.6b}$$

Then we have the following conditions for the weak minimum:

	Necessary condition	*Sufficient condition*	
Legendre test	$R(x) \geq 0$	$R(x) > 0$	(9.7.9)
Jacobi test	$\xi \geq x_2$	$\xi > x_2$, or no such ξ exists,	

where ξ is the *conjugate point* to be defined later. Both the conditions must be satisfied for sufficiency and both are necessary separately. Before we go on to prove the above assertion, it is perhaps helpful to have an intuitive understanding of why these two tests are relevant.

Let us consider the case in which $R(x)$ is positive at $x = a$, and negative at $x = b$, where a and b are both between x_1 and x_2. Let us first choose $\nu(x)$ to be the function as in Figure 9.3.
Note that $\nu(x)$ is of the order of ε, while $\nu'(x)$ is of the order $\sqrt{\varepsilon}$. If we choose ε to be sufficiently small, I_2 is positive. Next, we consider the same variation located at $x = b$. By the same consideration, I_2 is negative for this variation. Thus $y(x)$ does not extremize I. This shows that the *Legendre test* is a necessary condition.

The relevance of the *Jacobi test* is best illustrated by the problem of finding the shortest path between two points on the surface of a sphere. The solution is obtained by going along the great circle on which these two points lie. There are, however, two paths connecting these two points on the great circle. One of them is truly the shortest path, while the other is neither the shortest nor the longest. Take the circle on the surface of the sphere which passes one of the two points. The other point at which this circle and the great circle intersects lies on the one arc of the great circle which is neither the shortest nor the longest arc. This point is the conjugate point ξ of this problem.

We first discuss the *Legendre test*. We add to I_2, (9.7.6b), the following term which is zero,

$$\int_{x_1}^{x_2} \frac{d}{dx}(\nu^2(x)\omega(x))dx = \int_{x_1}^{x_2} (\nu^2(x)\omega'(x) + 2\nu(x)\nu'(x)\omega(x))dx. \tag{9.7.10}$$

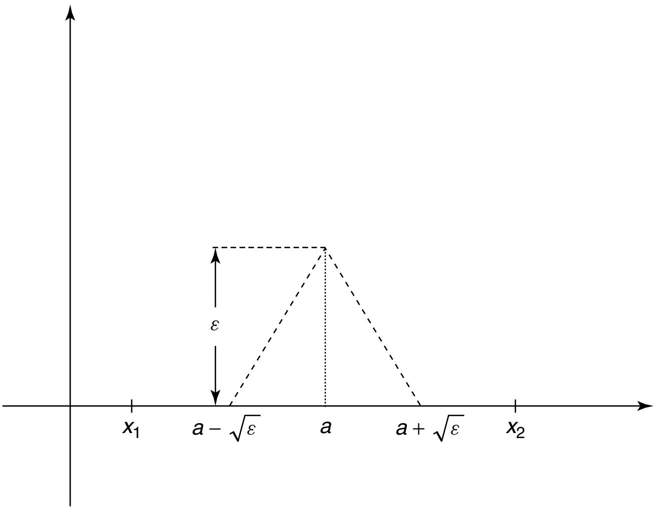

Fig. 9.3 The shape of $\nu(x)$ for the Legendre test.

Then we have I_2 to be

$$I_2 = \int_{x_1}^{x_2} \left[(P(x) + \omega'(x))\nu^2(x) + 2(Q(x) + \omega(x))\nu(x)\nu'(x) + R(x)\nu'^2(x) \right] dx. \tag{9.7.11}$$

We require the integrand to be a *complete square*, i.e.,

$$(Q(x) + \omega(x))^2 = (P(x) + \omega'(x))R(x). \tag{9.7.12}$$

Hence, if we can find $\omega(x)$ satisfying Eq. (9.7.12), we will have

$$I_2 = \int_{x_1}^{x_2} R(x) \left[\nu'(x) + \frac{Q(x) + \omega(x)}{R(x)} \nu(x) \right]^2 dx. \tag{9.7.13}$$

Now, it is not possible to have $R(x) < 0$ in any region between x_1 and x_2 for the minimum. If $R(x) < 0$ in some regions, we can solve the differential equation

$$\omega'(x) = -P(x) + \frac{(Q(x) + \omega(x))^2}{R(x)} \tag{9.7.14}$$

for $\omega(x)$ in such regions. Restricting the variation $\nu(x)$ such that

$$\begin{cases} \nu(x) \equiv 0, & \text{outside the region where } R(x) < 0, \\ \nu(x) \neq 0, & \text{inside the region where } R(x) < 0, \end{cases}$$

we can have $I_2 < 0$. Thus it is necessary to have $R(x) \geq 0$ for the entire region for the minimum. If

$$P(x)R(x) - Q^2(x) > 0, \quad P(x) > 0, \tag{9.7.15a}$$

then we have no need to go further to find $\omega(x)$. In many cases, however, we have

$$P(x)R(x) - Q^2(x) \leq 0, \tag{9.7.15b}$$

and we have to examine further. The differential equation (9.7.14) for $\omega(x)$ can be rewritten as

$$(Q(x) + \omega(x))' = -P(x) + Q'(x) + \frac{(Q(x) + \omega(x))^2}{R(x)}, \tag{9.7.16}$$

which is the *Riccatti differential equation*. Making the *Riccatti substitution*

$$Q(x) + \omega(x) = -R(x)\frac{u'(x)}{u(x)}, \tag{9.7.17}$$

we obtain the *second-order linear ordinary differential equation for* $u(x)$,

$$\frac{d}{dx}[R(x)\frac{d}{dx}u(x)] + (Q'(x) - P(x))u(x) = 0. \tag{9.7.18}$$

Expressing everything in Eq. (9.7.13) in terms of $u(x)$, I_2 becomes

$$I_2 = \int_{x_1}^{x_2} [R(x)(v'(x)u(x) - u'(x)v(x))^2 / u^2(x)]dx. \tag{9.7.19}$$

If we can find any $u(x)$ such that

$$u(x) \neq 0, \quad x_1 \leq x \leq x_2, \tag{9.7.20}$$

we will have

$$I_2 > 0 \quad \text{for} \quad R(x) > 0, \tag{9.7.21}$$

which is the sufficient condition for the minimum . This completes the derivation of the Legendre test. We further clarify the Legendre test after the discussion of the conjugate point ξ of the Jacobi test .

The ordinary differential equation (9.7.18) for $u(x)$ is related to the Euler equation through the *infinitesimal variation of the initial condition*. In the Euler equation,

$$f_y(x, y, y') - \frac{d}{dx}f_{y'}(x, y, y') = 0 \tag{9.7.22}$$

we make the following infinitesimal variation:

$$y(x) \to y(x) + \varepsilon u(x), \quad \varepsilon = \text{positive infinitesimal}. \tag{9.7.23}$$

Writing out this variation explicitly,

$$f_y(x, y + \varepsilon u, y' + \varepsilon u') - \frac{d}{dx} f_{y'}(x, y + \varepsilon u, y' + \varepsilon u') = 0,$$

we have, with the use of the Euler equation,

$$f_{yy}u + f_{yy'}u' - \frac{d}{dx}\left(f_{y'y}u + f_{y'y'}u'\right) = 0,$$

or,

$$Pu + Qu' - \frac{d}{dx}(Qu + Ru') = 0, \tag{9.7.24}$$

i.e.,

$$\frac{d}{dx}[R(x)\frac{d}{dx}u(x)] + (Q'(x) - P(x))u(x) = 0,$$

which is nothing but the ordinary differential equation (9.7.18). The solution to the differential equation (9.7.18) corresponds to the infinitesimal change of the initial condition. We further note that the differential equation (9.7.18) is of the *self-adjoint form*. Thus its Wronskian $W(u_1(x), u_2(x))$ is given by

$$W(x) \equiv u_1(x)u_2'(x) - u_1'(x)u_2(x) = C/R(x), \tag{9.7.25}$$

where $u_1(x)$ and $u_2(x)$ are the linearly independent solutions of Eq. (9.7.18) and C in Eq. (9.7.25) is some constant.

We now discuss the *conjugate point* ξ and the *Jacobi test*. We claim that the sufficient condition for the minimum is that $R(x) > 0$ on (x_1, x_2) and that the conjugate point ξ lies outside (x_1, x_2), i.e., both the Legendre test and the Jacobi test are satisfied. Suppose that

$$R(x) > 0 \quad \text{on} \quad (x_1, x_2), \tag{9.7.26}$$

which implies that the Wronskian has the same sign on (x_1, x_2). Suppose that $u_1(x)$ vanishes *only* at $x = \xi_1$ and $x = \xi_2$,

$$u_1(\xi_1) = u_1(\xi_2) = 0, \quad x_1 < \xi_1 < \xi_2 < x_2. \tag{9.7.27}$$

We claim that $u_2(x)$ must vanish at least once between ξ_1 and ξ_2. The Wronskian W evaluated at $x = \xi_1$ and $x = \xi_2$ are given, respectively, by

$$W(\xi_1) = -u_1'(\xi_1)u_2(\xi_1), \tag{9.7.28a}$$

and

$$W(\xi_2) = -u_1'(\xi_2)u_2(\xi_2). \tag{9.7.28b}$$

By the continuity of $u_1(x)$ on (x_1, x_2), $u_1'(\xi_1)$ has the opposite sign to $u_1'(\xi_2)$, i.e.,

$$u_1'(\xi_1)u_1'(\xi_2) < 0. \tag{9.7.29}$$

But, we have

$$W(\xi_1)W(\xi_2) = u_1'(\xi_1)u_1'(\xi_2)u_2(\xi_1)u_2(\xi_2) > 0, \tag{9.7.30}$$

from which we conclude that $u_2(\xi_1)$ has the opposite sign to $u_2(\xi_2)$, i.e.,

$$u_2(\xi_1)u_2(\xi_2) < 0. \tag{9.7.31}$$

Hence $u_2(x)$ must vanish at least once between ξ_1 and ξ_2.

Since $u(x)$ provide the infinitesimal variation of the initial condition, we choose $u_1(x)$ and $u_2(x)$ to be

$$u_1(x) \equiv \frac{\partial}{\partial \alpha} y(x, \alpha, \beta), \quad u_2(x) \equiv \frac{\partial}{\partial \beta} y(x, \alpha, \beta), \tag{9.7.32}$$

where $y(x, \alpha, \beta)$ is the solution of the Euler equation,

$$f_y - \frac{d}{dx} f_{y'} = 0, \tag{9.7.33a}$$

with the initial conditions,

$$y(x_1) = \alpha, \quad y(x_2) = \beta, \tag{9.7.33b}$$

and $u_i(x)$ $(i = 1, 2)$ satisfy the differential equation (9.7.18). We now claim that the sufficient condition for the weak minimum is that

$$R(x) > 0 \quad \text{and} \quad \xi > x_2. \tag{9.7.34}$$

We construct $U(x)$ by

$$U(x) \equiv u_1(x)u_2(x_1) - u_1(x_1)u_2(x), \tag{9.7.35}$$

which vanishes at $x = x_1$. We define the conjugate point ξ as the solution of the equation,

$$U(\xi) = 0, \quad \xi \neq x_1. \tag{9.7.36}$$

The function $U(x)$ represents another infinitesimal change in the solution to the Euler equation which would also pass through $x = x_1$. Since

$$U(x_1) = U(\xi) = 0, \tag{9.7.37}$$

there exists another solution $u(x)$ such that

$$u(x) = 0 \quad \text{for} \quad x \in (x_1, \xi). \tag{9.7.38}$$

We choose x_3 such that

$$x_2 < x_3 < \xi \tag{9.7.39}$$

and another solution $u(x)$ to be

$$u(x) = u_1(x)u_2(x_3) - u_1(x_3)u_2(x), \quad u(x_3) = 0. \tag{9.7.40}$$

We claim that

$$u(x) \neq 0 \quad \text{on} \quad (x_1, x_2). \tag{9.7.41}$$

Suppose $u(x) = 0$ in this interval. Then any other solution of the differential equation (9.7.18) must vanish between these points. But, $U(x)$ does not vanish. This completes the derivation of the Jacobi test.

We further clarify the Legendre test and the Jacobi test. We now assume that

$$\xi < x_2, \quad \text{and} \quad R(x) > 0 \quad \text{for} \quad x \in (x_1, x_2), \tag{9.7.42}$$

and show that

$$I_2 < 0 \quad \text{for some } \nu(x) \quad \text{such that} \quad \nu(x_1) = \nu(x_2) = 0. \tag{9.7.43}$$

We choose x_3 such that

$$\xi < x_3 < x_2.$$

We construct the solution $U(x)$ to Eq. (9.7.18) such that

$$U(x_1) = U(\xi) = 0.$$

Also, we construct the solution $\pm v(x)$, independent of $U(x)$, such that

$$v(x_3) = 0,$$

i.e., we choose x_3 such that

$$U(x_3) \neq 0.$$

We choose the sign of $v(x)$ such that the Wronskian $W(U(x), v(x))$ is given by

$$W(U(x), v(x)) = U(x)v'(x) - v(x)U'(x) = \frac{C}{R(x)}, \quad R(x) > 0, \quad C > 0.$$

The function $U(x) - v(x)$ solves the differential equation (9.7.18). It must vanish at least once between x_1 and ξ. We call this point $x = a$, so that

$$U(a) = v(a),$$

and

$$x_1 < a < \xi < x_3 < x_2.$$

We define $\nu(x)$ to be

$$\nu(x) \equiv \begin{cases} U(x), & \text{for} \quad x \in (x_1, a), \\ v(x), & \text{for} \quad x \in (a, x_3), \\ 0, & \text{for} \quad x \in (x_3, x_2). \end{cases}$$

In I_2, we rewrite the term involving $Q(x)$ as

$$2Q(x)\nu(x)\nu'(x) = Q(x)d(\nu^2(x)),$$

and we perform integration by parts in I_2 to obtain

$$I_2 = \left[Q(x)\nu^2(x) + R(x)\nu'(x)\nu(x)\right]\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \nu(x)\left[R(x)\nu''(x) + R'(x)\nu'(x)\right.$$

$$\left. + \left(Q'(x) - P(x)\right)\nu(x)\right] dx.$$

The integral in the second term of the right-hand side is broken up into three separate integrals, each of which vanishes identically since $\nu(x)$ satisfies the differential equation (9.7.18) in all the three regions. The $Q(x)$ term of the integrated part of I_2 also vanishes, i.e.,

$$Q(x)\nu^2(x)\Big|_{x_1}^{a} + Q(x)\nu^2(x)\Big|_{a}^{x_3} + Q(x)\nu^2(x)\Big|_{x_3}^{x_2} = 0.$$

We now consider the $R(x)$ term of the integrated part of I_2,

$$I_2 = R(x)\nu'(x)\nu(x)\Big|_{a-\varepsilon} - R(x)\nu'(x)\nu(x)\Big|_{a+\varepsilon}$$

$$= R(a)[U'(x)v(x) - v'(x)U(x)]_{x=a},$$

where the continuity of $U(x)$ and $v(x)$ at $x = a$ is used. Thus we have

$$I_2 = R(a)[U'(a)v(a) - v'(a)U(a)] = -R(a)W(U(a), v(a))$$

$$= -R(a)[\frac{C}{R(a)}] = -C < 0,$$

i.e.,

$$I_2 = -C < 0.$$

This ends the clarification of the Legendre test and the Jacobi test.

❑ **Example 9.22.** Catenary. Discuss the solution of soap film sustained between two circular wires.
Solution. The surface area is given by

$$I = \int_{-L}^{+L} 2\pi y\sqrt{1 + y'^2}dx.$$

Thus $f(x, y, y')$ is given by

$$f(x, y, y') = y\sqrt{1 + y'^2},$$

and is independent of x. Then we have

$$f - y'f_{y'} = \alpha,$$

where α is an arbitrary integration constant.
After a little algebra, we have

$$dy \Big/ \sqrt{\frac{y^2}{\alpha^2} - 1} = \pm dx.$$

We perform a change of variable as follows:

$$y = \alpha \cosh\theta, \quad dy = \alpha \sinh\theta \cdot d\theta.$$

Hence we have

$$\alpha d\theta = \pm dx,$$

or,

$$\theta = \pm(\frac{x - \beta}{\alpha}),$$

i.e.,

$$y = \alpha \cosh(\frac{x-\beta}{\alpha}),$$

where α and β are arbitrary constants of integration, to be determined from the boundary conditions at $x = \pm L$. We have, as the boundary conditions

$$R_1 = y(-L) = \alpha \cosh(\frac{L+\beta}{\alpha}), \quad R_2 = y(+L) = \alpha \cosh(\frac{L-\beta}{\alpha}).$$

For the sake of simplicity, we assume

$$R_1 = R_2 \equiv R.$$

Then we have

$$\beta = 0, \quad \frac{R}{L} = \frac{\alpha}{L} \cosh \frac{L}{\alpha},$$

and

$$y = \alpha \cosh \frac{x}{\alpha}.$$

Setting

$$v = \frac{L}{\alpha},$$

the boundary conditions at $x = \pm L$ read as

$$\frac{R}{L} = \frac{\cosh v}{v}.$$

Defining the function $G(v)$ by

$$G(v) \equiv \frac{\cosh v}{v}, \tag{9.7.44}$$

$G(v)$ is plotted in Figure 9.4.
If the geometry of the problem is such that

$$R/L > 1.5089,$$

then there exist *two candidates for the solution* at $v = v_<$ and $v = v_>$, where

$$0 < v_< < 1.1997 < v_>,$$

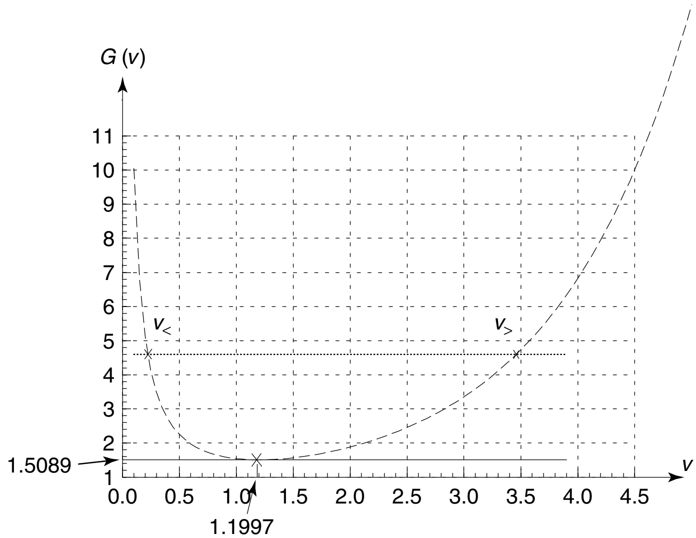

Fig. 9.4 Plot of $G(v)$.

and if

$$R/L < 1.5089,$$

then there exists *no solution*. If the geometry is such that

$$R/L = 1.5089,$$

then there exists *one candidate for the solution* at

$$v_< = v_> = v = \frac{L}{\alpha} = 1.1997.$$

We apply *two tests for the minimum*.

Legendre test:

$$f_{y'y'} = y/(1 + y'^2)^{3/2} = R > 0,$$

thus *passing the Legendre test*.

Jacobi test: Since

$$y(x, \alpha, \beta) = \alpha \cosh(\frac{x - \beta}{\alpha}),$$

we have

$$u_1(x) = \partial y/\partial \alpha = \cosh(\frac{x - \beta}{\alpha}) - (\frac{x - \beta}{\alpha}) \sinh(\frac{x - \beta}{\alpha}),$$

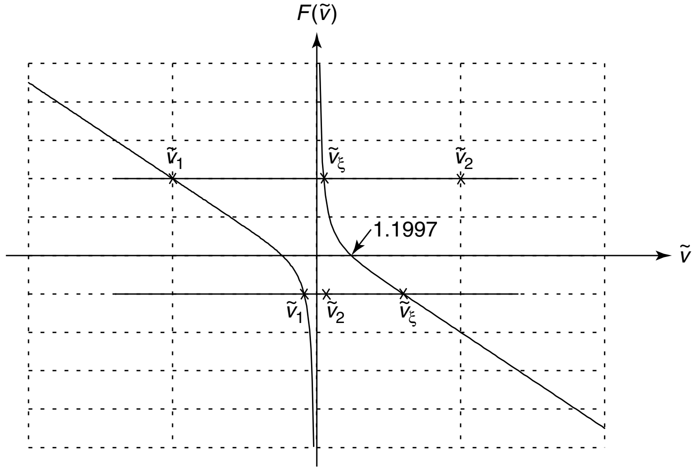

Fig. 9.5 Plot of $F(\tilde{v})$.

and

$$u_2(x) = -\partial\gamma/\partial\beta = \sinh(\frac{x-\beta}{\alpha}),$$

where the minus sign for $u_2(x)$ does not matter. We construct a solution $U(x)$ which vanishes at $x = x_1$,

$$U(x) = (\cosh\tilde{v} - \tilde{v}\sinh\tilde{v})\sinh\tilde{v}_1 - \sinh\tilde{v}(\cosh\tilde{v}_1 - \tilde{v}_1\sinh\tilde{v}_1),$$

where

$$\tilde{v} \equiv \frac{x-\beta}{\alpha}, \quad \tilde{v}_1 \equiv \frac{x_1-\beta}{\alpha}.$$

Then we have

$$U(x)/(\sinh\tilde{v}\sinh\tilde{v}_1) = (\coth\tilde{v} - \tilde{v}) - (\coth\tilde{v}_1 - \tilde{v}_1).$$

Defining the function $F(\tilde{v})$ by

$$F(\tilde{v}) \equiv \coth\tilde{v} - \tilde{v}, \tag{9.7.45}$$

$F(\tilde{v})$ is plotted in Figure 9.5.
We have

$$U(x)/(\sinh\tilde{v}\sinh\tilde{v}_1) = F(\tilde{v}) - F(\tilde{v}_1).$$

Note that $F(\tilde{v})$ is an odd function of $\tilde{v}$,

$$F(-\tilde{v}) = -F(\tilde{v}).$$

We set

$$\tilde{v}_1 \equiv -\frac{L}{\alpha}, \quad \tilde{v}_2 \equiv +\frac{L}{\alpha}, \quad \tilde{v}_\xi \equiv \frac{\xi}{\alpha},$$

where

$$\beta = 0,$$

is used. The equation,

$$U(\xi) = 0, \quad \xi \neq x_1,$$

which determines the conjugate point ξ, is equivalent to the following equation:

$$F(\tilde{v}_\xi) = F(\tilde{v}_1), \quad \tilde{v}_\xi \neq \tilde{v}_1.$$

If

$$F(\tilde{v}_1) > 0, \tag{9.7.46}$$

then, from Figure 9.5, we have

$$\tilde{v}_1 < \tilde{v}_\xi < \tilde{v}_2,$$

thus *failing the Jacobi test*. If, on the other hand,

$$F(\tilde{v}_1) < 0, \tag{9.7.47}$$

then, from Figure 9.5, we have

$$\tilde{v}_1 < \tilde{v}_2 < \tilde{v}_\xi,$$

thus *passing the Jacobi test*. The dividing line

$$F(\tilde{v}_1) = 0$$

corresponds to

$$\tilde{v}_1 < \tilde{v}_2 = \tilde{v}_\xi,$$

and thus we have *one solution* at

$$\tilde{v}_2 = -\tilde{v}_1 = \frac{L}{\alpha} = 1.1997.$$

Having derived the *particular statements of the Jacobi test* as applied to this example, Eqs. (9.7.46) and (9.7.47), we now test which of the two candidates for the solution, $v = v_<$ and $v = v_>$, is actually the minimizing solution. Two functions, $G(v)$ and $F(v)$, defined by Eqs. (9.7.44) and (9.7.45), are related to each other through

$$\frac{d}{dv}G(v) = -(\frac{\sinh v}{v^2})F(v). \tag{9.7.48}$$

At $v = v_<$, we know from Figure 9.4 that

$$\frac{d}{dv}G(v_<) < 0,$$

which implies

$$F(-v_<) = -F(v_<) < 0,$$

so that *one candidate, $v = v_<$, passes the Jacobi test* and is the *solution*, whereas at $v = v_>$, we know from Figure 9.4 that

$$\frac{d}{dv}G(v_>) > 0,$$

which implies

$$F(-v_>) = -F(v_>) > 0,$$

so that the *other candidate, $v = v_>$, fails the Jacobi test* and is *not the solution*.

When *two candidates, $v = v_<$ and $v = v_>$, coalesce to a single point,*

$$v = 1.1997,$$

where the first derivative of $G(v)$ vanishes, i.e.,

$$\frac{d}{dv}G(v) = 0,$$

$v_< = v_> = 1.1997$ is the *solution*.

We now consider a *strong variation* and the condition for the *strong minimum*. In the strong variation, since the varied derivatives behave very differently from the original derivatives, we cannot expand the integral I in Taylor series. Instead, we consider the *Weierstrass E function* defined by

$$E(x, y_0, y'_0, p) \equiv f(x, y_0, p) - \left[f(x, y_0, y'_0) + (p - y'_0)f_{y'}(x, y_0, y'_0)\right]. \tag{9.7.49}$$

Necessary and sufficient conditions for the strong minimum are given by

Necessary condition	*Sufficient condition*
(1) $f_{y'y'}(x, y_0, y_0') \geq 0$, where y_0 is the solution of the Euler equation	(1) $f_{y'y'}(x, y, p) > 0$, for every (x, y) close to (x, y_0), and every finite p
(2) $\xi \geq x_2$	(2) $\xi > x_2$
(3) $E(x, y_0, y_0', p) \geq 0$, for all finite p, and $x \in [x_1, x_2]$.	

(9.7.50)

We remark that if

$$p \sim y_0',$$

then we have

$$E \sim f_{y'y'}(x, y_0, y_0')\frac{(p - y_0')^2}{2!},$$

just like the *mean value theorem* of ordinary function $f(x)$, which is given by

$$f(x) - f(x_0) - f'(x_0) = \frac{(x - x_0)^2}{2!}f''(x_0 + \lambda(x - x_0)), \quad 0 < \lambda < 1.$$

9.8 Weierstrass–Erdmann Corner Relation

In this section, we consider the variational problem with the solutions which are the *piecewise continuous functions with corners*, i.e., the function itself is continuous, but the derivative is not. We maximize the integral

$$I = \int_{x_1}^{x_2} f(x, y, y')dx. \tag{9.8.1}$$

We put a point of corner at $x = a$. We write the solution for $x \leq a$, as $y(x)$ and, for $x > a$, as $Y(x)$. Then we have from the continuity of the solution,

$$y(a) = Y(a). \tag{9.8.2}$$

Now consider the variation of $y(x)$ and $Y(x)$ of the following forms:

$$y(x) \to y(x) + \varepsilon v(x), \quad Y(x) \to Y(x) + \varepsilon V(x), \tag{9.8.3a}$$

and

$$y'(x) \to y'(x) + \varepsilon v'(x), \quad Y'(x) \to Y'(x) + \varepsilon V'(x). \tag{9.8.3b}$$

Under these variations , we require I to be stationary,

$$I = \int_{x_1}^{a} f(x, y, y')dx + \int_{a}^{x_2} f(x, Y, Y')dx. \tag{9.8.4}$$

First, we consider the variation problem with the point of the discontinuity of the derivative at $x = a$ fixed. We have

$$\nu(a) = V(a). \tag{9.8.5}$$

Performing the above variations, we have

$$\delta I = \int_{x_1}^{a} \left[f_y \varepsilon \nu + f_{y'} \varepsilon \nu'\right] dx + \int_{a}^{x_2} \left[f_Y \varepsilon V + f_{Y'} \varepsilon V'\right] dx$$

$$= f_{y'} \varepsilon \nu\Big|_{x=x_1}^{x=a} + \int_{x_1}^{a} \varepsilon \nu \left[f_y - \frac{d}{dx} f_{y'}\right] dx$$

$$+ f_{Y'} \varepsilon V\Big|_{x=a}^{x=x_2} + \int_{a}^{x_2} \varepsilon V \left[f_Y - \frac{d}{dx} f_{Y'}\right] dx = 0.$$

If the variations vanish at both ends ($x = x_1$ and $x = x_2$), i.e.,

$$\nu(x_1) = V(x_2) = 0, \tag{9.8.6}$$

we have the following equations:

$$f_y - \frac{d}{dx} f_{y'} = 0, \quad x \in [x_1, x_2] \tag{9.8.7}$$

and

$$f_{y'}\Big|_{x=a^-} = f_{Y'}\Big|_{x=a^+},$$

namely

$$f_{y'} \text{ is continuous at } x = a. \tag{9.8.8}$$

Next, we consider the variation problem with the point of the discontinuity of the derivative at $x = a$ varied, i.e.,

$$a \to a + \Delta a. \tag{9.8.9}$$

The point of the discontinuity gets shifted, and yet the solutions are continuous at $x = a + \Delta a$, i.e.,

$$y(a) + y'(a)\Delta a + \varepsilon \nu(a) = Y(a) + Y'(a)\Delta a + \varepsilon V(a),$$

or,

$$\left[y'(a) - Y'(a)\right] \Delta a = \varepsilon \left[V(a) - \nu(a)\right], \tag{9.8.10}$$

which is the condition on Δa, $V(a)$, and $\nu(a)$.
The integral I gets changed into

$$I \to \int_{x_1}^{a+\Delta a} f(x, y + \varepsilon \nu, y' + \varepsilon \nu')dx + \int_{a+\Delta a}^{x_2} f(x, Y + \varepsilon V, Y' + \varepsilon V')dx$$

$$= \int_{x_1}^{a} f(x, y + \varepsilon \nu, y' + \varepsilon \nu')dx + f(x, y, y')\big|_{x=a^-} \Delta a$$

$$+ \int_{a}^{x_2} f(x, Y + \varepsilon V, Y' + \varepsilon V')dx - f(x, Y, Y')\big|_{x=a^+} \Delta a.$$

The integral parts after the integration by parts vanish due to the Euler equation in the respective region, and what remain are the integrated parts, i.e.,

$$\delta I = \varepsilon \nu f_{y'}\big|_{x=a^-} - \varepsilon V f_{Y'}\big|_{x=a^+} + \left(f\big|_{x=a^-} - f\big|_{x=a^+}\right) \Delta a$$

$$= \varepsilon(\nu - V)\, f_{y'}\big|_{x=a} + \left(f\big|_{x=a^-} - f\big|_{x=a^+}\right) \Delta a = 0, \tag{9.8.11}$$

where the first term of second line above follows from the continuity of $f_{y'}$. From the continuity condition (9.8.10) and expression (9.8.11), we obtain, by eliminating Δa,

$$-\left[y'(a) - Y'(a)\right] f_{y'} + \left(f\big|_{x=a^-} - f\big|_{x=a^+}\right) = 0,$$

or,

$$f - y'(a) f_{y'}\big|_{x=a^-} = f - Y'(a) f_{y'}\big|_{x=a^+},$$

i.e.,

$$f - y' f_{y'} \quad \text{continuous at } x = a. \tag{9.8.12}$$

For the solution of the variation problem with the discontinuous derivative, we have

$$\begin{aligned} &f_y - \tfrac{d}{dx} f_{y'} = 0, && \text{Euler equation,} \\ &f_{y'} && \text{continuous at } x = a, \\ &f - y' f_{y'} && \text{continuous at } x = a, \end{aligned} \tag{9.8.13}$$

which are called the *Weierstrass–Erdmann corner relation*.

❑ **Example 9.23.** Extremize

$$I = \int_0^1 (y' + 1)^2 y'^2 dx$$

with the end points fixed as below,

$$y(0) = 2, \quad y(1) = 0.$$

Are there solutions with discontinuous derivatives? Find the minimum value of I.

Solution. We have

$$f(x, y, y') = (y' + 1)^2 y'^2,$$

which is independent of x. Thus

$$f - y' f_{y'} = \text{constant},$$

where $f_{y'}$ is calculated to be

$$f_{y'} = 2(y' + 1)y'(2y' + 1).$$

Hence we have

$$f - y' f_{y'} = -(3y' + 1)(y' + 1)y'^2.$$

(1) If we want to have a *continuous* solution alone, we have $y' = a$(constant), from which, we conclude that $y = ax + b$. From the end point conditions, the solution is

$$y = 2(-x + 1), \quad y' = -2.$$

Thus the integral I is evaluated to be $I_{\text{cont.}} = 4$.

(2) Suppose that y' is *discontinuous* at $x = a$. Setting

$$p = y'_<, \quad p' = y'_>,$$

we have, from the corner relation at $x = a$,

$$\begin{cases} p(p+1)(2p+1) &= p'(p'+1)(2p'+1), \\ (3p+1)(p+1)p^2 &= (3p'+1)(p'+1)p'^2. \end{cases}$$

Setting

$$u = p + p', \quad v = p^2 + pp' + p'^2,$$

we have

$$\begin{cases} 3u + 2v + 1 = 0, \\ 3u(2v - u^2) + u + 4v = 0, \end{cases} \Rightarrow \begin{cases} p = 0, \\ p' = -1, \end{cases} \quad \text{or} \quad \begin{cases} p = -1, \\ p' = 0. \end{cases}$$

Thus the discontinuous solution gives $y' + 1 = 0$, or $y' = 0$. Then we have $I_{\text{disc.}} = 0 < I_{\text{cont.}} = 4$.

In the above example, the solution $y(x)$ itself became discontinuous. The discontinuous solution may not be present, depending on the given boundary conditions.

9.9 Problems for Chapter 9

9.1. (due to H. C.). Find an approximate value for the lowest eigenvalue E_0 for

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - x^2y^2\right]\psi(x,y) = -E\psi(x,y), \quad -\infty < x, y < \infty,$$

where $\psi(x,y)$ is normalized to unity,

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |\psi(x,y)|^2\, dxdy = 1.$$

9.2. (due to H. C.). A light ray in the x–y plane is incident on the lower half-plane ($y \le 0$) of the medium with the index of refraction $n(x,y)$ given by

$$n(x,y) = n_0(1 + ay), \quad y \le 0,$$

where n_0 and a are the positive constants.

(a) Find the path of a ray passing through $(0, 0)$ and $(l, 0)$.

(b) Find the apparent depth of the object located at $(0, 0)$ when looked through $(l, 0)$.

9.3. (due to H. C.). Estimate the lowest frequency of a circular drum of radius R with a rotationally symmetric trial function

$$\phi(r) = 1 - (\frac{r}{R})^n, \quad 0 \leq r \leq R.$$

Find n which gives the best estimate.
9.4. Minimize the integral

$$I \equiv \int_1^2 x^2 (y')^2 dx, \quad \text{with} \quad y(1) = 0, \quad y(2) = 1.$$

Apply the Legendre test and the Jacobi test to determine if your solution is a minimum. Is it a strong minimum or a weak minimum? Are there solutions with discontinuous derivatives?
9.5. Extremize

$$\int_0^1 \frac{(1 + y^2)^2}{(y')^2} dx, \quad \text{with} \quad y(0) = 0, \quad y(1) = 1.$$

Is your solution a weak minimum? Is it a strong minimum? Are there solutions with discontinuous derivatives?
9.6. Extremize

$$\int_0^1 (y'^2 - y'^4) dx, \quad \text{with} \quad y(0) = y(1) = 0.$$

Is your solution a weak minimum? Is it a strong minimum? Are there solutions with discontinuous derivatives?
9.7. Extremize

$$\int_0^2 (xy' + y'^2) dx, \quad \text{with} \quad y(0) = 1, \quad y(2) = 0.$$

Is your solution a weak minimum? Is it a strong minimum? Are there solutions with discontinuous derivatives?
9.8. Extremize

$$\int_1^2 \frac{x^3}{y'^2} dx, \quad \text{with} \quad y(1) = 1, \quad y(2) = 4.$$

Is your solution a weak minimum? Is it a strong minimum? Are there solutions with discontinuous derivatives?

9.9. (due to H. C.). An airplane with speed v_0 flies in a wind of speed a_x. What is the trajectory of the airplane if it is to enclose the greatest area in a given amount of time?
Hint: You may assume that the velocity of the airplane is

$$\frac{dx}{dt}\vec{e}_x + \frac{dy}{dt}\vec{e}_y = (v_x + a_x)\vec{e}_x + v_y\vec{e}_y,$$

with

$$v_x^2 + v_y^2 = v_0^2.$$

9.10. (due to H. C.). Find the shortest distance between two points on a cylinder. Apply the Jacobi test and the Legendre test to verify if your solution is a minimum.
9.11. The problem of isochronous oscillator is related to the problem of *Brachistochrone*. The point particle with mass m is smoothly moving along the cycloid,

$$x = 2R\int \sin^2(\frac{\theta}{2})d\theta = R(\theta - \sin\theta), \quad y = -2R\sin^2(\frac{\theta}{2}) = -R(1 - \cos\theta),$$

which is placed vertically in the uniform gravitational field pointing along the negative y-axis. We choose the origin of the arclength at the point $\theta = \pi$, or $(x_0, y_0) = (R\pi, -2R)$.
(a) Show that an infinitesimal arclength ds is given by

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2}d\theta = 2R\sin\frac{\theta}{2}d\theta,$$

and the arclength $s(\theta)$ as a function of θ is given by

$$s(\theta) = 4R(-\cos\frac{\theta}{2}).$$

(b) Show that the potential energy $V(s)$ of the particle is given by

$$V(s) = mgy = -mgR(1 - \cos\theta) = -2mgR + \frac{1}{2}\frac{mg}{4R}s^2,$$

and the motion of the point particle is simple harmonic with the period

$$T = 2\pi\sqrt{\frac{m}{(\frac{mg}{4R})}} = 4\pi\sqrt{\frac{R}{g}},$$

which is independent of the initial amplitude.

10
Calculus of Variations: Applications

10.1
Hamilton–Jacobi Equation and Quantum Mechanics

In this section, we will discuss, in some depth, canonical transformation theory and a "derivation" of quantum mechanics from the Hamilton–Jacobi equation following L. de Broglie and E. Schrödinger. We will also dwell on the optico-mechanical analogy of wave optics and wave mechanics.

We can discuss classical mechanics in terms of Hamilton's action principle

$$\delta I \equiv \delta \int_{t_1}^{t_2} L(q_r(t), \dot{q}_r(t), t)dt = 0, \quad r = 1, \ldots, f, \tag{10.1.1}$$

where

$$\delta q_r(t_1) = \delta q_r(t_2) = 0, \quad r = 1, \ldots, f, \tag{10.1.2}$$

where $L(q_r(t), \dot{q}_r(t), t)$ is the Lagrangian of the mechanical system and we obtained the Lagrange equation of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right) - \frac{\partial L}{\partial q_r} = 0, \quad r = 1, \ldots, f. \tag{10.1.3}$$

We can also formulate geometrical optics in terms of Fermat's principle

$$\delta T = \frac{1}{c}\delta \int_{x_1}^{x_2} dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2}\, n(x, y, z) = 0, \tag{10.1.4}$$

where

$$\delta x_1 = \delta x_2 = 0, \tag{10.1.5}$$

where $n(x, y, z)$ is the index of refraction. If $n(x, y, z)$ is independent of x, we obtain the equation that determines the path of the light ray in geometrical optics as

$$n(y, z) \Big/ \sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2} = \text{constant}. \tag{10.1.6}$$

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We first define the momentum $\{p_r(t)\}_{r=1}^{f}$ canonically conjugate to the generalized coordinate $\{q_r(t)\}_{r=1}^{f}$ by the following equation:

$$p_r(t) \equiv \frac{\partial L(q_s(t), \dot{q}_s(t), t)}{\partial \dot{q}_r(t)}, \quad r, s = 1, \ldots, f. \tag{10.1.7}$$

We solve Eq. (10.1.7) for $\dot{q}_r(t)$ as a function of $\{q_s(t), p_s(t)\}_{s=1}^{f}$ and t. We define the Hamiltonian $H(q_r(t), p_r(t), t)$ as the Legendre transform of the Lagrangian $L(q_r(t), \dot{q}_r(t), t)$ by the following equation:

$$H(q_r(t), p_r(t), t) \equiv \sum_{s=1}^{f} p_s(t)\dot{q}_s(t) - L(q_r(t), \dot{q}_r(t), t), \tag{10.1.8}$$

where we substitute $\dot{q}_r(t)$, expressed as a function of $\{q_s(t), p_s(t)\}_{s=1}^{f}$ and t, into the right-hand side of Eq. (10.1.8). Then, we take the independent variations of $q_r(t)$ and $p_r(t)$ in Eq. (10.1.8). We obtain Hamilton's canonical equations of motion

$$\frac{d}{dt}q_r(t) = \frac{\partial H(q_s(t), p_s(t), t)}{\partial p_r(t)}, \quad \frac{d}{dt}p_r(t) = -\frac{\partial H(q_s(t), p_s(t), t)}{\partial q_r(t)}, \quad r = 1, \ldots, f. \tag{10.1.9}$$

We move on to the discussion of canonical transformation theory. We rewrite the definition of the Hamiltonian $H(q_r(t), p_r(t), t)$, (10.1.8), as follows:

$$L(q_r(t), \dot{q}_r(t), t) = \sum_{s=1}^{f} p_s(t)\dot{q}_s(t) - H(q_r(t), p_r(t), t). \tag{10.1.10}$$

We now consider the transformation of the pair of the canonical variables from $\{q_r(t), p_r(t)\}_{r=1}^{f}$ to $\{Q_r(t), P_r(t)\}_{r=1}^{f}$. This transformation is said to be canonical transformation if the following condition holds:

$$\begin{aligned} &\delta \int_{t_1}^{t_2} dt \left(\sum_{s=1}^{f} p_s(t)\dot{q}_s(t) - H(q_r(t), p_r(t), t) \right) \\ &= \delta \int_{t_1}^{t_2} dt \left(\sum_{s=1}^{f} P_s(t)\dot{Q}_s(t) - \overline{H}(Q_r(t), P_r(t), t) \right) = 0. \end{aligned} \tag{10.1.11a}$$

We have the end-point conditions:

$$\delta q_r(t_{1,2}) = \delta p_r(t_{1,2}) = \delta Q_r(t_{1,2}) = \delta P_r(t_{1,2}) = 0, \quad r = 1, \ldots, f. \tag{10.1.11b}$$

New canonical pair $\{Q_r(t), P_r(t)\}_{r=1}^{f}$ satisfies Hamilton's canonical equation of motion, (10.1.9), with $\overline{H}(Q_r(t), P_r(t), t)$ as the new Hamiltonian. With Eq. (10.1.11b), Eq. (10.1.11a) is equivalent to the following equation:

$$\sum_{s=1}^{f} p_s(t)\dot{q}_s(t) - H(q_r(t), p_r(t), t)$$
$$= \sum_{s=1}^{f} P_s(t)\dot{Q}_s(t) - \overline{H}(Q_r(t), P_r(t), t) + \frac{d}{dt}U. \qquad (10.1.12)$$

U is the function of arbitrary pair of $\{q_r(t), p_r(t), Q_r(t), P_r(t)\}$ and t, and is the single-valued continuous function. We call U the generator of canonical transformation. The generator U can assume only the following four forms:

$$F_1(q_r(t), Q_r(t), t),\ \ F_2(q_r(t), P_r(t), t),\ \ F_3(p_r(t), Q_r(t), t),\ \ F_4(p_r(t), P_r(t), t). \quad (10.1.13)$$

(1) We now choose U to be the function of $\{q_r(t),\ Q_r(t)\}_{r=1}^{f}$ and t, $U = F_1(q_r(t), Q_r(t), t)$. Since we have

$$\frac{d}{dt}U = \sum_{s=1}^{f}\left\{\frac{\partial F_1(q_r(t), Q_r(t), t)}{\partial q_s(t)}\dot{q}_s(t) + \frac{\partial F_1(q_r(t), Q_r(t), t)}{\partial Q_s(t)}\dot{Q}_s(t)\right\} + \frac{\partial}{\partial t}F_1, \qquad (10.1.14)$$

we obtain the transformation formula from Eqs. (10.1.12) and (10.1.14),

$$p_r(t) = \frac{\partial F_1(q_s(t), Q_s(t), t)}{\partial q_r(t)}, \quad P_r(t) = -\frac{\partial F_1(q_s(t), Q_s(t), t)}{\partial Q_r(t)}, \quad r = 1, \ldots, f, \qquad (10.1.15a)$$

$$\overline{H}(Q_r(t), P_r(t), t) = H(q_r(t), p_r(t), t) + \frac{\partial}{\partial t}F_1(q_r(t), Q_r(t), t). \qquad (10.1.15b)$$

(2) We now choose U to be the function of $\{q_r(t), P_r(t)\}_{r=1}^{f}$ and t, $U = F_2(q_r(t), P_r(t), t)$. In view of the transformation formula (10.1.15a),

$$\frac{\partial F_1(q_s(t), Q_s(t), t)}{\partial Q_r(t)} = -P_r(t),$$

we can define $F_2(q_r(t), P_r(t), t)$ suitably in terms of $F_1(q_r(t), Q_r(t), t)$ by the method of Legendre transform much as in thermodynamics,

$$F_2(q_r(t), P_r(t), t) = F_1(q_r(t), Q_r(t), t) + \sum_{r=1}^{f} P_r(t)Q_r(t). \qquad (10.1.16)$$

By solving Eq. (10.1.16) for $F_1(q_r(t), Q_r(t), t)$ and substituting in Eq. (10.1.12), we obtain

$$\sum_{s=1}^{f} p_s(t)\dot{q}_s(t) - H(q_r(t), p_r(t), t)$$

$$= -\sum_{r=1}^{f} Q_r(t)\dot{P}_r(t) - \overline{H}(Q_r(t), P_r(t), t) + \frac{d}{dt}F_2(q_r(t), P_r(t), t).$$

Carrying out the total time derivative of $F_2(q_r(t), P_r(t), t)$ and equating the coefficients of $\dot{q}_r(t)$ and $\dot{P}_r(t)$, we obtain the transformation formula

$$p_r(t) = \frac{\partial F_2(q_r(t), P_r(t), t)}{\partial q_r(t)}, \quad Q_r(t) = \frac{\partial F_2(q_r(t), P_r(t), t)}{\partial P_r(t)}, \quad r = 1, \ldots, f, \tag{10.1.17a}$$

$$\overline{H}(Q_r(t), P_r(t), t) = H(q_r(t), p_r(t), t) + \frac{\partial}{\partial t}F_2(q_r(t), P_r(t), t). \tag{10.1.17b}$$

(3) We now choose U to be the function of $\{p_r(t), \ Q_r(t)\}_{r=1}^{f}$ and t, $U = F_3(p_r(t), Q_r(t), t)$. In view of the transformation formula (10.1.15a), we can define $F_3(p_r(t), Q_r(t), t)$ suitably in terms of $F_1(q_r(t), Q_r(t), t)$ by the method of Legendre transform much as in thermodynamics,

$$F_3(p_r(t), Q_r(t), t) = F_1(q_r(t), Q_r(t), t) - \sum_{r=1}^{f} p_r(t)q_r(t). \tag{10.1.18}$$

Equation (10.1.12) now reads as

$$-\sum_{s=1}^{f} q_s(t)\dot{p}_s(t) - H(q_r(t), p_r(t), t)$$
$$= \sum_{s=1}^{f} P_s(t)\dot{Q}_s(t) - \overline{H}(Q_r(t), P_r(t), t) + \frac{d}{dt}F_3(p_r(t), Q_r(t), t).$$

Equating the coefficients of $\dot{p}_r(t)$ and $\dot{Q}_r(t)$, we obtain the transformation formula,

$$q_r(t) = -\frac{\partial F_3(p_r(t), Q_r(t), t)}{\partial p_r(t)}, \quad P_r(t) = -\frac{\partial F_3(p_r(t), Q_r(t), t)}{\partial Q_r(t)}, \quad r = 1, \ldots, f, \tag{10.1.19a}$$

$$\overline{H}(Q_r(t), P_r(t), t) = H(q_r(t), p_r(t), t) + \frac{\partial}{\partial t}F_3(p_r(t), Q_r(t), t). \tag{10.1.19b}$$

(4) We now choose U to be the function of $\{p_r(t), P_r(t)\}_{r=1}^{f}$ and t, $U = F_4(p_r(t), P_r(t), t)$. We can also define $F_4(p_r(t), P_r(t), t)$ suitably in terms of $F_1(q_r(t), Q_r(t), t)$ by a double Legendre transform as in thermodynamics,

$$F_4(p_r(t), P_r(t), t) = F_1(q_r(t), Q_r(t), t) + \sum_{r=1}^{f} P_r(t)Q_r(t) - \sum_{r=1}^{f} p_r(t)q_r(t). \tag{10.1.20}$$

Equation (10.1.12) reduces to

$$-\sum_{s=1}^{f} q_s(t)\dot{p}_s(t) - H(q_r(t), p_r(t), t)$$
$$= -\sum_{r=1}^{f} Q_r(t)\dot{P}_r(t) - \overline{H}(Q_r(t), P_r(t), t) + \frac{d}{dt}F_4(p_r(t), P_r(t), t).$$

Equating the coefficients of $\dot{p}_r(t)$ and $\dot{P}_r(t)$, we obtain the transformation formula,

$$q_r(t) = -\frac{\partial F_4(p_r(t), P_r(t), t)}{\partial p_r(t)}, \quad Q_r(t) = \frac{\partial F_4(p_r(t), P_r(t), t)}{\partial P_r(t)}, \quad r = 1, \ldots, f, \tag{10.1.21a}$$

$$\overline{H}(Q_r(t), P_r(t), t) = H(q_r(t), p_r(t), t) + \frac{\partial}{\partial t}F_4(p_r(t), P_r(t), t). \tag{10.1.21b}$$

In Case (2), we now try to choose $F_2(q_r(t), P_r(t), t) = S(q_r(t), P_r(t), t)$ such that

$$\overline{H}(Q_r(t), P_r(t), t) \equiv 0. \tag{10.1.22}$$

To satisfy the condition (10.1.22), the generator S must satisfy the following equation in view of Eqs. (10.1.17a), (10.1.17b), and (10.1.22),

$$H\left(q_1(t), \ldots, q_f(t), \frac{\partial S}{\partial q_1}, \ldots, \frac{\partial S}{\partial q_f}, t\right) + \frac{\partial S}{\partial t} = 0. \tag{10.1.23}$$

Equation (10.1.23) is called the Hamilton–Jacobi equation .

We consider the single particle in three-dimensional space with Cartesian components. From Eq. (10.1.17a), we have

$$\vec{p}(t) = \vec{\nabla}_{\vec{q}}S(\vec{q}(t), \vec{P}(t), t),$$

which states that the direction of the motion of the particle is always normal to the surface with

$$S(\vec{q}(t), \vec{P}(t), t) = \text{constant}.$$

The situation is similar to geometrical optics. Now the Hamilton–Jacobi equation provides us the interpretation of classical mechanics as "geometrical" mechanics much like geometrical optics. We recall that geometrical optics was derived from wave optics by Eikonal approximation. It is quite natural then to search for the governing equation of "wave" mechanics by applying the inverse of Eikonal approximation to the Hamilton–Jacobi equation. This step was taken by de Broglie and Schrödinger in 1924–1925 and the governing equation of "wave" mechanics is the celebrated Schrödinger wave equation. The time-independent Schrödinger equation was first deduced as

$$H\left(\vec{q}, \frac{\hbar}{i}\vec{\nabla}_{\vec{q}}\right)\phi(\vec{q}) = E\phi(\vec{q}), \tag{10.1.24}$$

and then, from the observation that the time-dependence of the optical scalar wave is given by $\exp[-i\omega t]$ and with the assumption that the time-dependence of the wavefunction for the stationary state must be given by $\exp[-iEt/\hbar]$, the time-dependent Schrödinger equation is deduced as

$$i\hbar\frac{\partial}{\partial t}\psi(\vec{q}, t) = H\left(\vec{q}, \frac{\hbar}{i}\vec{\nabla}_{\vec{q}}, t\right)\psi(\vec{q}, t). \tag{10.1.25}$$

The "derivation" of Eq. (10.1.24) is possible in the following manner. We assume that the Hamiltonian $H(\vec{q}(t), \vec{p}(t), t)$ is independent of time t and is equal to the total energy E. We separate the generator $S(\vec{q}(t), \vec{P}(t), t)$ as

$$S(\vec{q}(t), \vec{P}(t), t) = W(\vec{q}(t), \vec{P}(t)) - Et. \tag{10.1.26}$$

We observe that the Hamilton–Jacobi equation becomes

$$H(\vec{q}, \vec{\nabla}_{\vec{q}}W(\vec{q})) = \frac{1}{2m}\{\vec{\nabla}_{\vec{q}}W(\vec{q})\}^2 + V(\vec{q}) = E. \tag{10.1.27}$$

Namely, we have

$$\{\vec{\nabla}_{\vec{q}}W(\vec{q})\}^2 = \vec{p}^2 = 2m\{E - V(\vec{q})\}, \tag{10.1.28}$$

which is nothing but the mechanical analog of the Eikonal equation for geometrical optics,

$$\{\vec{\nabla}_{\vec{q}}L(\vec{q})\}^2 = n^2(\vec{q}), \tag{10.1.29}$$

where $L(\vec{q})$ is the Eikonal of the optical scalar wave $\phi(\vec{q}, t)$ defined by

$$\phi(\vec{q}, t) = A(\vec{q}, t)\ \exp[i(L(\vec{q}) - \omega t)]. \tag{10.1.30}$$

From Eqs. (10.1.28) and (10.1.29), we find that the magnitude of $\vec{p}$ in classical mechanics corresponds to the index of refraction $n(\vec{q})$ in geometrical optics. We recall that the time-independent optical scalar wave equation is given by

$$\vec{\nabla}_{\vec{q}}^2\phi(\vec{q}) + \frac{4\pi^2}{\lambda^2}\phi(\vec{q}) = 0. \tag{10.1.31}$$

According to de Broglie's matter wave hypothesis, we have

$$\frac{\lambda}{2\pi} = \frac{\hbar}{|\vec{p}|}, \tag{10.1.32}$$

and, substituting Eq. (10.1.32) into Eq. (10.1.31) and using Eq. (10.1.28), we obtain the time-independent Schrödinger equation,

$$\vec{\nabla}_{\vec{q}}^2\phi(\vec{q}) + \frac{2m\{E - V(\vec{q})\}}{\hbar^2}\phi(\vec{q}) = 0,$$

or

$$H\left(\vec{q}, \frac{\hbar}{i}\vec{\nabla}_{\vec{q}}\right)\phi(\vec{q}) = E\phi(\vec{q}). \tag{10.1.33}$$

We invoked de Broglie's matter wave hypothesis instead of the inverse of Eikonal approximation to derive the time-independent Schrödinger equation.

We point out that the optical–mechanical analogy is not complete. We observe that the time-dependent optical scalar wave equation is given by

$$\vec{\nabla}_{\vec{q}}^2\phi(\vec{q}, t) - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\phi(\vec{q}, t) = 0. \tag{10.1.34}$$

This is second order in partial time derivative. As the initial value problem, we need to specify the value of $\phi(\vec{q}, t)$ and $\partial\phi(\vec{q}, t)/\partial t$ at some time for the complete specification of $\phi(\vec{q}, t)$ at a later time. On the other hand, we observe that the time-dependent Schrödinger equation, Eq. (10.1.25), is first order in the partial time derivative. As the initial value problem, we need to specify only the value of $\psi(\vec{q}, t)$ at some time for the complete specification of $\psi(\vec{q}, t)$ at a later time. We remark that this is Huygens' principle in wave mechanics.

We note that Hamilton was one step short of arriving at wave mechanics as early as 1834, although he did not have any experimentally compelling reason to take such a step. On the other hand, by 1924, L. de Broglie and E. Schrödinger had sufficient experimentally compelling reasons to take such a step.

In the summer of 1925, M. Born, W. Heisenberg, and P. Jordan deduced matrix mechanics from the consistency of the Ritz combination principle, the Bohr quantization condition, the Fourier analysis of physical quantity in classical physics, and Hamilton's canonical equation of motion . We will state the basic principles of matrix mechanics below.

Assumption 1. All physical quantities are represented by matrices. If the physical quantities are *real*, the corresponding matrices are *Hermitian*.

Assumption 2. The time dependence of the (a, b) element of the physical quantity is of the form given by $\exp[2\pi i\nu_{a,b}t]$.

Assumption 3. The frequency $\nu_{a,b}$ follows the Ritz combination principle,

$$\nu_{a,b} + \nu_{b,c} = \nu_{a,c}, \quad \nu_{a,a} = 0, \quad \nu_{a,b} = -\nu_{b,a}. \tag{10.1.35}$$

Assumption 4. The time derivative of the physical quantity is represented by the time derivative of the corresponding matrix.

Assumption 5. The sum $A + B$ of the two physical quantities, A and B, is represented by the sum of the corresponding two matrices.

Assumption 6. The product AB of the two physical quantities, A and B, is represented by the product of the corresponding two matrices.
Assumption 7. The coordinate q satisfies the equation of the motion of the mechanical system under consideration.
Assumption 8. The canonical momentum p conjugate to the coordinate q is defined for the system with one degree of freedom and they satisfy

$$(pq)_{a,b} - (qp)_{a,b} = \frac{\hbar}{i}\delta_{a,b}. \tag{10.1.36}$$

Assumption 9. The canonical momenta $\{p_r\}_{r=1}^{f}$ conjugate to the coordinates $\{q_s\}_{s=1}^{f}$ are defined for the system with f degrees of freedom and they satisfy

$$\begin{aligned} q_r q_s - q_s q_r &= 0, \\ p_r p_s - p_s p_r &= 0, \\ (p_r q_s)_{a,b} - (q_s p_r)_{a,b} &= (\hbar/i)\delta_{r,s}\delta_{a,b}. \end{aligned} \tag{10.1.37}$$

Assumption 10. The equation of motion of the system is given by Hamilton's canonical equation of motion,

$$\frac{d}{dt}q_r = \frac{\partial H}{\partial p_r}, \quad \frac{d}{dt}p_r = -\frac{\partial H}{\partial q_r}, \tag{10.1.38}$$

where the mechanical energy of the system is given by $H(\{q_r\}_{r=1}^{f}, \{p_r\}_{r=1}^{f})$.

With these assumptions, we can prove the following theorem with some mathematical preliminary.

Theorem. *Given the function $F(\{q_r\}_{r=1}^{f}, \{p_r\}_{r=1}^{f})$ of $\{q_r\}_{r=1}^{f}$ and $\{p_r\}_{r=1}^{f}$, we have*

$$\frac{\partial F}{\partial q_r} = \frac{i}{\hbar}(p_r F - F p_r), \quad \frac{\partial F}{\partial p_r} = -\frac{i}{\hbar}(q_r F - F q_r). \tag{10.1.39}$$

Hence it follows that

$$i\hbar\frac{d}{dt}q_r = q_r H - H q_r, \quad i\hbar\frac{d}{dt}p_r = p_r H - H p_r. \tag{10.1.40}$$

P.A.M. Dirac developed the transformation theory of quantum mechanics with the abstract notion of the bra-vector and the ket-vector in 1926. With the transformation theory, the distinction between the Schrödinger approach and the Heisenberg approach is that the state vector is time dependent and the operators are time-independent for the former, and the state vector is time-independent and the operators are time dependent for the latter.

Through 1940s and 1950s, two radically different approaches to quantum theory were developed. One is Feynman's action principle and the other is Schwinger's action principle. Schwinger's action principle proposed in the early 1950s is the differential formalism of quantum action principle to be compared with Feynman's action principle proposed in the early 1940s, which is the integral formalism of quantum action principle. These two quantum action principles are equivalent to each other and are essential in carrying out the computation of electrodynamic level shifts of the atomic energy level.

10.2
Feynman's Action Principle in Quantum Theory

Feynman's Action Principle

We note that the operator $\hat{q}(t)$ at all time t (the operator $\hat{\phi}(x)$ at all space–time indices x on space-like hypersurface σ) forms a complete set of the operators. In other words, the quantum theoretical state vector can be expressed by the complete set of eigenket $|q, t\rangle$ of the commuting operators $\hat{q}(t)$ (the complete set of eigenket $|\phi, \sigma\rangle$ of the commuting operators $\hat{\phi}(x)$). Feynman's action principle states that the transformation function $\langle q'', t'' | q', t'\rangle$ ($\langle \phi'', \sigma'' | \phi', \sigma'\rangle$) is given by

$$\langle q'', t'' | q', t'\rangle = \frac{1}{N(\Omega)} \int_{q(t')=q'}^{q(t'')=q''} \mathcal{D}[q] \, \exp\left[\frac{i}{\hbar}\int_{t'}^{t''} dt L(q(t), \dot{q}(t))\right], \tag{10.2.1M}$$

$$\langle \phi'', \sigma'' | \phi', \sigma'\rangle = \frac{1}{N(\Omega)} \int_{\phi',\sigma'}^{\phi'',\sigma''} \mathcal{D}[\phi] \, \exp\left[\frac{i}{\hbar}\int_{\Omega(\sigma'',\sigma')} d^4x \mathcal{L}(\phi(x), \partial_\mu \phi(x))\right]. \tag{10.2.1F}$$

The space–time region Ω is given by that of space–time region sandwiched between t'' and t' (σ'' and σ'). We state here three assumptions involved:
(A-1) The principle of superposition.

$$\int_{q(t')=q'}^{q(t'')=q''} \mathcal{D}[q] = \int_{t=t'''} dq''' \int_{q_{\text{II}}(t''')=q'''}^{q_{\text{II}}(t'')=q''} \mathcal{D}[q_{\text{II}}] \int_{q_{\text{I}}(t')=q'}^{q_{\text{I}}(t''')=q'''} \mathcal{D}[q_{\text{I}}], \tag{10.2.2M}$$

$$\int_{\phi',\sigma'}^{\phi'',\sigma''} \mathcal{D}[\phi] = \int_{\sigma'''} d\phi''' \int_{\phi''',\sigma'''}^{\phi'',\sigma''} \mathcal{D}[\phi_{\text{II}}] \int_{\phi',\sigma'}^{\phi''',\sigma'''} \mathcal{D}[\phi_{\text{I}}]. \tag{10.2.2F}$$

(A-2) Functional integral by parts is allowed.
(A-3) Resolution of identity.

$$\int dq' \, |q', t'\rangle\langle q', t'| = 1, \quad \int d\phi' \, |\phi', \sigma'\rangle\langle \phi', \sigma'| = 1. \tag{10.2.3M,F}$$

From the consistency of the three assumptions, the normalization constant $N(\Omega)$ must satisfy

$$N(\Omega_1 + \Omega_2) = N(\Omega_1)N(\Omega_2), \tag{10.2.4M,F}$$

which also originates from the additivity of the action functional,

$$I[q;\, t'', t'] = I[q;\, t'', t'''] + I[q;\, t''', t'], \quad I[\phi;\, \sigma'', \sigma'] = I[\phi;\, \sigma'', \sigma'''] + I[\phi;\, \sigma''', \sigma']. \tag{10.2.5M,F}$$

The action functional is defined by

$$I[q;\, t'', t'] = \int_{t'}^{t''} dt L(q(t), \dot{q}(t)), \quad I[\phi;\, \sigma'', \sigma'] = \int_{\Omega(\sigma'', \sigma')} d^4x \mathcal{L}(\phi(x), \partial_\mu \phi(x)). \tag{10.2.6M,F}$$

Operator, Equation of Motion, and Time-Ordered Product: We note that in Feynman's action principle, the operator is defined by its matrix elements.

$$\left\langle q'', t'' \left| \hat{q}(t) \right| q', t' \right\rangle = \frac{1}{N(\Omega)} \int_{q(t')=q'}^{q(t'')=q''} \mathcal{D}\,[q] q(t) \exp\left[\frac{i}{\hbar} I[q;\, t'', t'] \right], \tag{10.2.7M}$$

$$\left\langle \phi'', \sigma'' \left| \hat{\phi}(x) \right| \phi', \sigma' \right\rangle = \frac{1}{N(\Omega)} \int_{\phi', \sigma'}^{\phi'', \sigma''} \mathcal{D}\,[\phi] \phi(x) \exp\left[\frac{i}{\hbar} I[\phi;\, \sigma'', \sigma'] \right]. \tag{10.2.7F}$$

If t lies on the t''-surface (x lies on the σ''-surface), Eqs. (10.2.7M) and (10.2.7F) can be rewritten on the basis of (10.2.1M) and (10.2.1F) as

$$\left\langle q'', t'' \left| \hat{q}(t'') \right| q', t' \right\rangle = q'' \left\langle q'', t'' \right| \left. q', t' \right\rangle, \tag{10.2.8M}$$

$$\left\langle \phi'', \sigma'' \left| \hat{\phi}(x'') \right| \phi', \sigma' \right\rangle = \phi''(x'') \left\langle \phi'', \sigma'' \right| \left. \phi', \sigma' \right\rangle, \tag{10.2.8F}$$

for all $\left| q', t' \right\rangle$ (for all $\left| \phi', \sigma' \right\rangle$) which form a complete set.

Then we have

$$\left\langle q'', t'' \right| \hat{q}(t'') = q'' \left\langle q'', t'' \right|, \tag{10.2.9M}$$

$$\left\langle \phi'', \sigma'' \right| \hat{\phi}(x'') = \phi''(x'') \left\langle \phi'', \sigma'' \right|, \tag{10.2.9F}$$

which are the defining equations of the eigenbras, $\left\langle q'', t'' \right|$ and $\left\langle \phi'', \sigma'' \right|$.

Next we consider the variation of the action functional,

$$\delta I\,[q;\, t'', t'] = \int_{t'}^{t''} dt \left\{ \frac{\partial L(q(t), \dot{q}(t))}{\partial q(t)} - \frac{d}{dt} \left(\frac{\partial L(q(t), \dot{q}(t))}{\partial \dot{q}(t)} \right) \right\} \delta q(t)$$
$$+ \int_{t'}^{t''} dt \frac{d}{dt} \left\{ \frac{\partial L(q(t), \dot{q}(t))}{\partial \dot{q}(t)} \delta q(t) \right\}, \tag{10.2.10M}$$

$$\delta I\,[\phi;\sigma'',\sigma'] = \int_\Omega d^4x \left\{ \frac{\partial\mathcal{L}(\phi(x),\partial_\mu\phi(x))}{\partial\phi(x)} - \partial_\mu\left(\frac{\partial\mathcal{L}(\phi(x),\partial_\mu\phi(x))}{\partial(\partial_\mu\phi(x))}\right)\right\}\delta\phi(x)$$
$$+\int_\Omega d^4x\partial_\mu\left\{\frac{\partial\mathcal{L}(\phi(x),\partial_\mu\phi(x))}{\partial(\partial_\mu\phi(x))}\delta\phi(x)\right\}. \qquad (10.2.10F)$$

We consider a particular variations in which the end points are fixed,

$$\delta q(t') = \delta q(t'') = 0; \quad \delta\phi(x' \text{ on } \sigma') = \delta\phi(x'' \text{ on } \sigma'') = 0. \qquad (10.2.11M,F)$$

Then the second terms in (10.2.10M) and (10.2.10F) vanish. We obtain the Euler derivatives,

$$\frac{\delta I[q;t'',t']}{\delta q(t)} = \frac{\partial L(q(t),\dot q(t))}{\partial q(t)} - \frac{d}{dt}\left(\frac{\partial L(q(t),\dot q(t))}{\partial \dot q(t)}\right),$$
$$\frac{\delta I[\phi;\sigma'',\sigma']}{\delta\phi(x)} = \frac{\partial\mathcal{L}(\phi(x),\partial_\mu\phi(x))}{\partial\phi(x)} - \partial_\mu\left(\frac{\partial\mathcal{L}(\phi(x),\partial_\mu\phi(x))}{\partial(\partial_\mu\phi(x))}\right).$$

The (Euler-) Lagrange equation of motion follows from the identities:

$$\mathcal{D}\,[q+\varepsilon] = \mathcal{D}\,[q]; \quad \text{and} \quad \mathcal{D}\,[\phi+\varepsilon] = \mathcal{D}\,[\phi].$$

Under these identities, we have the following equations:

$$\frac{1}{N(\Omega)}\int_{q(t')=q'}^{q(t'')=q''}\mathcal{D}\,[q]\exp\left[\frac{i}{\hbar}I[q;t'',t']\right]$$
$$= \frac{1}{N(\Omega)}\int_{q(t')=q'}^{q(t'')=q''}\mathcal{D}\,[q+\varepsilon]\exp\left[\frac{i}{\hbar}I[q+\varepsilon;t'',t']\right]$$
$$= \frac{1}{N(\Omega)}\int_{q(t')=q'}^{q(t'')=q''}\mathcal{D}\,[q]\exp\left[\frac{i}{\hbar}I[q+\varepsilon;t'',t']\right],$$
$$\frac{1}{N(\Omega)}\int_{\phi',\sigma'}^{\phi'',\sigma''}\mathcal{D}\,[\phi]\exp\left[\frac{i}{\hbar}I[\phi;\sigma'',\sigma']\right]$$
$$= \frac{1}{N(\Omega)}\int_{\phi',\sigma'}^{\phi'',\sigma''}\mathcal{D}\,[\phi+\varepsilon]\exp\left[\frac{i}{\hbar}I[\phi+\varepsilon;\sigma'',\sigma']\right]$$
$$= \frac{1}{N(\Omega)}\int_{\phi',\sigma'}^{\phi'',\sigma''}\mathcal{D}\,[\phi]\exp\left[\frac{i}{\hbar}I[\phi+\varepsilon;\sigma'',\sigma']\right].$$

Expanding the above equations in the powers of $\varepsilon(t)$ and $\varepsilon(x)$, and picking up the lowest order terms, we obtain

$$\frac{1}{N(\Omega)}\int_{q(t')=q'}^{q(t'')=q''}\mathcal{D}\,[q]\frac{\hbar}{i}\frac{\delta}{\delta q(t)}\exp\left[\frac{i}{\hbar}I[q;t'',t']\right] = 0,$$
$$\frac{1}{N(\Omega)}\int_{\phi',\sigma'}^{\phi'',\sigma''}\mathcal{D}\,[\phi]\frac{\hbar}{i}\frac{\delta}{\delta\phi(x)}\exp\left[\frac{i}{\hbar}I[\phi;\sigma'',\sigma']\right] = 0.$$

We thus have, in accordance with Feynman's action principle,

$$\left\langle q'', t'' \left| \frac{\delta I[\hat{q}; t'', t']}{\delta \hat{q}(t)} \right| q', t' \right\rangle = \frac{1}{N(\Omega)} \int_{q(t')=q'}^{q(t'')=q''} \mathcal{D}[q] \frac{\delta I}{\delta q(t)} \exp\left[\frac{i}{\hbar} I[q; t'', t']\right]$$

$$= \frac{1}{N(\Omega)} \int_{q(t')=q'}^{q(t'')=q''} \mathcal{D}[q] \frac{\hbar}{i} \frac{\delta}{\delta q(t)} \exp\left[\frac{i}{\hbar} I[q; t'', t']\right] = 0, \tag{10.2.12M}$$

$$\left\langle \phi'', \sigma'' \left| \frac{\delta I[\hat{\phi}; \sigma'', \sigma']}{\delta \hat{\phi}(x)} \right| \phi', \sigma' \right\rangle = \frac{1}{N(\Omega)} \int_{\phi', \sigma'}^{\phi'', \sigma''} \mathcal{D}[\phi] \frac{\delta I}{\delta \phi(x)} \exp\left[\frac{i}{\hbar} I[\phi; \sigma'', \sigma']\right]$$

$$= \frac{1}{N(\Omega)} \int_{\phi', \sigma'}^{\phi'', \sigma''} \mathcal{D}[\phi] \frac{\hbar}{i} \frac{\delta}{\delta \phi(x)} \exp\left[\frac{i}{\hbar} I[\phi; \sigma'', \sigma']\right] = 0. \tag{10.2.12F}$$

By the assumption that the eigenkets $|q, t\rangle$ ($|\phi, \sigma\rangle$) form the complete set, we obtain the (Euler-) Lagrange equation of motion at the operator level from (10.2.12M) and (10.2.12F) as

$$-\frac{\delta I[\hat{q}]}{\delta \hat{q}(t)} = \frac{d}{dt}\left(\frac{\partial L(\hat{q}(t), \dot{\hat{q}}(t))}{\partial \dot{\hat{q}}(t)}\right) - \frac{\partial L(\hat{q}(t), \dot{\hat{q}}(t))}{\partial \hat{q}(t)} = 0, \tag{10.2.13M}$$

$$-\frac{\delta I[\hat{\phi}]}{\delta \hat{\phi}(x)} = \partial_\mu \left(\frac{\partial \mathcal{L}(\hat{\phi}(x), \partial_\mu \hat{\phi}(x))}{\partial(\partial_\mu \hat{\phi}(x))}\right) - \frac{\partial \mathcal{L}(\hat{\phi}(x), \partial_\mu \hat{\phi}(x))}{\partial \hat{\phi}(x)} = 0. \tag{10.2.13F}$$

As for the time-ordered product, we have

$$\frac{1}{N(\Omega)} \int_{q(t')=q'}^{q(t'')=q''} \mathcal{D}[q] q(t_1) \cdots q(t_n) \exp\left[\frac{i}{\hbar} I\right] = \left\langle q'', t'' \left| \mathrm{T}(\hat{q}(t_1) \cdots \hat{q}(t_n)) \right| q', t' \right\rangle, \tag{10.2.14M}$$

$$\frac{1}{N(\Omega)} \int_{\phi', \sigma'}^{\phi'', \sigma''} \mathcal{D}[\phi] \phi(x_1) \cdots \phi(x_n) \exp\left[\frac{i}{\hbar} I\right]$$

$$= \left\langle \phi'', \sigma'' \left| \mathrm{T}(\hat{\phi}(x_1) \cdots \hat{\phi}(x_n)) \right| \phi', \sigma' \right\rangle. \tag{10.2.14F}$$

We can prove the above by mathematical induction starting from $n = 2$. The left-hand sides of (10.2.14M) and (10.2.14F) are the matrix elements of the canonical T*-product,

$$\mathrm{T}^*(\partial_\mu^{x_1} \hat{O}_1(x_1) \cdots \hat{O}_n(x_n)) \equiv \partial_\mu^{x_1} \mathrm{T}^*(\hat{O}_1(x_1) \cdots \hat{O}_n(x_n)),$$

with

$$\mathrm{T}^*(\hat{\phi}_{r_1}(x_1) \cdots \hat{\phi}_{r_n}(x_n)) \equiv \mathrm{T}(\hat{\phi}_{r_1}(x_1) \cdots \hat{\phi}_{r_n}(x_n)).$$

Canonical Momentum and Equal-Time Canonical Commutators. We define the momentum operator as the displacement operator,

$$\left\langle q'', t'' \left| \hat{p}(t'') \right| q', t' \right\rangle = \frac{\hbar}{i}\frac{\partial}{\partial q''} \left\langle q'', t'' \middle| q', t' \right\rangle, \tag{10.2.15M}$$

$$\left\langle \phi'', \sigma'' \left| \hat{\pi}(x'') \right| \phi', \sigma' \right\rangle = \frac{\hbar}{i}\frac{\delta}{\delta \phi''} \left\langle \phi'', \sigma'' \middle| \phi', \sigma' \right\rangle. \tag{10.2.15F}$$

In order to express the right-hand sides of (10.2.15M) and (10.2.15F) in the form in which we can use Feynman's action principle, we consider the following variation of the action functional;
(1) Inside Ω, we consider the infinitesimal variations of $q(t)$ and $\phi(x)$,

$$q(t) \longrightarrow q(t) + \delta q(t) \quad \text{and} \quad \phi(x) \longrightarrow \phi(x) + \delta\phi(x), \tag{10.2.16M,F}$$

where, as $\delta q(t)$ and $\delta\phi(x)$, we take

$$\delta q(t') = 0,\ \delta q(t) = \xi(t),\ \delta q(t'') = \xi'';\ \delta\phi(x') = 0,\ \delta\phi(x) = \xi(x),\ \delta\phi(x'') = \xi''. \tag{10.2.17M,F}$$

(2) Inside Ω, the physical system evolves in time in accordance with the (Euler-) Lagrange equation of motion.
As the response of the action functional to a particular variations, (1) and (2), we have

$$\delta I[q; t'', t'] = \frac{\partial L(q(t''), \dot{q}(t''))}{\partial \dot{q}(t'')}\xi'', \tag{10.2.18M}$$

$$\delta I[\phi; \sigma'', \sigma'] = \int_{\sigma''} d\sigma''_{\mu} \left\{ \frac{\partial \mathcal{L}(\phi(x''), \partial_{\mu}\phi(x''))}{\partial(\partial_{\mu}\phi(x''))}\xi'' \right\}. \tag{10.2.18F}$$

Thus we obtain

$$\frac{\delta I[q; t'', t']}{\delta q(t'')} = \frac{\partial L(q(t''), \dot{q}(t''))}{\partial \dot{q}(t'')}, \qquad \frac{\delta I[\phi; \sigma'', \sigma']}{\delta \phi(x'')} = n_{\mu}(x'')\frac{\partial \mathcal{L}(\phi(x''), \partial_{\mu}\phi(x''))}{\partial(\partial_{\mu}\phi(x''))}, \tag{10.2.19M,F}$$

where $n_{\mu}(x'')$ is the unit normal vector at point x'' on the space-like surface σ''. With this, we obtain

$$\frac{\hbar}{i}\frac{\partial}{\partial q(t'')} \left\langle q'', t'' \middle| q', t' \right\rangle = \frac{\hbar}{i} \lim_{\xi'' \to 0} \frac{\left\langle q'' + \xi'', t'' \middle| q', t' \right\rangle - \left\langle q'', t'' \middle| q', t' \right\rangle}{\xi''}$$

$$= \frac{\hbar}{i} \lim_{\xi'' \to 0} \frac{1}{N(\Omega)} \int_{q(t')=q'}^{q(t'')=q''} \mathcal{D}[q] \exp\left[\frac{i}{\hbar} I[q; t'', t']\right]$$

$$\times \frac{1}{\xi''}\left\{\exp\left[\frac{i}{\hbar}(I[q+\xi;t'',t']-I[q;t'',t'])\right]-1\right\}$$
$$=\lim_{\xi''\to 0}\frac{1}{N(\Omega)}\int_{q(t')=q'}^{q(t'')=q''}\mathcal{D}[q]\exp\left[\frac{i}{\hbar}I[q;t'',t']\right]$$
$$\times \frac{1}{\xi''}\{I[q+\xi;t'',t']-I[q;t'',t']+0(\xi''2)\}$$
$$=\frac{1}{N(\Omega)}\int_{q(t')=q'}^{q(t'')=q''}\mathcal{D}[q]\exp\left[\frac{i}{\hbar}I[q;t'',t']\right]\frac{\delta I[q;t'',t']}{\delta q(t'')}$$
$$=\frac{1}{N(\Omega)}\int_{q(t')=q'}^{q(t'')=q''}\mathcal{D}[q]\exp\left[\frac{i}{\hbar}I[q;t'',t']\right]\frac{\partial L(q(t''),\dot{q}(t''))}{\partial \dot{q}(t'')}$$
$$=\left\langle q'',t''\left|\frac{\partial L(\hat{q}(t''),\dot{\hat{q}}(t''))}{\partial \dot{\hat{q}}(t'')}\right|q',t'\right\rangle. \qquad (10.2.20M)$$

Likewise, we obtain

$$\frac{\hbar}{i}\frac{\delta}{\delta\phi''(x'')}\langle\phi'',\sigma''|\,\phi',\sigma'\rangle=\frac{1}{N(\Omega)}\int_{\phi',\sigma'}^{\phi'',\sigma''}\mathcal{D}[\phi]\exp\left[\frac{i}{\hbar}I[\phi;\sigma'',\sigma']\right]$$
$$\times\frac{\delta I[\phi;\sigma'',\sigma']}{\delta\phi(x'')}$$
$$=\frac{1}{N(\Omega)}\int_{\phi',\sigma'}^{\phi'',\sigma''}\mathcal{D}[\phi]\exp\left[\frac{i}{\hbar}I[\phi;\sigma'',\sigma']\right]n_\mu(x'')\frac{\partial\mathcal{L}(\phi(x''),\partial_\mu\phi(x''))}{\partial(\partial_\mu\phi(x''))}$$
$$=\left\langle\phi'',\sigma''\left|n_\mu(x'')\frac{\partial\mathcal{L}(\hat{\phi}(x''),\partial_\mu\hat{\phi}(x''))}{\partial(\partial_\mu\hat{\phi}(x''))}\right|\phi',\sigma'\right\rangle. \qquad (10.2.20F)$$

From (10.2.15M) and (10.2.15F), and (10.2.20M) and (10.2.20F), we obtain the identities

$$\langle q'',t''\,|\hat{p}(t'')|\,q',t'\rangle=\left\langle q'',t''\left|\frac{\partial L(\hat{q}(t''),\dot{\hat{q}}(t''))}{\partial\dot{\hat{q}}(t'')}\right|q',t'\right\rangle,$$
$$\langle\phi'',\sigma''\,|\hat{\pi}(x'')|\,\phi',\sigma'\rangle=\left\langle\phi'',\sigma''\left|n_\mu(x'')\frac{\partial\mathcal{L}(\hat{\phi}(x''),\partial_\mu\hat{\phi}(x''))}{\partial(\partial_\mu\hat{\phi}(x''))}\right|\phi',\sigma'\right\rangle,$$

or

$$\hat{p}(t)=\frac{\partial L(\hat{q}(t),\dot{\hat{q}}(t))}{\partial\dot{\hat{q}}(t)}\quad\text{and}\quad\hat{\pi}(x)=n_\mu(x)\frac{\partial\mathcal{L}(\hat{\phi}(x),\partial_\mu\hat{\phi}(x))}{\partial(\partial_\mu\hat{\phi}(x))}. \qquad (10.2.21M, F)$$

Equation (10.2.21M) is the definition of the canonical momentum $\hat{p}(t)$. With a choice of the unit normal vector as $n_\mu(x)=(1,0,0,0)$, Eq. (10.2.21F) is also the definition of the canonical momentum $\hat{\pi}(x)$. A noteworthy point is that $\hat{\pi}(x)$ is a normal dependent quantity.

Lastly, for quantum mechanics of the Bose particle system, from

$$\langle q'', t'' | \hat{q}_{\mathrm{B}}(t'') | q', t' \rangle = q''_{\mathrm{B}} \langle q'', t'' | q', t' \rangle \tag{10.2.22M}$$

and from (10.2.15M), we have

$$\begin{aligned} \langle q'', t'' | \hat{p}_{\mathrm{B}}(t'') \hat{q}_{\mathrm{B}}(t'') | q', t' \rangle &= \frac{\hbar}{i} \frac{\partial}{\partial q''_{\mathrm{B}}} \left(q''_{\mathrm{B}} \langle q'', t'' | q', t' \rangle \right) \\ &= \frac{\hbar}{i} \langle q'', t'' | q', t' \rangle + q''_{\mathrm{B}} \langle q'', t'' | \hat{p}_{\mathrm{B}}(t'') | q', t' \rangle \\ &= \frac{\hbar}{i} \langle q'', t'' | q', t' \rangle + \langle q'', t'' | \hat{q}_{\mathrm{B}}(t'') \hat{p}_{\mathrm{B}}(t'') | q', t' \rangle , \end{aligned} \tag{10.2.23M}$$

i.e., we have

$$[\hat{p}_{\mathrm{B}}(t), \hat{q}_{\mathrm{B}}(t)] = \frac{\hbar}{i}, \tag{10.2.24M}$$

where the commutator, $[A, B]$, is given by

$$[A, B] = AB - BA.$$

For quantum mechanics of the Fermi particle system, we have a minus sign in front of the second terms of the second line and the third line of Eq. (10.2.23M) which originates from the anticommuting Fermion number, so that we obtain

$$\{\hat{p}_{\mathrm{F}}(t), \hat{q}_{\mathrm{F}}(t)\} = \frac{\hbar}{i}, \tag{10.2.25M}$$

where the anticommutator, $\{A, B\}$, is given by

$$\{A, B\} = AB + BA.$$

Equations (10.2.24M) and (10.2.25M) are the natural consequences of the choice of the momentum operator as the displacement operator, (10.2.15M).

For quantum field theory of the Bose field, from

$$\left\langle \phi'', \sigma'' \middle| \hat{\phi}_{\mathrm{B}}(x'') \middle| \phi', \sigma' \right\rangle = \phi''_{\mathrm{B}}(x'') \langle \phi'', \sigma'' | \phi', \sigma' \rangle \tag{10.2.22F}$$

and from (10.2.15F), we have

$$\begin{aligned} \left\langle \phi'', \sigma'' \middle| \hat{\pi}_{\mathrm{B}}(x''_1) \hat{\phi}_{\mathrm{B}}(x''_2) \middle| \phi', \sigma' \right\rangle &= \frac{\hbar}{i} \frac{\delta}{\delta \phi''_{\mathrm{B}}(x''_1)} \left(\phi''_{\mathrm{B}}(x''_2) \left\langle \phi'', \sigma'' \middle| \phi', \sigma' \right\rangle \right) \\ &= \frac{\hbar}{i} \delta_{\sigma''}(x''_1 - x''_2) \left\langle \phi'', \sigma'' \middle| \phi', \sigma' \right\rangle + \phi''_{\mathrm{B}}(x''_2) \left\langle \phi'', \sigma'' \middle| \hat{\pi}_{\mathrm{B}}(x''_1) \middle| \phi', \sigma' \right\rangle \\ &= \frac{\hbar}{i} \delta_{\sigma''}(x''_1 - x''_2) \left\langle \phi'', \sigma'' \middle| \phi', \sigma' \right\rangle + \left\langle \phi'', \sigma'' \middle| \hat{\phi}_{\mathrm{B}}(x''_2) \hat{\pi}_{\mathrm{B}}(x''_1) \middle| \phi', \sigma' \right\rangle . \end{aligned} \tag{10.2.23F}$$

We have

$$[\hat{\pi}_{\mathrm{B}}(x_1), \hat{\phi}_{\mathrm{B}}(x_2)] = \frac{\hbar}{i}\delta_\sigma(x_1 - x_2). \tag{10.2.24F}$$

For quantum field theory of the Fermi field, for the same reason as in (10.2.25M), we obtain

$$\{\hat{\pi}_{\mathrm{F}}(x_1), \hat{\phi}_{\mathrm{F}}(x_2)\} = \frac{\hbar}{i}\delta_\sigma(x_1 - x_2). \tag{10.2.25F}$$

Equations (10.2.24F) and (10.2.25F) are the natural consequences of the choice of the momentum operator as the displacement operator, (10.2.15F).

From Feynman's action principle, (10.2.1M,F), the assumptions (A.1), (A-2), and (A-3), the path integral definition of the operator, (10.2.7M,F), and the definition of the momentum operator as the displacement operator, (10.2.15M,F), we deduced the following four statements:
(a) the (Euler-) Lagrange equation of motion, (10.2.13M,F),
(b) the definition of the time-ordered product, (10.2.14M,F),
(c) the definition of canonical conjugate momentum, (10.2.21M,F),
(d) the equal time canonical commutator, (10.2.24M), (10.2.24F), and (10.2.25M,F).

Thus we demonstrated the equivalence of canonical quantization and path integral quantization for the nonsingular Lagrangian (density). In the next section, we will establish the equivalence of Schwinger's action principle and Feynman's action principle for the nonsingular Lagrangian (density).

10.3 Schwinger's Action Principle in Quantum Theory

In this section, we will discuss Schwinger's action principle in quantum theory and demonstrate its equivalence to Feynman's action principle. We first state Schwinger's action principle and then obtain the transition probability amplitude in the form of Feynman's action principle by the method of the functional Fourier transform.

Schwinger's action principle asserts that the variation of the transition probability amplitude $\langle\infty|-\infty\rangle$ results from variation of action functional, $I[\hat{\phi}^i, \hat{\psi}^{\mathrm{i}}]$, which assumes the following form:

$$\delta\left\langle\infty\middle|-\infty\right\rangle = i\left\langle\infty\left|\delta I[\hat{\phi}^i, \hat{\psi}^{\mathrm{i}}]\right|-\infty\right\rangle. \tag{10.3.1}$$

Here $|-\infty\rangle$ ($\langle\infty|$) is any eigenket (any eigenbra) of any dynamical quantity which lies in the remote past (future), $\hat{\phi}_i$ and $\hat{\psi}_{\mathrm{i}}$ generically represent Boson variables and Fermion variables, respectively, and the indices i and i represent both the space–time degrees of freedom and the internal degrees of freedom, respectively.

In order to visualize what we mean by the variation $\delta I[\hat{\phi}^i, \hat{\psi}^{\mathrm{i}}]$ of the action functional $I[\hat{\phi}^i, \hat{\psi}^{\mathrm{i}}]$, it is convenient to introduce the external hook coupling as

$$I[\hat{\phi}^i, \hat{\psi}^{\mathrm{i}}] + J_i \hat{\phi}^i + J_{\mathrm{i}} \hat{\psi}^{\mathrm{i}}, \tag{10.3.2}$$

and consider the variation of the c-number external hooks, J_i and J_{i}.

Applying Schwinger's action principle to the action functional, (10.3.2), we have the following equations for $\langle \infty | - \infty \rangle$,

$$(\delta / i\delta J_i) \left\langle \infty \middle| - \infty \right\rangle = \left\langle \infty \middle| \hat{\phi}^i \middle| - \infty \right\rangle, \tag{10.3.3B}$$

$$(\delta / i\delta J_{\mathrm{i}}) \left\langle \infty \middle| - \infty \right\rangle = \left\langle \infty \middle| \hat{\psi}^{\mathrm{i}} \middle| - \infty \right\rangle. \tag{10.3.3F}$$

We express the right-hand sides of (10.3.3B) and (10.3.3F) as

$$\begin{cases} \left\langle \infty \middle| \hat{\phi}^i \middle| - \infty \right\rangle &= \sum \left\langle \infty \middle| \phi^{i'} \right\rangle \phi^{i'} \left\langle \phi^{i'} \middle| - \infty \right\rangle, \\ \left\langle \infty \middle| \hat{\psi}^{\mathrm{i}} \middle| - \infty \right\rangle &= \sum \left\langle \infty \middle| \psi^{\mathrm{i}'} \right\rangle \psi^{\mathrm{i}'} \left\langle \psi^{\mathrm{i}'} \middle| - \infty \right\rangle. \end{cases} \tag{10.3.4}$$

Here the summation is over the eigenvectors $\left| \phi^{i'} \right\rangle$ and $\left| \psi^{\mathrm{i}'} \right\rangle$ of complete sets of commuting and anticommuting operators $\hat{\phi}^i$ and $\hat{\psi}^{\mathrm{i}}$, respectively. Taking the functional derivative once more, we have

$$\begin{cases} (\delta / i\delta J_j)(\delta / i\delta J_i) \left\langle \infty \middle| - \infty \right\rangle &= \left\langle \infty \middle| \mathrm{T}(\hat{\phi}^j \hat{\phi}^i) \middle| - \infty \right\rangle, \\ (\delta / i\delta J_{\mathrm{j}})(\delta / i\delta J_i) \left\langle \infty \middle| - \infty \right\rangle &= \left\langle \infty \middle| \mathrm{T}(\hat{\psi}^{\mathrm{j}} \hat{\phi}^i) \middle| - \infty \right\rangle, \\ (\delta / i\delta J_{\mathrm{j}})(\delta / i\delta J_{\mathrm{i}}) \left\langle \infty \middle| - \infty \right\rangle &= \left\langle \infty \middle| \mathrm{T}(\hat{\psi}^{\mathrm{j}} \hat{\psi}^{\mathrm{i}}) \middle| - \infty \right\rangle. \end{cases} \tag{10.3.5}$$

Here the time-ordered products are defined by

$$\begin{cases} \mathrm{T}(\hat{\phi}^j \hat{\phi}^i) &\equiv \theta(j, i) \hat{\phi}^j \hat{\phi}^i + \theta(i, j) \hat{\phi}^i \hat{\phi}^j, \\ \mathrm{T}(\hat{\psi}^{\mathrm{j}} \hat{\phi}^i) &\equiv \theta(\mathrm{j}, i) \hat{\psi}^{\mathrm{j}} \hat{\phi}^i + \theta(i, \mathrm{j}) \hat{\phi}^i \hat{\psi}^{\mathrm{j}}, \\ \mathrm{T}(\hat{\psi}^{\mathrm{j}} \hat{\psi}^{\mathrm{i}}) &\equiv \theta(\mathrm{j}, \mathrm{i}) \hat{\psi}^{\mathrm{j}} \hat{\psi}^{\mathrm{i}} - \theta(\mathrm{i}, \mathrm{j}) \hat{\psi}^{\mathrm{i}} \hat{\psi}^{\mathrm{j}}. \end{cases} \tag{10.3.6}$$

The multiple time-ordered products are defined by the successive application of the functional derivatives upon $\left\langle \infty \middle| - \infty \right\rangle$. We employ the abbreviation

$$\mathrm{T}^{ij\cdots \mathrm{kl}\cdots} \equiv \mathrm{T}(\hat{\phi}^i \hat{\phi}^j \cdots \hat{\psi}^{\mathrm{k}} \hat{\psi}^{\mathrm{l}} \cdots). \tag{10.3.7}$$

Then, for the matrix element of the multiple time-ordered products , we have

$$\left\langle \infty \middle| \mathrm{T}^{ij\cdots \mathrm{kl}\cdots} \middle| - \infty \right\rangle = \frac{\delta}{i\delta J_i} \frac{\delta}{i\delta J_j} \cdots \frac{\delta}{i\delta J_{\mathrm{k}}} \frac{\delta}{i\delta J_{\mathrm{l}}} \cdots \left\langle \infty \middle| - \infty \right\rangle. \tag{10.3.8}$$

In order to remove the operator ordering ambiguity, we write (10.3.1) as

$$\delta \left\langle \infty \middle| -\infty \right\rangle = i\left\langle \infty \middle| \mathrm{T}(\delta I[\hat{\phi}^i, \hat{\psi}^{\mathrm{i}}]) \middle| -\infty \right\rangle. \tag{10.3.9}$$

Then the operator dynamical equations are written as

$$\mathrm{T}\left(\frac{\delta I[\hat{\phi}^i, \hat{\psi}^{\mathrm{i}}]}{\delta \hat{\phi}^i}\right) = -J_i, \tag{10.3.10}$$

$$\mathrm{T}\left(\frac{\delta I[\hat{\phi}^i, \hat{\psi}^{\mathrm{i}}]}{\delta \hat{\psi}^{\mathrm{i}}}\right) = -J_{\mathrm{i}}. \tag{10.3.11}$$

We now take the matrix elements of (10.3.10) and (10.3.11) between the states, $|-\infty\rangle$ and $|\infty\rangle$, with the results

$$\left(\frac{\delta I[\delta/i\delta J]}{\delta \phi^i}\right)\left\langle \infty \middle| -\infty \right\rangle = -J_i \left\langle \infty \middle| -\infty \right\rangle, \tag{10.3.12}$$

$$\left(\frac{\delta I[\delta/i\delta J]}{\delta \psi^{\mathrm{i}}}\right)\left\langle \infty \middle| -\infty \right\rangle = -J_{\mathrm{i}} \left\langle \infty \middle| -\infty \right\rangle. \tag{10.3.13}$$

We note that $\delta I[\delta/i\delta J]/\delta\phi^i$ and $\delta I[\delta/i\delta J]/\delta\psi^{\mathrm{i}}$ are obtained by expanding $\delta I[\phi^i, \psi^{\mathrm{i}}]/\delta\phi^i$ and $\delta I[\phi^i, \psi^{\mathrm{i}}]/\delta\psi^{\mathrm{i}}$ in powers of the ϕ^i and ψ^{i} and replacing the ϕ^i and ψ^{i} in the expansion with the $\delta/i\delta J_i$ and $\delta/i\delta J_{\mathrm{i}}$, respectively. In order to solve (10.3.12) and (10.3.13) for the transition probability amplitude $\langle\infty|-\infty\rangle$, we introduce the functional Fourier transform of the transition probability amplitude $\langle\infty|-\infty\rangle$ as

$$\left\langle \infty \middle| -\infty \right\rangle = \int \mathcal{D}[\phi]\mathcal{D}[\psi]F[\phi,\psi]\exp[i(J_i\phi^i + J_{\mathrm{i}}\psi^{\mathrm{i}})]. \tag{10.3.14}$$

Here $\mathcal{D}[\phi]$ and $\mathcal{D}[\psi]$ are formally defined by

$$\mathcal{D}[\phi] \equiv \prod_i d\phi^i, \quad \mathcal{D}[\psi] \equiv \prod_{\mathrm{i}} d\psi^{\mathrm{i}}. \tag{10.3.15}$$

Inserting (10.3.14) into (10.3.12) and (10.3.13), we get

$$\begin{aligned} 0 &= \int \mathcal{D}[\phi]\mathcal{D}[\psi]F[\phi,\psi](\delta I[\phi,\psi]/\delta\phi^i + J_i)\exp[i(J_i\phi^i + J_{\mathrm{i}}\psi^{\mathrm{i}})] \\ &= \int \mathcal{D}[\phi]\mathcal{D}[\psi]F[\phi,\psi](\delta I[\phi,\psi]/\delta\phi^i + \delta/i\delta\phi^i)\exp[i(J_i\phi^i + J_{\mathrm{i}}\psi^{\mathrm{i}})], \end{aligned} \tag{10.3.16}$$

$$\begin{aligned} 0 &= \int \mathcal{D}[\phi]\mathcal{D}[\psi]F[\phi,\psi](\delta I[\phi,\psi]/\delta\psi^{\mathrm{i}} + J_{\mathrm{i}})\exp[i(J_i\phi^i + J_{\mathrm{i}}\psi^{\mathrm{i}})] \\ &= \int \mathcal{D}[\phi]\mathcal{D}[\psi]F[\phi,\psi](\delta I[\phi,\psi]/\delta\psi^{\mathrm{i}} + \delta/i\delta\psi^{\mathrm{i}})\exp[i(J_i\phi^i + J_{\mathrm{i}}\psi^{\mathrm{i}})]. \end{aligned} \tag{10.3.17}$$

Integrating by parts, we obtain the differential equation for the functional Fourier transform $F[\phi, \psi]$ as

$$\left(\frac{\delta I[\phi, \psi]}{\delta \phi^i} - \frac{\delta}{i\delta \phi^i}\right) F[\phi, \psi] = 0, \tag{10.3.18}$$

$$\left(\frac{\delta I[\phi, \psi]}{\delta \psi^{\mathrm{i}}} - \frac{\delta}{i\delta \psi^{\mathrm{i}}}\right) F[\phi, \psi] = 0. \tag{10.3.19}$$

Differential equations, (10.3.18) and (10.3.19), can be immediately integrated with the result,

$$F[\phi, \psi] = \frac{1}{N} \exp[iI[\phi, \psi]], \tag{10.3.20}$$

where N is the normalization constant. Hence finally we have

$$\left\langle \infty \middle| -\infty \right\rangle = \frac{1}{N} \int \mathcal{D}[\phi]\mathcal{D}[\psi] \exp[i(I[\phi, \psi] + J_i\phi^i + J_{\mathrm{i}}\psi^{\mathrm{i}})]. \tag{10.3.21}$$

This is Feynman's action principle. If we apply the identity, (10.3.8), to (10.3.21), we have

$$\left\langle \infty \middle| \mathrm{T}^{ij\cdots \mathrm{kl}\cdots} \middle| -\infty \right\rangle = \frac{1}{N} \int \mathcal{D}[\phi]\mathcal{D}[\psi]\phi^i\phi^j \cdots \psi^{\mathrm{k}}\psi^{\mathrm{l}} \cdots \\ \times \exp[i(I[\phi, \psi] + J_i\phi^i + J_{\mathrm{i}}\psi^{\mathrm{i}})]. \tag{10.3.22}$$

We thus have established the equivalence of Schwinger's action principle and Feynman's action principle, at least for the nonsingular system in the abstract notation without committing ourselves to quantum mechanics or quantum field theory.

When the infinite-dimensional invariance group is present, our discussion becomes invalid, since the action functional $I[\phi, \psi]$ remains constant under the action of the infinite-dimensional invariance group. The appearance of the external hooks, J_i and J_{i}, coupled linearly to the dynamical variables, $(\phi^i, \psi^{\mathrm{i}})$, violates the infinite-dimensional group invariance .

The systems with the infinite-dimensional invariance group , for example, the Abelian and non-Abelian gauge fields and the gravitational field , are the subject matter of Section 10.9.

10.4
Schwinger–Dyson Equation in Quantum Field Theory

In quantum field theory, we also have the notion of Green's functions, which is quite distinct from the ordinary Green's functions in mathematical physics in one important aspect: the governing equation of motion of the connected part of the two-point "full" Green's function in quantum field theory is the closed system of the coupled nonlinear integro-differential equations. We illustrate this point in some detail for the Yukawa coupling of the fermion field and the boson field.

We shall establish the relativistic notation. We employ the natural unit system in which

$$\hbar = c = 1. \tag{10.4.1}$$

We define the Minkowski space–time metric tensor $\eta_{\mu\nu}$ by

$$\eta_{\mu\nu} \equiv \text{diag}(1; -1, -1, -1) \equiv \eta^{\mu\nu}, \quad \mu, \nu = 0, 1, 2, 3. \tag{10.4.2}$$

We define the contravariant components and the covariant components of the space–time coordinates x by

$$x^{\mu} \equiv (x^0, x^1, x^2, x^3), \tag{10.4.3a}$$

$$x_{\mu} \equiv \eta_{\mu\nu} x^{\nu} = (x^0, -x^1, -x^2, -x^3). \tag{10.4.3b}$$

We define the differential operators ∂_μ and ∂^μ by

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3} \right), \quad \partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \eta^{\mu\nu} \partial_\nu. \tag{10.4.4}$$

We define the four-scalar product by

$$x \cdot y \equiv x^\mu y_\mu = \eta_{\mu\nu} x^\mu y^\nu = x^0 \cdot y^0 - \vec{x} \cdot \vec{y}. \tag{10.4.5}$$

We adopt the convention that the Greek indices μ, ν,. . ., run over 0, 1, 2, and 3, the Latin indices i, j, . . ., run over 1, 2, and 3, and the repeated indices are summed over.

We consider the quantum field theory described by the total Lagrangian density $\mathcal{L}_{\text{tot}}$ of the form

$$\mathcal{L}_{\text{tot}} = \frac{1}{4} \left[\widehat{\overline{\psi}}_\alpha(x), D_{\alpha\beta}(x) \hat{\psi}_\beta(x) \right] + \frac{1}{4} \left[D^{\text{T}}_{\beta\alpha}(-x) \widehat{\overline{\psi}}_\alpha(x), \hat{\psi}_\beta(x) \right]$$
$$+ \frac{1}{2} \hat{\phi}(x) K(x) \hat{\phi}(x) + \mathcal{L}_{\text{int}}(\hat{\phi}(x), \hat{\psi}(x), \widehat{\overline{\psi}}(x)), \tag{10.4.6}$$

where we have

$$D_{\alpha\beta}(x) = (i\gamma_\mu \partial^\mu - m + i\varepsilon)_{\alpha\beta}, \quad D^{\text{T}}_{\beta\alpha}(-x) = (-i\gamma^{\text{T}}_\mu \partial^\mu - m + i\varepsilon)_{\beta\alpha}, \tag{10.4.7a}$$

$$K(x) = -\partial^2 - \kappa^2 + i\varepsilon, \tag{10.4.7b}$$

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \quad (\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0, \quad \widehat{\overline{\psi}}_\alpha(x) = (\hat{\psi}^\dagger(x) \gamma^0)_\alpha, \tag{10.4.8}$$

$$I_{\text{tot}}[\hat{\phi}, \hat{\psi}, \widehat{\overline{\psi}}] = \int d^4x \mathcal{L}_{\text{tot}}((10.4.9)), \quad I_{\text{int}}[\hat{\phi}, \hat{\psi}, \widehat{\overline{\psi}}] = \int d^4x \mathcal{L}_{\text{int}}((10.4.6)). \tag{10.4.9}$$

We have Euler–Lagrange equations of motion for the field operators:

$$\hat{\psi}_\alpha(x): \quad \frac{\delta \hat{I}_{\text{tot}}}{\delta \widehat{\overline{\psi}}_\alpha(x)} = 0 \quad \text{or} \quad D_{\alpha\beta}(x)\hat{\psi}_\beta(x) + \frac{\delta \hat{I}_{\text{int}}}{\delta \widehat{\overline{\psi}}_\alpha(x)} = 0, \tag{10.4.10a}$$

$$\widehat{\overline{\psi}}_\beta(x): \quad \frac{\delta \hat{I}_{\text{tot}}}{\delta \hat{\psi}_\beta(x)} = 0 \quad \text{or} \quad -D^{\mathrm{T}}_{\beta\alpha}(-x)\widehat{\overline{\psi}}_\alpha(x) + \frac{\delta \hat{I}_{\text{int}}}{\delta \hat{\psi}_\beta(x)} = 0, \tag{10.4.10b}$$

$$\hat{\phi}(x): \quad \frac{\delta I_{\text{tot}}}{\delta \hat{\phi}(x)} = 0 \quad \text{or} \quad K(x)\hat{\phi}(x) + \frac{\delta \hat{I}_{\text{int}}}{\delta \hat{\phi}(x)} = 0. \tag{10.4.10c}$$

We have the equal time canonical (anti-)commutators:

$$\delta(x^0 - y^0)\{\hat{\psi}_\beta(x), \widehat{\overline{\psi}}_\alpha(x)\} = \gamma^0_{\beta\alpha}\delta^4(x-y), \tag{10.4.11a}$$

$$\delta(x^0 - y^0)\{\hat{\psi}_\beta(x), \hat{\psi}_\alpha(y)\} = \delta(x^0 - y^0)\{\widehat{\overline{\psi}}_\beta(x), \widehat{\overline{\psi}}_\alpha(x)\} = 0, \tag{10.4.11b}$$

$$\delta(x^0 - y^0)[\hat{\phi}(x), \partial^y_0\hat{\phi}(y)] = i\delta^4(x-y), \tag{10.4.11c}$$

$$\delta(x^0 - y^0)[\hat{\phi}(x), \hat{\phi}(y)] = \delta(x^0 - y^0)[\partial^x_0\hat{\phi}(x), \partial^y_0\hat{\phi}(y)] = 0, \tag{10.4.11d}$$

and the rest of the equal time mixed canonical commutators are equal to 0. We define the *generating functional of the full Green's functions* by

$$\begin{aligned}
&Z[J, \overline{\eta}, \eta] \\
&\equiv \langle 0, \text{out}|\, \mathrm{T}(\exp[i\{(J\hat{\phi}) + (\overline{\eta}\hat{\psi}) + (\widehat{\overline{\psi}}\eta)\}])\, |0, \text{in}\rangle \\
&\equiv \sum_{l,m,n=0}^{\infty} \frac{i^{l+m+n}}{l!m!n!} \langle 0, \text{out}|\, \mathrm{T}((\overline{\eta}\hat{\psi})^l (J\hat{\phi})^m (\widehat{\overline{\psi}}\eta)^n)\, |0, \text{in}\rangle \\
&\equiv \sum_{l,m,n=0}^{\infty} \frac{i^{l+m+n}}{l!m!n!} J(y_1)\cdots J(y_m)\overline{\eta}_{\alpha_l}(x_l)\cdots\overline{\eta}_{\alpha_1}(x_1) \\
&\quad \times \langle 0, \text{out}|\, \mathrm{T}\{\hat{\psi}_{\alpha_1}(x_1)\cdots\hat{\psi}_{\alpha_l}(x_l)\hat{\phi}(y_1)\cdot\cdot\hat{\phi}(y_m) \\
&\quad \times \widehat{\overline{\psi}}_{\beta_1}(z_1)\cdots\widehat{\overline{\psi}}_{\beta_n}(z_n)\}\, |0, \text{in}\rangle\ \eta_{\beta_n}(z_n)\cdots\eta_{\beta_1}(z_1) \\
&\equiv \sum_{n=0}^{\infty} \frac{i^n}{n!} \langle 0, \text{out}|\, \mathrm{T}(\overline{\eta}\hat{\psi} + J\hat{\phi} + \widehat{\overline{\psi}}\eta)^n\, |0, \text{in}\rangle\ .
\end{aligned} \tag{10.4.12}$$

Here the repeated continuous space–time indices x_1 through z_n are to be integrated over and we introduced the abbreviations in Eq. (10.4.12),

$$J\hat{\phi} \equiv \int d^4y J(y)\hat{\phi}(y), \quad \overline{\eta}\hat{\psi} \equiv \int d^4x \overline{\eta}(x)\hat{\psi}(x), \quad \widehat{\overline{\psi}}\eta \equiv \int d^4z \widehat{\overline{\psi}}(z)\eta(z).$$

The time-ordered product is defined by

$$\begin{aligned}
&\mathrm{T}\{\hat{\Psi}(x_1)\cdots\hat{\Psi}(x_n)\} \\
&\equiv \sum_{\substack{\text{all possible} \\ \text{permutations } P}} \delta_P\theta(x^0_{P1} - x^0_{P2})\cdots\theta(x^0_{P(n-1)} - x^0_{Pn})\hat{\Psi}(x_{P1})\cdots\hat{\Psi}(x_{Pn}),
\end{aligned}$$

with

$$\delta_P = \begin{cases} 1 & P \text{ even and odd} & \text{for Boson,} \\ 1 & P \text{ even} & \text{for Fermion,} \\ -1 & P \text{ odd} & \text{for Fermion.} \end{cases}$$

We observe that

$$\frac{\delta}{\delta\overline{\eta}_\beta(x)}(\overline{\eta}\hat{\psi})^l = l\hat{\psi}_\beta(x)(\overline{\eta}\hat{\psi})^{l-1}, \tag{10.4.13a}$$

$$\frac{\delta}{\delta\eta_\alpha(x)}(\widehat{\overline{\psi}}\eta)^n = -n\widehat{\overline{\psi}}_\alpha(x)(\widehat{\overline{\psi}}\eta)^{n-1}, \tag{10.4.13b}$$

$$\frac{\delta}{\delta J(x)}(J\hat{\phi})^m = m\hat{\phi}(x)(J\hat{\phi})^{m-1}. \tag{10.4.13c}$$

We also observe from the definition of $Z[J, \overline{\eta}, \eta]$,

$$\frac{1}{i}\frac{\delta}{\delta\overline{\eta}_\beta(x)} Z[J, \overline{\eta}, \eta] = \left\langle 0, \text{out}\middle| \mathrm{T}\left(\hat{\psi}_\beta(x) \exp\left[i \int d^4z \{ J(z)\hat{\phi}(z) + \overline{\eta}_\alpha(z)\hat{\psi}_\alpha(z) + \widehat{\overline{\psi}}_\beta(z)\eta_\beta(z)\}\right]\right)\middle| 0, \text{in}\right\rangle, \tag{10.4.14a}$$

$$i\frac{\delta}{\delta\eta_\alpha(x)} Z[J, \overline{\eta}, \eta] = \left\langle 0, \text{out}\middle| \mathrm{T}\left(\widehat{\overline{\psi}}_\alpha(x) \exp\left[i \int d^4z \{ J(z)\hat{\phi}(z) + \overline{\eta}_\alpha(z)\hat{\psi}_\alpha(z) + \widehat{\overline{\psi}}_\beta(z)\eta_\beta(z)\}\right]\right)\middle| 0, \text{in}\right\rangle, \tag{10.4.14b}$$

$$\frac{1}{i}\frac{\delta}{\delta J(x)} Z[J, \overline{\eta}, \eta] = \left\langle 0, \text{out}\middle| \mathrm{T}\left(\hat{\phi}(x) \exp\left[i \int d^4z \{ J(z)\hat{\phi}(z) + \overline{\eta}_\alpha(z)\hat{\psi}_\alpha(z) + \widehat{\overline{\psi}}_\beta(z)\eta_\beta(z)\}\right]\right)\middle| 0, \text{in}\right\rangle. \tag{10.4.14c}$$

From the definition of the time-ordered product and the equal time canonical (anti-)commutators, Eqs. (10.4.11a)–((10.4.11d), we have at the operator level,

Fermion:

$$\begin{aligned} &D_{\alpha\beta}(x)\mathrm{T}(\hat{\psi}_\beta(x)\exp[i\{J\hat{\phi} + \overline{\eta}_\alpha\hat{\psi}_\alpha + \widehat{\overline{\psi}}_\beta\eta_\beta\}]) \\ &\quad = \mathrm{T}(D_{\alpha\beta}(x)\hat{\psi}_\beta(x)\exp[i\{J\hat{\phi} + \overline{\eta}\hat{\psi} + \widehat{\overline{\psi}}\eta\}]) \\ &\quad - \eta_\alpha(x)\mathrm{T}(\exp[i\{J\hat{\phi} + \overline{\eta}\hat{\psi} + \widehat{\overline{\psi}}\eta\}]), \end{aligned} \tag{10.4.15}$$

Antifermion:

$$\begin{aligned} &-D^{\mathrm{T}}_{\beta\alpha}(-x)\mathrm{T}(\widehat{\overline{\psi}}_\alpha(x)\exp[i\{J\hat{\phi} + \overline{\eta}_\alpha\hat{\psi}_\alpha + \widehat{\overline{\psi}}_\beta\eta_\beta\}]) \\ &= \mathrm{T}(-D^{\mathrm{T}}_{\beta\alpha}(-x)\widehat{\overline{\psi}}_\alpha(x)\exp[i\{J\hat{\phi} + \overline{\eta}\hat{\psi} + \widehat{\overline{\psi}}\eta\}]) \\ &+ \overline{\eta}_\beta(x)\mathrm{T}(\exp[i\{J\hat{\phi} + \overline{\eta}\hat{\psi} + \widehat{\overline{\psi}}\eta\}]), \end{aligned} \tag{10.4.16}$$

Boson:

$$
\begin{aligned}
&K(x)\mathrm{T}(\hat{\phi}(x)\exp[i\{J\hat{\phi}+\overline{\eta}_\alpha\hat{\psi}_\alpha+\widehat{\overline{\psi}}_\beta\eta_\beta\}]) \\
&\quad = \mathrm{T}(K(x)\hat{\phi}(x)\exp[i\{J\hat{\phi}+\overline{\eta}\hat{\psi}+\widehat{\overline{\psi}}\eta\}]) \\
&\quad - J(x)\mathrm{T}(\exp[i\{J\hat{\phi}+\overline{\eta}\hat{\psi}+\widehat{\overline{\psi}}\eta\}]).
\end{aligned} \tag{10.4.17}
$$

Applying Euler–Lagrange equations of motion, Eqs. (10.4.10a through c), to the first terms of the right-hand sides of Eqs. (10.4.15), (10.4.16), and (10.4.17), and taking the vacuum expectation values , we obtain the equations of motion of the generating functional $Z[J,\overline{\eta},\eta]$ of the "full" Green's functions as

$$
\left\{ D_{\alpha\beta}(x)\frac{1}{i}\frac{\delta}{\delta\overline{\eta}_\beta(x)} + \frac{\delta I_{\mathrm{int}}\left[\frac{1}{i}\frac{\delta}{\delta J},\frac{1}{i}\frac{\delta}{\delta\overline{\eta}},i\frac{\delta}{\delta\eta}\right]}{\delta\left(i\frac{\delta}{\delta\eta_\alpha(x)}\right)} + \eta_\alpha(x)\right\} Z[J,\overline{\eta},\eta]=0, \tag{10.4.18a}
$$

$$
\left\{ -D^{\mathrm{T}}_{\beta\alpha}(-x)i\frac{\delta}{\delta\eta_\alpha(x)} + \frac{\delta I_{\mathrm{int}}\left[\frac{1}{i}\frac{\delta}{\delta J},\frac{1}{i}\frac{\delta}{\delta\overline{\eta}},i\frac{\delta}{\delta\eta}\right]}{\delta\left(\frac{1}{i}\frac{\delta}{\delta\overline{\eta}_\beta(x)}\right)} - \overline{\eta}_\beta(x)\right\} Z[J,\overline{\eta},\eta]=0, \tag{10.4.18b}
$$

$$
\left\{ K(x)\frac{1}{i}\frac{\delta}{\delta J(x)} + \frac{\delta I_{\mathrm{int}}\left[\frac{1}{i}\frac{\delta}{\delta J},\frac{1}{i}\frac{\delta}{\delta\overline{\eta}},i\frac{\delta}{\delta\eta}\right]}{\delta\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)} + J(x)\right\} Z[J,\overline{\eta},\eta]=0. \tag{10.4.18c}
$$

Equivalently, from Eqs. (10.4.10a), (10.4.10b), and (10.4.10c), we can write, respectively, Eqs. (10.4.18a), (10.4.18b), and (10.4.18c) as

$$
\left\{ \frac{\delta I_{\mathrm{tot}}\left[\frac{1}{i}\frac{\delta}{\delta J},\frac{1}{i}\frac{\delta}{\delta\overline{\eta}},i\frac{\delta}{\delta\eta}\right]}{\delta\left(i\frac{\delta}{\delta\eta_\alpha(x)}\right)} + \eta_\alpha(x)\right\} Z[J,\overline{\eta},\eta]=0, \tag{10.4.19a}
$$

$$
\left\{ \frac{\delta I_{\mathrm{tot}}\left[\frac{1}{i}\frac{\delta}{\delta J},\frac{1}{i}\frac{\delta}{\delta\overline{\eta}},i\frac{\delta}{\delta\eta}\right]}{\delta\left(\frac{1}{i}\frac{\delta}{\delta\overline{\eta}_\beta(x)}\right)} - \overline{\eta}_\beta(x)\right\} Z[J,\overline{\eta},\eta]=0, \tag{10.4.19b}
$$

$$
\left\{ \frac{\delta I_{\mathrm{tot}}\left[\frac{1}{i}\frac{\delta}{\delta J},\frac{1}{i}\frac{\delta}{\delta\overline{\eta}},i\frac{\delta}{\delta\eta}\right]}{\delta\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)} + J(x)\right\} Z[J,\overline{\eta},\eta]=0. \tag{10.4.19c}
$$

We note that the coefficients of the external hook terms, $\eta_\alpha(x)$, $\overline{\eta}_\beta(x)$, and $J(x)$, in Eqs. (10.4.19a)–(10.4.19c) are ± 1, which is the reflection of the fact that we are dealing with canonical quantum field theory and originates from the equal time canonical (anti-)commutators .

With this preparation, we shall discuss the Schwinger theory of Green's function with the interaction Lagrangian density $\mathcal{L}_{\mathrm{int}}(\hat{\phi}(x),\hat{\psi}(x),\widehat{\overline{\psi}}(x))$ of the Yukawa coupling in mind,

$$
\mathcal{L}_{\mathrm{int}}(\hat{\phi}(x),\hat{\psi}(x),\widehat{\overline{\psi}}(x)) = -G_0\widehat{\overline{\psi}}_\alpha(x)\gamma_{\alpha\beta}(x)\hat{\psi}_\beta(x)\hat{\phi}(x), \tag{10.4.20}
$$

with

$$G_0 = \begin{cases} g_0 \\ f \\ e \end{cases}, \gamma(x) = \begin{cases} i\gamma_5 \\ i\gamma_5\tau_k \\ \gamma_\mu \end{cases}, \hat{\psi}(x) = \begin{cases} \hat{\psi}_\alpha(x) \\ \hat{\psi}_{\mathrm{N},\alpha}(x) \\ \hat{\psi}_\alpha(x) \end{cases},$$

$$\phi(x) = \begin{cases} \hat{\phi}(x) \\ \hat{\phi}_k(x) \\ \hat{A}_\mu(x) \end{cases}. \tag{10.4.21}$$

We define the *vacuum expectation values* , $\langle F\rangle^{J,\overline{\eta},\eta}$ *and* $\langle F\rangle^J$, *of the operator function* $F(\hat{\phi}(x), \hat{\psi}(x), \hat{\overline{\psi}}(x))$ *in the presence of the external hook terms* $\{J, \overline{\eta}, \eta\}$ by

$$\langle F\rangle^{J,\overline{\eta},\eta} \equiv \frac{1}{Z[J,\overline{\eta},\eta]} F\left(\frac{1}{i}\frac{\delta}{\delta J(x)}, \frac{1}{i}\frac{\delta}{\delta\overline{\eta}(x)}, i\frac{\delta}{\delta\eta(x)}\right) Z[J,\overline{\eta},\eta], \tag{10.4.22a}$$

$$\langle F\rangle^J = \langle F\rangle^{J,\overline{\eta},\eta}\Big|_{\overline{\eta}=\eta=0}. \tag{10.4.22b}$$

We define the connected parts of the two-point "full" Green's functions in the presence of the external hook $J(x)$ by **Fermion**:

$$\begin{aligned} S'^J_{\mathrm{F},\alpha\beta}(x_1,x_2) &\equiv \frac{1}{i}\frac{\delta}{\delta\overline{\eta}_\alpha(x_1)} i\frac{\delta}{\delta\eta_\beta(x_2)} \frac{1}{i}\ln Z[J,\overline{\eta},\eta]\Big|_{\overline{\eta}=\eta=0} \\ &= \frac{1}{i}\left(\frac{1}{i}\frac{\delta}{\delta\overline{\eta}_\alpha(x_1)}\right)\langle\hat{\overline{\psi}}_\beta(x_2)\rangle^{J,\overline{\eta},\eta}\Big|_{\overline{\eta}=\eta=0} \\ &= \frac{1}{i}\Big\{\langle\hat{\psi}_\alpha(x_1)\hat{\overline{\psi}}_\beta(x_2)\rangle^{J,\overline{\eta},\eta}\Big|_{\overline{\eta}=\eta=0} \\ &\quad - \langle\hat{\psi}_\alpha(x_1)\rangle^{J,\overline{\eta},\eta}\langle\hat{\overline{\psi}}_\beta(x_2)\rangle^{J,\overline{\eta},\eta}\Big|_{\overline{\eta}=\eta=0}\Big\} \\ &= \frac{1}{i}\langle\hat{\psi}_\alpha(x_1)\hat{\overline{\psi}}_\beta(x_2)\rangle^J \equiv \frac{1}{i}\ \langle 0,\mathrm{out}|\,\mathrm{T}(\hat{\psi}_\alpha(x_1)\hat{\overline{\psi}}_\beta(x_2))\,|0,\mathrm{in}\rangle^J_{\mathrm{C}}, \end{aligned} \tag{10.4.23}$$

$$\langle\hat{\psi}_\alpha(x_1)\rangle^{J,\overline{\eta},\eta}\Big|_{\overline{\eta}=\eta=0} = \langle\hat{\overline{\psi}}_\beta(x_2)\rangle^{J,\overline{\eta},\eta}\Big|_{\overline{\eta}=\eta=0} = 0, \tag{10.4.24}$$

and

Boson:

$$\begin{aligned} D'_{\mathrm{F}}J(x_1,x_2) &\equiv \frac{1}{i}\frac{\delta}{\delta J(x_1)}\frac{1}{i}\frac{\delta}{\delta J(x_2)}\ \frac{1}{i}\ln Z[J,\overline{\eta},\eta]\Big|_{\overline{\eta}=\eta=0} \\ &= \frac{1}{i}\left(\frac{1}{i}\frac{\delta}{\delta J(x_1)}\langle\hat{\phi}(x_2)\rangle^J\right) = \frac{1}{i}\{\langle\hat{\phi}(x_1)\hat{\phi}(x_2)\rangle^J - \langle\hat{\phi}(x_1)\rangle^J\langle\hat{\phi}(x_2)\rangle^J\} \\ &\equiv \frac{1}{i}\left\langle 0,\mathrm{out}\middle|\,\mathrm{T}(\hat{\phi}(x_1)\hat{\phi}(x_2))\,\middle|0,\mathrm{in}\right\rangle^J_{\mathrm{C}}, \end{aligned} \tag{10.4.25}$$

$$\langle\hat{\phi}(x)\rangle^J\Big|_{J=0} = 0. \tag{10.4.26}$$

We have the equations of motion of $Z[J, \overline{\eta}, \eta]$, Eqs. (10.4.18a)–(10.4.18c), when the interaction Lagrangian density $\mathcal{L}_{\mathrm{int}}(\hat{\phi}(x), \hat{\psi}(x), \hat{\overline{\psi}}(x))$ is given by Eq. (10.4.20) as

$$\left\{ D_{\alpha\beta}(x)\left(\frac{1}{i}\frac{\delta}{\delta\overline{\eta}_\beta(x)}\right) - G_0\gamma_{\alpha\beta}(x)\left(\frac{1}{i}\frac{\delta}{\delta\overline{\eta}_\beta(x)}\right)\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\right\} Z[J, \overline{\eta}, \eta]$$
$$= -\eta_\alpha(x) Z[J, \overline{\eta}, \eta], \tag{10.4.27a}$$
$$\left\{ -D^{\mathrm{T}}_{\beta\alpha}(-x)\left(i\frac{\delta}{\delta\eta_\alpha(x)}\right) + G_0\left(i\frac{\delta}{\delta\eta_\alpha(x)}\right)\gamma_{\alpha\beta}(x)\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\right\} Z[J, \overline{\eta}, \eta]$$
$$= +\overline{\eta}_\beta(x) Z[J, \overline{\eta}, \eta], \tag{10.4.27b}$$
$$\left\{ K(x)\left(\frac{1}{i}\frac{\delta}{\delta JJ(x)}\right) - G_0\left(i\frac{\delta}{\delta\eta_\alpha(x)}\right)\gamma_{\alpha\beta}(x)\left(\frac{1}{i}\frac{\delta}{\delta\overline{\eta}_\beta(x)}\right)\right\} Z[J, \overline{\eta}, \eta]$$
$$= -J(x) Z[J, \overline{\eta}, \eta]. \tag{10.4.27c}$$

Dividing Eqs. (10.4.27a)–(10.4.27c) by $Z[J, \overline{\eta}, \eta]$, and referring to Eqs. (10.4.22a) and (10.4.22b), we obtain the equations of motion for

$$\langle\hat{\psi}_\beta(x)\rangle^{J,\overline{\eta},\eta}, \ \langle\hat{\overline{\psi}}_\alpha(x)\rangle^{J,\overline{\eta},\eta} \text{ and } \langle\hat{\phi}(x)\rangle^{J,\overline{\eta},\eta},$$

as

$$D_{\alpha\beta}(x)\langle\hat{\psi}_\beta(x)\rangle^{J,\overline{\eta},\eta} - G_0\gamma_{\alpha\beta}(x)\langle\hat{\psi}_\beta(x)\hat{\phi}(x)\rangle^{J,\overline{\eta},\eta} = -\eta_\alpha(x), \tag{10.4.28}$$
$$-D^{\mathrm{T}}_{\beta\alpha}(-x)\langle\hat{\overline{\psi}}_\alpha(x)\rangle + G_0\gamma_{\alpha\beta}(x)\langle\hat{\overline{\psi}}_\alpha(x)\hat{\phi}(x)\rangle^{J,\overline{\eta},\eta} = +\overline{\eta}_\beta(x), \tag{10.4.29}$$
$$K(x)\langle\hat{\phi}(x)\rangle^{J,\overline{\eta},\eta} - G_0\gamma_{\alpha\beta}(x)\langle\hat{\overline{\psi}}_\alpha(x)\hat{\psi}_\beta(x)\rangle^{J,\overline{\eta},\eta} = -J(x). \tag{10.4.30}$$

We take the following functional derivatives:
$i\frac{\delta}{\delta\eta_\varepsilon(y)}\text{Eq. (10.4.28)}\Big|_{\overline{\eta}=\eta=0}$:

$$D_{\alpha\beta}(x)\langle\hat{\overline{\psi}}_\varepsilon(y)\hat{\psi}_\beta(x)\rangle^J$$
$$-G_0\gamma_{\alpha\beta}(x)\langle\hat{\overline{\psi}}_\varepsilon(y)\hat{\psi}_\beta(x)\hat{\phi}(x)\rangle^J = -i\delta_{\alpha\varepsilon}\delta^4(x-y),$$

$\frac{1}{i}\frac{\delta}{\delta\overline{\eta}_\varepsilon(y)}\text{Eq. (10.4.29)}\Big|_{\overline{\eta}=\eta=0}$:

$$-D^{\mathrm{T}}_{\beta\alpha}(-x)\langle\hat{\psi}_\varepsilon(y)\hat{\overline{\psi}}_\alpha(x)\rangle^J$$
$$+G_0\gamma_{\alpha\beta}(x)\langle\hat{\psi}_\varepsilon(y)\hat{\overline{\psi}}_\alpha(x)\hat{\phi}(x)\rangle^J = -i\delta_{\beta\varepsilon}\delta^4(x-y),$$

$\frac{1}{i}\frac{\delta}{\delta J(y)}\text{Eq. (10.4.30)}\Big|_{\overline{\eta}=\eta=0}$:

$$K(x)\{\langle\hat{\phi}(y)\hat{\phi}(x)\rangle^J - \langle\phi(y)\rangle^J\langle\phi(x)\rangle^J\}$$
$$-G_0\gamma_{\alpha\beta}(x)\frac{1}{i}\frac{\delta}{\delta J(y)}\langle\hat{\overline{\psi}}_\alpha(x)\hat{\psi}_\beta(x)\rangle^J = i\delta^4(x-y).$$

These equations are a part of the infinite system of coupled equations. We observe the following identities:

$$\langle \widehat{\overline{\psi}}_{\varepsilon}(y)\hat{\psi}_{\beta}(x)\rangle^{J} = -iS'^{J}_{\mathrm{F},\beta\varepsilon}(x,y),$$
$$\langle \hat{\psi}_{\varepsilon}(y)\widehat{\overline{\psi}}_{\alpha}(x)\rangle^{J} = iS'^{J}_{\mathrm{F},\varepsilon\alpha}(y,x),$$
$$\langle \widehat{\overline{\psi}}_{\varepsilon}(y)\hat{\psi}_{\beta}(x)\hat{\phi}(x)\rangle^{J} = -i\left(\langle\hat{\phi}(x)\rangle^{J} + \frac{1}{i}\frac{\delta}{\delta J(x)}\right)S'^{J}_{\mathrm{F},\beta\varepsilon}(x,y),$$
$$\langle \hat{\psi}_{\varepsilon}(y)\widehat{\overline{\psi}}_{\alpha}(x)\hat{\phi}(x)\rangle^{J} = i\left(\langle\hat{\phi}(x)\rangle^{J} + \frac{1}{i}\frac{\delta}{\delta J(x)}\right)S'^{J}_{\mathrm{F},\varepsilon\alpha}(y,x).$$

With these identities, we obtain the equations of motion of the connected parts of the two-point "full" Green's functions in the presence of the external hook $J(x)$,

$$\left\{D_{\alpha\beta}(x) - G_0\gamma_{\alpha\beta}(x)\left(\langle\hat{\phi}(x)\rangle^{J} + \frac{1}{i}\frac{\delta}{\delta J(x)}\right)\right\}S'^{J}_{\mathrm{F},\beta\varepsilon}(x,y) = \delta_{\alpha\varepsilon}\delta^{4}(x-y), \tag{10.4.31a}$$

$$\left\{D^{\mathrm{T}}_{\beta\alpha}(-x) - G_0\gamma_{\alpha\beta}(x)\left(\langle\hat{\phi}(x)\rangle^{J} + \frac{1}{i}\frac{\delta}{\delta J(x)}\right)\right\}S'^{J}_{\mathrm{F},\varepsilon\alpha}(y,x) = \delta_{\beta\varepsilon}\delta^{4}(x-y), \tag{10.4.32a}$$

$$K(x)D'^{J}_{\mathrm{F}}(x,y) + G_0\gamma_{\alpha\beta}(x)\frac{1}{i}\frac{\delta}{\delta J(y)}S'^{J}_{\mathrm{F},\beta\alpha}(x,x_{\pm}) = \delta^{4}(x-y). \tag{10.4.33a}$$

Since the transpose of Eq. (10.4.32a) is Eq. (10.4.31a), we have to consider only Eqs. (10.4.31a) and (10.4.33a). We may get the impression that we have the equations of motion of the two-point "full" Green's functions, $S'^{J}_{\mathrm{F},\alpha\beta}(x,y)$ and $D'^{J}_{\mathrm{F}}(x,y)$, in closed form at first sight. Because of the presence of the functional derivatives $\delta/i\delta J(x)$, and $\delta/i\delta J(y)$, however, Eqs. (10.4.31a), (10.4.32a) and (10.4.33a) involve the three-point "full" Green's functions and are merely a part of the infinite system of the coupled nonlinear equations of motion of the "full" Green's functions.

From this point onward, we use the variables, "1", "2", "3", ..., to represent the continuous space–time indices, $x, y, z, \ldots$, the spinor indices, $\alpha, \beta, \gamma, \ldots$, as well as other internal indices, $i, j, k, \ldots$,.

With the use of the "free" Green's functions, $S^{\mathrm{F}}_{0}(1-2)$ and $D^{\mathrm{F}}_{0}(1-2)$, defined by

$$D(1)S^{\mathrm{F}}_{0}(1-2) = 1, \tag{10.4.34a}$$
$$K(1)D^{\mathrm{F}}_{0}(1-2) = 1, \tag{10.4.34b}$$

we rewrite the functional differential equations satisfied by the "full" Green's functions, $S'^{J}_{\mathrm{F}}(1,2)$ and $D'^{J}_{\mathrm{F}}(1,2)$, Eqs. (10.4.31a) and (10.4.33a), into the integral equations,

$$S'^{J}_{\mathrm{F}}(1,2) = S^{\mathrm{F}}_{0}(1-2) + S^{\mathrm{F}}_{0}(1-3)(G_0\gamma(3))\left(\langle\hat{\phi}(3)\rangle^{J} + \frac{1}{i}\frac{\delta}{\delta J(3)}\right)S'^{J}_{\mathrm{F}}(3,2), \tag{10.4.31b}$$

$$D_{\mathrm{F}}^{\prime J}(1,2)=D_0^{\mathrm{F}}(1-2)+D_0^{\mathrm{F}}(1-3)\left(-G_0\mathrm{tr}\gamma(3)\frac{1}{i}\frac{\delta}{\delta J(2)}S_{\mathrm{F}}^{\prime J}(3,3_\pm)\right). \quad (10.4.33\mathrm{b})$$

We compare Eqs. (10.4.31b) and (10.4.33b) with the defining integral equations of the *proper self-energy parts* , $\mathbf{\Sigma}^*$ and $\mathbf{\Pi}^*$, due to Dyson, in the presence of the external hook $J(x)$,

$$\begin{aligned}S_{\mathrm{F}}^{\prime J}(1,2)&=S_0^{\mathrm{F}}(1-2)+S_0^{\mathrm{F}}(1-3)(G_0\gamma(3)\langle\phi(3)\rangle^J)S_{\mathrm{F}}^{\prime J}(3,2)\\&\quad+S_0^{\mathrm{F}}(1-3)\mathbf{\Sigma}^*(3,4)S_{\mathrm{F}}^{\prime J}(4,2),\end{aligned} \quad (10.4.35)$$

$$D_{\mathrm{F}}^{\prime J}(1,2)=D_0^{\mathrm{F}}(1-2)+D_0^{\mathrm{F}}(1-3)\mathbf{\Pi}^*(3,4)D_{\mathrm{F}}^{\prime J}(4,2), \quad (10.4.36)$$

obtaining

$$G_0\gamma(1)\frac{1}{i}\frac{\delta}{\delta J(1)}S_{\mathrm{F}}^{\prime J}(1,2)=\mathbf{\Sigma}^*(1,3)S_{\mathrm{F}}^{\prime J}(3,2)\equiv\mathbf{\Sigma}^*(1)S_{\mathrm{F}}^{\prime J}(1,2), \quad (10.4.37)$$

$$-G_0\mathrm{tr}\gamma(1)\frac{1}{i}\frac{\delta}{\delta J(2)}S_{\mathrm{F}}^{\prime J}(1,1_\pm)=\mathbf{\Pi}^*(1,3)D_{\mathrm{F}}^{\prime J}(3,2)\equiv\mathbf{\Pi}^*(1)D_{\mathrm{F}}^{\prime J}(1,2). \quad (10.4.38)$$

Thus we can write the functional differential equations, Eqs. (10.4.31a) and (10.4.33a), compactly as

$$\{D(1)-G_0\gamma(1)\langle\hat{\phi}(1)\rangle^J-\mathbf{\Sigma}^*(1)\}S_{\mathrm{F}}^{\prime J}(1,2)=\delta(1-2), \quad (10.4.39)$$

$$\{K(1)-\mathbf{\Pi}^*(1)\}D_{\mathrm{F}}^{\prime J}(1,2)=\delta(1-2). \quad (10.4.40)$$

Defining the *nucleon differential operator* and the *meson differential operator* by

$$D_{\mathrm{N}}(1,2)\equiv\{D(1)-G_0\gamma(1)\langle\hat{\phi}(1)\rangle^J\}\delta(1-2)-\mathbf{\Sigma}^*(1,2), \quad (10.4.41)$$

and

$$D_{\mathrm{M}}(1,2)\equiv K(1)\delta(1-2)-\mathbf{\Pi}^*(1,2), \quad (10.4.42)$$

we can write the differential equations, Eqs. (10.4.39) and (10.4.40), as

$$D_{\mathrm{N}}(1,3)S_{\mathrm{F}}^{\prime J}(3,2)=\delta(1-2),\quad\text{or}\quad D_{\mathrm{N}}(1,2)=(S_{\mathrm{F}}^{\prime J}(1,2))^{-1}, \quad (10.4.43)$$

and

$$D_{\mathrm{M}}(1,3)D_{\mathrm{F}}^{\prime J}(3,2)=\delta(1-2),\quad\text{or}\quad D_{\mathrm{M}}(1,2)=(D_{\mathrm{F}}^{\prime J}(1,2))^{-1}. \quad (10.4.44)$$

Next, we take the functional derivative of Eq. (10.4.39), $\frac{1}{i}\frac{\delta}{\delta J(3)}$Eq. (10.4.39):

$$\begin{aligned}&\{D(1)-G_0\gamma(1)\langle\hat{\phi}(1)\rangle^J-\mathbf{\Sigma}^*(1)\}\frac{1}{i}\frac{\delta}{\delta J(3)}S_{\mathrm{F}}^{\prime J}(1,2)\\&\quad=\left\{iG_0\gamma(1)D_{\mathrm{F}}^{\prime J}(1,3)+\frac{1}{i}\frac{\delta}{\delta J(3)}\mathbf{\Sigma}^*(1)\right\}S_{\mathrm{F}}^{\prime J}(1,2).\end{aligned} \quad (10.4.45)$$

Solving Eq. (10.4.45) for $\delta S_F'^J(1,2)/i\delta J(3)$ and with the use of Eqs. (10.4.39), (10.4.41), and (10.4.43), we obtain

$$\begin{aligned}\frac{1}{i}\frac{\delta}{\delta J(3)}S_F'^J(1,2) &= S_F'^J(1,4)\left\{iG_0\gamma(4)D_F'^J(4,3)+\frac{1}{i}\frac{\delta}{\delta J(3)}\boldsymbol{\Sigma}^*(4)\right\}S_F'^J(4,2)\\ &= iG_0S_F'^J(1,4)\Bigg\{\gamma(4)\delta(4-5)\delta(4-6)\\ &\quad +\frac{1}{G_0}\frac{\delta}{\delta\langle\hat{\phi}(6)\rangle^J}\boldsymbol{\Sigma}^*(4,5)\Bigg\}S_F'^J(5,2)D_F'^J(6,3). \end{aligned} \tag{10.4.46}$$

Comparing Eq. (10.4.46) with the definition of the *vertex operator* $\boldsymbol{\Gamma}(4,5;6)$ of Dyson,

$$\frac{1}{i}\frac{\delta}{\delta J(3)}S_F'^J(1,2) \equiv iG_0S_F'^J(1,4)\boldsymbol{\Gamma}(4,5;6)S_F'^J(5,2)D_F'^J(6,3), \tag{10.4.47}$$

we obtain

$$\boldsymbol{\Gamma}(1,2;3) = \gamma(1)\delta(1-2)\delta(1-3)+\frac{1}{G_0}\frac{\delta}{\delta\langle\hat{\phi}(3)\rangle^J}\boldsymbol{\Sigma}^*(1,2), \tag{10.4.48}$$

while we can write the left-hand side of Eq. (10.4.47) as

$$\frac{1}{i}\frac{\delta}{\delta J(3)}S_F'^J(1,2) = iD_F'^J(6,3)\frac{\delta}{\delta\langle\hat{\phi}(6)\rangle^J}S_F'^J(1,2). \tag{10.4.49}$$

From this, we have

$$-\frac{1}{G_0}\frac{\delta}{\delta\langle\hat{\phi}(6)\rangle^J}S_F'^J(1,2) = -S_F'^J(1,4)\boldsymbol{\Gamma}(4,5;6)S_F'^J(5,2),$$

and we obtain the compact representation of $\boldsymbol{\Gamma}(1,2;3)$,

$$\begin{aligned}\boldsymbol{\Gamma}(1,2;3) &= -\frac{1}{G_0}\frac{\delta}{\delta\langle\hat{\phi}(3)\rangle^J}(S_F'^J(1,2))^{-1} = -\frac{1}{G_0}\frac{\delta}{\delta\langle\hat{\phi}(3)\rangle^J}D_N(1,2)\\ &= (10.4.48).\end{aligned} \tag{10.4.50}$$

Lastly, from Eqs. (10.4.38) and (10.4.39), which define $\boldsymbol{\Sigma}^*(1,2)$ and $\boldsymbol{\Pi}^*(1,2)$ indirectly and the defining equation of $\boldsymbol{\Gamma}(1,2;3)$, Eq. (10.4.47), we have

$$\boldsymbol{\Sigma}^*(1,3)S_F'^J(3,2) = -iG_0^2\gamma(1)S_F'^J(1,4)\boldsymbol{\Gamma}(4,5;6)S_F'^J(5,2)D_F'^J(6,1), \tag{10.4.51}$$

$$\boldsymbol{\Pi}^*(1,3)D_F'^J(3,2) = iG_0^2\mathrm{tr}\gamma(1)S_F'^J(1,4)\boldsymbol{\Gamma}(4,5;6)S_F'^J(5,1)D_F'^J(6,2). \tag{10.4.52}$$

Namely, we obtain

$$\boldsymbol{\Sigma}^*(1,2) = -iG_0^2\gamma(1)S_F'^J(1,3)\boldsymbol{\Gamma}(3,2;4)D_F'^J(4,1), \tag{10.4.53}$$

$$\boldsymbol{\Pi}^*(1,2) = iG_0^2\mathrm{tr}\gamma(1)S_F'^J(1,3)\boldsymbol{\Gamma}(3,4;2)S_F'^J(4,1). \tag{10.4.54}$$

Equation (10.4.30) can be expressed after setting $\eta = \overline{\eta} = 0$ as

$$K(1)\langle\hat{\phi}(1)\rangle^J + iG_0\mathrm{tr}(\gamma(1)S_{\mathrm{F}}^{\prime J}(1,1)) = -J(1). \tag{10.4.55}$$

System of equations, (10.4.41), (10.4.42), (10.4.43), (10.4.44), (10.4.48), (10.4.53), (10.4.54), and (10.4.55), is called the Schwinger–Dyson equation. This system of the nonlinear coupled integro-differential equations is exact and closed. Starting from the 0th-order term of $\mathbf{\Gamma}(1, 2; 3)$, we can develop the covariant perturbation theory by iteration. In the first-order approximation, after setting $J = 0$, we have the following expressions:

$$\mathbf{\Sigma}^*(1-2) \cong -iG_0^2\gamma(1)S_0^{\mathrm{F}}(1-2)\gamma(2)D_0^{\mathrm{F}}(2-1), \tag{10.4.56}$$

$$\mathbf{\Pi}^*(1-2) \cong iG_0^2\mathrm{tr}\{\gamma(1)S_0^{\mathrm{F}}(1-2)\gamma(2)S_0^{\mathrm{F}}(2-1)\}, \tag{10.4.57}$$

and

$$\mathbf{\Gamma}(1,2;3) \cong \gamma(1)\delta(1-2)\delta(1-3) - iG_0^2\gamma(1)S_0^{\mathrm{F}}(1-3)\gamma(3) \times S_0^{\mathrm{F}}(3-2)\gamma(2)D_0^{\mathrm{F}}(2-1). \tag{10.4.58}$$

We point out that the covariant perturbation theory based on the Schwinger–Dyson equation is somewhat different in spirit from the standard covariant perturbation theory due to Feynman and Dyson. The former is capable of dealing with the bound-state problem in general as will be shown shortly. Its power is demonstrated in the positronium problem.

Summary of Schwinger–Dyson Equation

$$D_{\mathrm{N}}(1,3)S_{\mathrm{F}}^{\prime J}(3,2) = \delta(1-2), \quad D_{\mathrm{M}}(1,3)D_{\mathrm{F}}^{\prime J}(3,2) = \delta(1-2),$$

$$D_{\mathrm{N}}(1,2) \equiv \{D(1) - G_0\gamma(1)\langle\hat{\phi}(1)\rangle^J\}\delta(1-2) - \mathbf{\Sigma}^*(1,2),$$

$$D_{\mathrm{M}}(1,2) \equiv K(1)\delta(1-2) - \mathbf{\Pi}^*(1,2),$$

$$K(1)\langle\hat{\phi}(1)\rangle^J + iG_0\mathrm{tr}(\gamma(1)S_{\mathrm{F}}^{\prime J}(1,1)) = -J(1),$$

$$\mathbf{\Sigma}^*(1,2) \equiv -iG_0^2\gamma(1)S_{\mathrm{F}}^{\prime J}(1,3)\mathbf{\Gamma}(3,2;4)D_{\mathrm{F}}^{\prime J}(4,1),$$

$$\mathbf{\Pi}^*(1,2) \equiv iG_0^2\mathrm{tr}\{\gamma(1)S_{\mathrm{F}}^{\prime J}(1,3)\mathbf{\Gamma}(3,4;2)S_{\mathrm{F}}^{\prime J}(4,1)\},$$

$$\mathbf{\Gamma}(1,2;3) \equiv -\frac{1}{G_0}\frac{\delta}{\delta\langle\hat{\phi}(3)\rangle^J}(S_{\mathrm{F}}^{\prime J}(1,2))^{-1}$$

$$= \gamma(1)\delta(1-2)\delta(1-3) + \frac{1}{G_0}\frac{\delta}{\delta\langle\hat{\phi}(3)\rangle^J}\mathbf{\Sigma}^*(1,2).$$

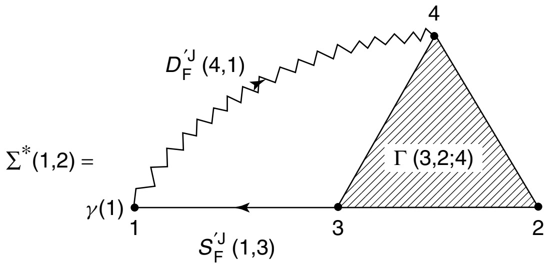

Fig. 10.1 Graphical representation of the proper self-energy part, $\boldsymbol{\Sigma}^*(1, 2)$.

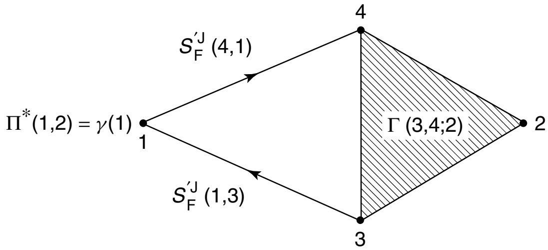

Fig. 10.2 Graphical representation of the proper self-energy part, $\boldsymbol{\Pi}^*(1, 2)$.

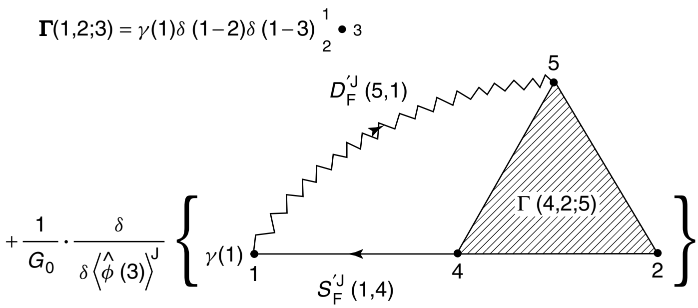

Fig. 10.3 Graphical representation of the vertex operator, $\boldsymbol{\Gamma}(1, 2; 3)$.

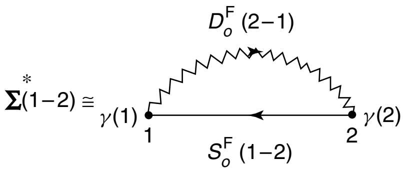

Fig. 10.4 The first-order approximation to the proper self-energy part, $\boldsymbol{\Sigma}^*(1, 2)$.

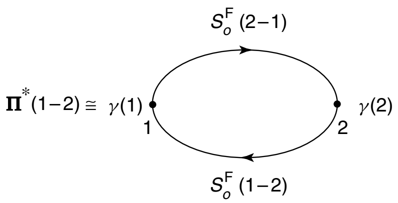

Fig. 10.5 The first-order approximation to the proper self-energy part, $\mathbf{\Pi}^*(1,2)$.

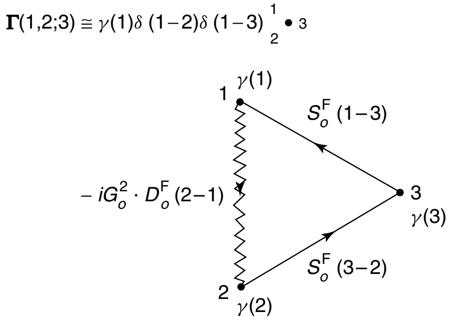

Fig. 10.6 The first-order approximation to the vertex operator, $\mathbf{\Gamma}(1,2;3)$.

We consider the two-body (four-point) nucleon "full" Green's function $S_{\mathrm{F}}^{\prime J}(1,2;3,4)$ with the Yukawa coupling, Eqs. (10.4.20) and (10.4.21), in mind, defined by

$$
\begin{aligned}
S_{\mathrm{F}}^{\prime J}(1,2;3,4) &\equiv \frac{1}{i}\frac{\delta}{\delta\overline{\eta}(1)}\frac{1}{i}\frac{\delta}{\delta\overline{\eta}(2)}i\frac{\delta}{\delta\eta(4)}i\frac{\delta}{\delta\eta(3)}\;\frac{1}{i}\ln Z[J,\overline{\eta},\eta]\bigg|_{\overline{\eta}=\eta=0} \\
&= \left(\frac{1}{i}\right)^2\{\langle\hat{\psi}(1)\hat{\psi}(2)\hat{\overline{\psi}}(3)\hat{\overline{\psi}}(4)\rangle^J - \langle\hat{\psi}(1)\hat{\psi}(2)\rangle^J\langle\hat{\overline{\psi}}(3)\hat{\overline{\psi}}(4)\rangle^J \\
&\quad + \langle\hat{\psi}(1)\hat{\overline{\psi}}(3)\rangle^J\langle\hat{\psi}(2)\hat{\overline{\psi}}(4)\rangle^J - \langle\hat{\psi}(1)\hat{\overline{\psi}}(4)\rangle^J\langle\hat{\psi}(2)\hat{\overline{\psi}}(3)\rangle^J\} \\
&\equiv \left(\frac{1}{i}\right)^2\left\langle 0,\text{out}\left|\mathrm{T}(\hat{\psi}(1)\hat{\psi}(2)\hat{\overline{\psi}}(3)\hat{\overline{\psi}}(4))\right|0,\text{in}\right\rangle_{\mathrm{C}}^{J}. \qquad (10.4.59)
\end{aligned}
$$

Operating the differential operators, $D_{\mathrm{N}}(1,5)$ and $D_{\mathrm{N}}(2,6)$, on $S_{\mathrm{F}}^{\prime J}(5,6;3,4)$ and using the Schwinger–Dyson equation derived above, we obtain

$$
\begin{aligned}
&\{D_{\mathrm{N}}(1,5)D_{\mathrm{N}}(2,6) - I(1,2;5,6)\}S_{\mathrm{F}}^{\prime J}(5,6;3,4) \\
&\quad = \delta(1-3)\delta(2-4) - \delta(1-4)\delta(2-3), \qquad (10.4.60)
\end{aligned}
$$

where the operator $I(1,2;3,4)$ is called the proper interaction kernel and satisfies the following integral equations:

$$
\begin{aligned}
&I(1,2;5,6)S_{\mathrm{F}}^{\prime J}(5,6;3,4) \\
&\quad = ig_0^2\mathrm{tr}^{(\mathrm{M})}[\gamma(1)\Gamma(2)D_{\mathrm{F}}^{\prime J}(5,6)]S_{\mathrm{F}}^{\prime J}(5,6;3,4) \\
&\qquad + ig_0^2\mathrm{tr}^{(\mathrm{M})}\left[\gamma(1)S_{\mathrm{F}}^{\prime J}(1,5)\frac{1}{i}\frac{\delta}{\delta J(5)}\right]I(5,2;6,7)S_{\mathrm{F}}^{\prime J}(6,7;3,4) \qquad (10.4.61)
\end{aligned}
$$

$$= ig_0^2 \mathrm{tr}^{(M)}[\gamma(2)\Gamma(1)D_F'^J(5,6)]S_F'^J(5,6;3,4)$$
$$+ ig_0^2 \mathrm{tr}^{(M)}\left[\gamma(2)S_F'^J(2,5)\frac{1}{i}\frac{\delta}{\delta J(5)}\right] I(1,5;6,7)S_F'^J(6,7;3,4). \tag{10.4.62}$$

Here $\mathrm{tr}^{(M)}$ indicates that the trace should be taken only over the meson coordinate. Equation (10.4.60) can be cast into the integral equation after a little algebra as

$$S_F'^J(1,2;3,4) = S_F'^J(1,3)S_F'^J(2,4) - S_F'^J(1,4)S_F'^J(2,3)$$
$$+ S_F'^J(1,5)S_F'^J(2,6)I(5,6;7,8)S_F'^J(7,8;3,4). \tag{10.4.63}$$

System of equations, (10.4.60) (or (10.4.63)) and (10.4.61) (or (10.4.62)), is called the Bethe–Salpeter equation . If we set $t_1 = t_2 = t > t_3 = t_4 = t'$ in Eq. (10.4.59), $S_F'^J(1,2;3,4)$ represents the transition probability amplitude whereby the nucleons originally located at $\vec{x}_3$ and $\vec{x}_4$ at time t' are to be found at $\vec{x}_1$ and $\vec{x}_2$ at the later time t. In the integral equations for $I(1,2;3,4)$, Eqs. (10.4.61) and (10.4.62), the first terms of the right-hand sides represent the scattering state and the second terms represent the bound state. The bound state problem is formulated by dropping the first terms of the right-hand sides of Eqs. (10.4.61) and (10.4.62). The proper interaction kernel

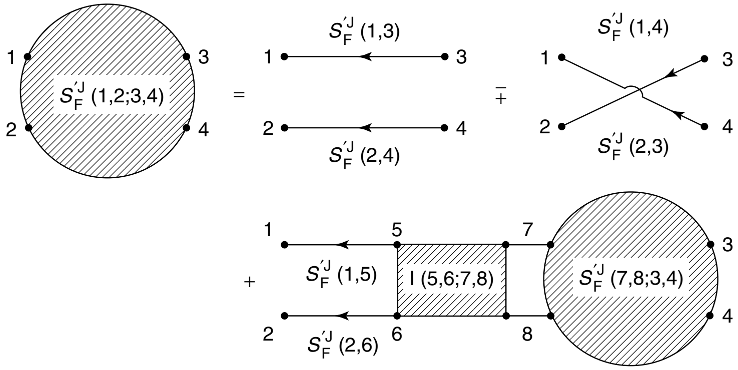

Fig. 10.7 Graphical representation of the Bethe–Salpeter equation.

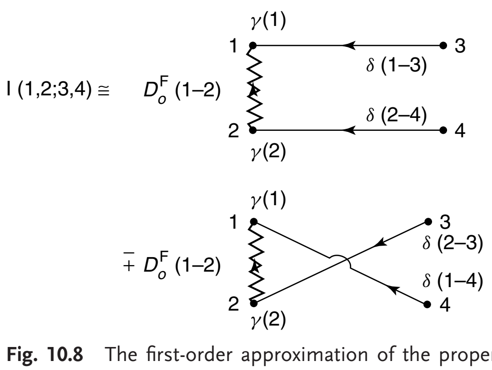

Fig. 10.8 The first-order approximation of the proper interaction kernel, $I(1,2;3,4)$.

$I(1,2;3,4)$ assumes the following form in the first-order approximation:

$$I(1,2;3,4) \cong ig_0^2\gamma(1)\gamma(2)D_0^{\mathrm{F}}(1-2)(\delta(1-3)\delta(2-4)-\delta(1-4)\delta(2-3)). \tag{10.4.64}$$

10.5 Schwinger–Dyson Equation in Quantum Statistical Mechanics

We consider the grand canonical ensemble of the Fermion (mass m) and the Boson (mass κ) with Euclidean Lagrangian density in contact with the particle source μ,

$$\begin{aligned}
&\mathcal{L}'_{\mathrm{E}}(\psi_{\mathrm{E}\alpha}(\tau,\vec{x}),\partial_\mu\psi_{\mathrm{E}\alpha}(\tau,\vec{x}),\phi(\tau,\vec{x}),\partial_\mu\phi(\tau,\vec{x}))\\
&=\overline{\psi}_{\mathrm{E}\,\alpha}(\tau,\vec{x})\left\{i\gamma^k\partial_k+i\gamma^4\left(\frac{\partial}{\partial\tau}-\mu\right)-m\right\}_{\alpha,\beta}\psi_{\mathrm{E}\,\beta}(\tau,\vec{x})\\
&\quad+\frac{1}{2}\phi(\tau,\vec{x})\left(\frac{\partial^2}{\partial x_\nu^2}-\kappa^2\right)\phi(\tau,\vec{x})+\frac{1}{2}g\mathrm{Tr}\{\gamma[\overline{\psi}_{\mathrm{E}}(\tau,\vec{x}),\psi_{\mathrm{E}}(\tau,\vec{x})]\}\phi(\tau,\vec{x}).
\end{aligned} \tag{10.5.1}$$

The density matrix $\hat{\rho}_{\mathrm{GC}}(\beta)$ of the grand canonical ensemble in the Schrödinger Picture is given by

$$\hat{\rho}_{\mathrm{GC}}(\beta)=\exp[-\beta(\hat{H}-\mu\hat{N})],\quad \beta=\frac{1}{k_{\mathrm{B}}T}, \tag{10.5.2}$$

where the total Hamiltonian $\hat{H}$ is split into two parts,

$$\begin{aligned}
\hat{H}_0 &= \text{ free Hamiltonian for Fermion (mass } m\text{) and Boson (mass } \kappa\text{)}.\\
\hat{H}_1 &= -\int d^3\vec{x}\,\hat{j}(\vec{x})\hat{\phi}(\vec{x}),
\end{aligned} \tag{10.5.3}$$

with the "current" given by

$$\hat{j}(\vec{x})=\frac{1}{2}g\mathrm{Tr}\{\gamma[\widehat{\overline{\psi}}_{\mathrm{E}}(\vec{x}),\hat{\psi}_{\mathrm{E}}(\vec{x})]\}, \tag{10.5.4}$$

and

$$\hat{N}=\frac{1}{2}\int d^3\vec{x}\,\mathrm{Tr}\{-\gamma^4[\widehat{\overline{\psi}}_{\mathrm{E}}(\vec{x}),\hat{\psi}_{\mathrm{E}}(\vec{x})]\}. \tag{10.5.5}$$

By the standard method of quantum field theory, we use the Interaction Picture with $\hat{N}$ included in the free part, and obtain

$$\hat{\rho}_{\mathrm{GC}}(\beta)=\hat{\rho}_0(\beta)\hat{S}(\beta), \tag{10.5.6a}$$

$$\hat{\rho}_0(\beta)=\exp[-\beta(\hat{H}_0-\mu\hat{N})], \tag{10.5.7}$$

$$\hat{S}(\beta)=\mathrm{T}_\tau\left\{\exp\left[-\int_0^\beta d\tau\int d^3\vec{x}\,\hat{\mathcal{H}}_1(\tau,\vec{x})\right]\right\} \tag{10.5.8a}$$

and

$$\hat{\mathcal{H}}_1(\tau, \vec{x}) = -\hat{\jmath}^{(\mathrm{I})}(\tau, \vec{x})\hat{\phi}^{(\mathrm{I})}(\tau, \vec{x}). \tag{10.5.9}$$

We know that the Interaction Picture operator $\hat{f}^{(\mathrm{I})}(\tau, \vec{x})$ is related to the Schrödinger Picture operator $\hat{f}(\vec{x})$ through

$$\hat{f}^{(\mathrm{I})}(\tau, \vec{x}) = \hat{\rho}_0^{-1}(\tau) \cdot \hat{f}(\vec{x}) \cdot \hat{\rho}_0(\tau). \tag{10.5.10}$$

We introduce the external hook $\{J(\tau, \vec{x}), \overline{\eta}_\alpha(\tau, \vec{x}), \eta_\beta(\tau, \vec{x})\}$ in the Interaction Picture, and obtain

$$\begin{aligned}\hat{\mathcal{H}}_1^{\mathrm{int}}(\tau, \vec{x}) = -\{&[\hat{\jmath}^{(\mathrm{I})}(\tau, \vec{x}) + J(\tau, \vec{x})]\hat{\phi}^{(\mathrm{I})}(\tau, \vec{x}) \\ &+ \overline{\eta}_\alpha(\tau, \vec{x})\hat{\psi}^{(\mathrm{I})}_{\mathrm{E}\,\alpha}(\tau, \vec{x}) + \widehat{\overline{\psi}}_{\mathrm{E}\,\beta}(\tau, \vec{x})\eta_\beta(\tau, \vec{x}))\}.\end{aligned} \tag{10.5.11}$$

We replace Eqs. (10.5.6a), (10.5.7), and (10.5.8a) with

$$\hat{\rho}_{\mathrm{GC}}(\beta; [J, \overline{\eta}, \eta]) = \hat{\rho}_0(\beta)\hat{S}(\beta; [J, \overline{\eta}, \eta]), \tag{10.5.6b}$$

$$\hat{\rho}_0(\beta) = \exp[-\beta(\hat{H}_0 - \mu\hat{N})], \tag{10.5.7}$$

$$\hat{S}(\beta; [J, \overline{\eta}, \eta]) = \mathrm{T}_\tau \left\{\exp\left[-\int_0^\beta d\tau \int d^3\vec{x}\hat{\mathcal{H}}_1^{\mathrm{int}}(\tau, \vec{x})\right]\right\}. \tag{10.5.8b}$$

Here we have
(a) $0 \le \tau \le \beta$.

$$\begin{aligned}&\frac{\delta}{\delta J(\tau, \vec{x})}\hat{\rho}_{\mathrm{GC}}(\beta; [J, \overline{\eta}, \eta])\bigg|_{J=\overline{\eta}=\eta=0} \\ &= \hat{\rho}_0(\beta)\ \mathrm{T}_\tau \left\{\hat{\phi}^{(\mathrm{I})}(\tau, \vec{x}) \exp\left[-\int_0^\beta d\tau \int d^3\vec{x}]\hat{\mathcal{H}}_1^{\mathrm{int}}(\tau, \vec{x})\right]\right\}\bigg|_{J=\overline{\eta}=\eta=0} \\ &= \hat{\rho}_0(\beta)\mathrm{T}_\tau \left\{\exp\left[-\int_\tau^\beta d\tau \int d^3\vec{x}\hat{\mathcal{H}}_1^{\mathrm{int}}\right]\right\} \\ &\quad \times \hat{\phi}^{(\mathrm{I})}(\tau, \vec{x})\ \mathrm{T}_\tau \left\{\exp\left[-\int_0^\tau d\tau \int d^3\vec{x}\hat{\mathcal{H}}_1^{\mathrm{int}}\right]\right\}\bigg|_{J=\overline{\eta}=\eta=0} \\ &= \hat{\rho}_0(\beta)\ \hat{S}(\beta; [J, \overline{\eta}, \eta])\hat{S}(-\tau; [J, \overline{\eta}, \eta])\hat{\phi}^{(\mathrm{I})}(\tau, \vec{x})\hat{S}(\tau; [J, \overline{\eta}, \eta])\bigg|_{J=\overline{\eta}=\eta=0} \\ &= \hat{\rho}_{\mathrm{GC}}(\beta; [J, \overline{\eta}, \eta])\{\hat{\rho}_0(\tau)\hat{S}(\tau; [J, \overline{\eta}, \eta])\}^{-1}\hat{\phi}(\vec{x})\ \{\hat{\rho}_0(\tau)\hat{S}(\tau; [J, \overline{\eta}, \eta])\}\bigg|_{J=\overline{\eta}=\eta=0} \\ &= \hat{\rho}_{\mathrm{GC}}(\beta)\hat{\phi}(\tau, \vec{x}).\end{aligned}$$

Thus we obtain

$$\frac{\delta}{\delta J(\tau, \vec{x})}\hat{\rho}_{\mathrm{GC}}(\beta; [J, \overline{\eta}, \eta])\bigg|_{J=\overline{\eta}=\eta=0} = \hat{\rho}_{\mathrm{GC}}(\beta)\hat{\phi}(\tau, \vec{x}). \tag{10.5.12}$$

Likewise we obtain

$$\left.\frac{\delta}{\delta\overline{\eta}_\alpha(\tau,\vec{x})}\hat{\rho}_{\mathrm{GC}}(\beta;[J,\overline{\eta},\eta])\right|_{J=\overline{\eta}=\eta=0} = \hat{\rho}_{\mathrm{GC}}(\beta)\hat{\psi}_{\mathrm{E}\,\alpha}(\tau,\vec{x}), \tag{10.5.13}$$

$$\left.\frac{\delta}{\delta\eta_\beta(\tau,\vec{x})}\hat{\rho}_{\mathrm{GC}}(\beta;[J,\overline{\eta},\eta])\right|_{J=\overline{\eta}=\eta=0} = -\hat{\rho}_{\mathrm{GC}}(\beta)\widehat{\overline{\psi}}_{\mathrm{E}\,\beta}(\tau,\vec{x}), \tag{10.5.14}$$

and

$$\left.\frac{\delta^2}{\delta\overline{\eta}_\alpha(\tau,\vec{x})\delta\eta_\beta(\tau',\vec{x})}\hat{\rho}_{\mathrm{GC}}(\beta;[J,\overline{\eta},\eta])\right|_{J=\overline{\eta}=\eta=0}$$
$$= -\hat{\rho}_{\mathrm{GC}}(\beta)\mathrm{T}_\tau\{\hat{\psi}_{\mathrm{E}\,\alpha}(\tau,\vec{x})\widehat{\overline{\psi}}_{\mathrm{E}\,\beta}(\tau,\vec{x})\}. \tag{10.5.15}$$

The Heisenberg Picture operator $\hat{f}(\tau,\vec{x})$ is related to the Schrödinger Picture operator $\hat{f}(\vec{x})$ by

$$\hat{f}(\tau,\vec{x}) = \hat{\rho}_{\mathrm{GC}}^{-1}(\tau)\cdot\hat{f}(\vec{x})\cdot\hat{\rho}_{\mathrm{GC}}(\tau). \tag{10.5.16}$$

(b) $\tau \notin [0,\beta]$.

As for $\tau \notin [0,\beta]$, the functional derivative of $\hat{\rho}_{\mathrm{GC}}(\beta;[J,\overline{\eta},\eta])$ with respect to $\{J,\overline{\eta},\eta\}$ vanishes.

In order to derive the equation of motion for the partition function, we use the "equation of motion" of $\hat{\psi}_{\mathrm{E}\,\alpha}(\tau,\vec{x})$ and $\hat{\phi}(\tau,\vec{x})$,

$$\left\{i\gamma^k\partial_k + i\gamma^4\left(\frac{\partial}{\partial\tau}-\mu\right) - m + g\gamma\hat{\phi}(\tau,\vec{x})\right\}_{\beta,\alpha}\hat{\psi}_{\mathrm{E}\,\alpha}(\tau,\vec{x}) = 0, \tag{10.5.17}$$

$$\widehat{\overline{\psi}}_{\mathrm{E}\,\beta}(\tau,\vec{x})\left\{i\gamma^k\partial_k + i\gamma^4\left(\frac{\partial}{\partial\tau}-\mu\right) - m + g\gamma\hat{\phi}(\tau,\vec{x})\right\}^{\mathrm{T}}_{\beta,\alpha} = 0, \tag{10.5.18}$$

$$\left(\frac{\partial^2}{\partial x_\nu^2}-\kappa^2\right)\hat{\phi}(\tau,\vec{x}) + g\mathrm{Tr}\{\gamma\widehat{\overline{\psi}}_{\mathrm{E}}(\tau,\vec{x})\hat{\psi}_{\mathrm{E}}(\tau,\vec{x})\} = 0, \tag{10.5.19}$$

and the equal "time" canonical (anti-)commuters,

$$\delta(\tau-\tau')\{\hat{\psi}_{\mathrm{E}\,\alpha}(\tau,\vec{x}),\widehat{\overline{\psi}}_{\mathrm{E}\,\beta}(\tau,\vec{x})\} = \delta_{\alpha\beta}\delta(\tau-\tau')\delta(\vec{x}-\vec{x}'), \tag{10.5.20a}$$

$$\delta(\tau-\tau')\left[\hat{\phi}(\tau,\vec{x}),\frac{\partial}{\partial\tau'}\hat{\phi}(\tau',\vec{x}')\right] = \delta(\tau-\tau')\delta^3(\vec{x}-\vec{y}), \tag{10.5.20b}$$

with all the rest of equal "time" (anti-)commutators equal to 0. We obtain the equations of motion of the partition function of the grand canonical ensemble

$$Z_{\mathrm{GC}}(\beta;[J,\overline{\eta},\eta]) = \mathrm{Tr}\hat{\rho}_{\mathrm{GC}}(\beta;[J,\overline{\eta},\eta]) \tag{10.5.21}$$

in the presence of the external hook $\{J,\overline{\eta},\eta\}$ from Eqs. (10.5.12), (10.5.13), and (10.5.14) as

$$\left\{i\gamma^k\partial_k + i\gamma^4\left(\frac{\partial}{\partial\tau} - \mu\right) - m + g\gamma\frac{1}{i}\frac{\delta}{\delta J(\tau,\vec{x})}\right\}_{\beta,\alpha}\frac{1}{i}\frac{\delta}{\delta\overline{\eta}_\alpha(\tau,\vec{x})}Z_{\mathrm{GC}}(\beta;[J,\overline{\eta},\eta])$$
$$= -\eta_\beta(\tau,\vec{x})Z_{\mathrm{GC}}(\beta;[J,\overline{\eta},\eta]), \tag{10.5.22}$$

$$\left\{i\gamma^k\partial_k + i\gamma^4\left(\frac{\partial}{\partial\tau} + \mu\right) + m - g\gamma\frac{1}{i}\frac{\delta}{\delta J(\tau,\vec{x})}\right\}^{\mathrm{T}}_{\beta,\alpha} i\frac{\delta}{\delta\eta_\beta(\tau,\vec{x})}Z_{\mathrm{GC}}(\beta;[J,\overline{\eta},\eta])$$
$$= \overline{\eta}_\alpha(\tau,\vec{x})Z_{\mathrm{GC}}(\beta;[J,\overline{\eta},\eta]), \tag{10.5.23}$$

$$\left\{\left(\frac{\partial^2}{\partial x_\nu^2} - \kappa^2\right)\frac{1}{i}\frac{\delta}{\delta J(\tau,\vec{x})} - g\gamma_{\beta\alpha}\frac{\delta^2}{\delta\overline{\eta}_\alpha(\tau,\vec{x})\delta\eta_\beta(\tau,\vec{x})}\right\}Z_{\mathrm{GC}}(\beta;[J,\overline{\eta},\eta])$$
$$= J(\tau,\vec{x})Z_{\mathrm{GC}}(\beta;[J,\overline{\eta},\eta]). \tag{10.5.24}$$

We can solve the functional differential equations, (10.5.22), (10.5.23), and (10.5.24), by the method discussed in Section 10.3. As in Section 10.3, we define the functional Fourier transform $\tilde{Z}_{\mathrm{GC}}(\beta;[\phi,\psi_{\mathrm{E}},\overline{\psi}_{\mathrm{E}}])$ of $Z_{\mathrm{GC}}(\beta;[J,\overline{\eta},\eta])$ by

$$Z_{\mathrm{GC}}(\beta;[J,\overline{\eta},\eta]) \equiv \int\mathcal{D}[\overline{\psi}_{\mathrm{E}}]\mathcal{D}[\psi_{\mathrm{E}}]\mathcal{D}[\phi]\tilde{Z}_{\mathrm{GC}}(\beta;[\phi,\psi_{\mathrm{E}},\overline{\psi}_{\mathrm{E}}])$$
$$\times\exp\left[i\int_0^\beta d\tau\int d^3\vec{x}\{J(\tau,\vec{x})\phi(\tau,\vec{x}) + \overline{\eta}_\alpha(\tau,\vec{x})\psi_{\mathrm{E}\,\alpha}(\tau,\vec{x}) + \overline{\psi}_{\mathrm{E}\,\beta}(\tau,\vec{x})\eta_\beta(\tau,\vec{x})\}\right].$$

We obtain the equations of motion satisfied by the functional Fourier transform $\tilde{Z}_{\mathrm{GC}}(\beta;[\phi,\psi_{\mathrm{E}},\overline{\psi}_{\mathrm{E}}])$ from Eqs. (10.5.22), (10.5.23), and (10.5.24), after the functional integral by parts on the right-hand sides involving $\overline{\eta}_\alpha$, η_β, and J as

$$\frac{\delta}{\delta\psi_{\mathrm{E}\,\alpha}(\tau,\vec{x})}\ln\tilde{Z}_{\mathrm{GC}}(\beta;[\phi,\psi_{\mathrm{E}},\overline{\psi}_{\mathrm{E}}]) = \frac{\delta}{\delta\psi_{\mathrm{E}\,\alpha}(\tau,\vec{x})}\int_0^\beta d\tau\int d^3\vec{x}\mathcal{L}'_{\mathrm{E}}((10.5.1)),$$
$$\frac{\delta}{\delta\overline{\psi}_{\mathrm{E}\,\beta}(\tau,\vec{x})}\ln\tilde{Z}_{\mathrm{GC}}(\beta;[\phi,\psi_{\mathrm{E}},\overline{\psi}_{\mathrm{E}}]) = \frac{\delta}{\delta\overline{\psi}_{\mathrm{E}\,\beta}(\tau,\vec{x})}\int_0^\beta d\tau\int d^3\vec{x}\mathcal{L}'_{\mathrm{E}}((10.5.1)),$$
$$\frac{\delta}{\delta\phi(\tau,\vec{x})}\ln\tilde{Z}_{\mathrm{GC}}(\beta;[\phi,\psi_{\mathrm{E}},\overline{\psi}_{\mathrm{E}}]) = \frac{\delta}{\delta\phi(\tau,\vec{x})}\int_0^\beta d\tau\int d^3\vec{x}\mathcal{L}'_{\mathrm{E}}((10.5.1)),$$

which we can immediately integrate to obtain

$$\tilde{Z}_{\mathrm{GC}}(\beta;[\phi,\psi_{\mathrm{E}},\overline{\psi}_{\mathrm{E}}]) = C\exp\left[\int_0^\beta d\tau\int d^3\vec{x}\mathcal{L}'_{\mathrm{E}}((10.5.1))\right]. \tag{10.5.25a}$$

Thus we have the path integral representation of $Z_{\mathrm{GC}}(\beta;[J,\overline{\eta},\eta])$ as

$$Z_{\mathrm{GC}}(\beta;[J,\overline{\eta},\eta]) = C\int\mathcal{D}[\overline{\psi}_{\mathrm{E}}]\mathcal{D}[\psi_{\mathrm{E}}]\mathcal{D}[\phi]\exp\left[\int_0^\beta d\tau\int d^3\vec{x}\{\mathcal{L}'_{\mathrm{E}}((10.5.1))\right.$$
$$\left. + iJ(\tau,\vec{x})\phi(\tau,\vec{x}) + i\overline{\eta}_\alpha(\tau,\vec{x})\psi_{\mathrm{E}\,\alpha}(\tau,\vec{x}) + i\overline{\psi}_{\mathrm{E}\,\beta}(\tau,\vec{x})\eta_\beta(\tau,\vec{x})\}\right]$$

$$= Z_0 \exp\left[-g\gamma_{\beta\alpha} \int_0^\beta d\tau \int d^3\vec{x} i \frac{\delta}{\delta\eta_\beta(\tau,\vec{x})} \frac{1}{i} \frac{\delta}{\delta\overline{\eta}_\alpha(\tau,\vec{x})} \frac{1}{i} \frac{\delta}{\delta J(\tau,\vec{x})} \right]$$
$$\times \exp\left[\int_0^\beta d\tau \int d^3\vec{x} \int_0^\beta d\tau \int d^3\vec{x}' \left\{ -\frac{1}{2} J(\tau,\vec{x}) D_0(\tau-\tau',\vec{x}-\vec{x}') J(\tau',\vec{x}') \right.\right.$$
$$\left.\left. +\overline{\eta}_\alpha(\tau,\vec{x}) S_{\underset{\alpha\beta}{0}}(\tau-\tau',\vec{x}-\vec{x}')\eta_\beta(\tau',\vec{x}') \right\} \right]. \qquad (10.5.25b)$$

The normalization constant Z_0 is so chosen that

$$Z_0 = Z_{GC}(\beta; J = \overline{\eta} = \eta = 0, g = 0)$$
$$= \prod_{|\vec{p}|,|\vec{k}|} \{1 + \exp[-\beta(\varepsilon_{\vec{p}} - \mu)]\}\{1 + \exp[-\beta(\varepsilon_{\vec{p}} + \mu)]\}\{1 - \exp[-\beta\omega_{\vec{k}}]\}^{-1}, \qquad (10.5.26)$$

with

$$\varepsilon_{\vec{p}} = (\vec{p}^2 + m^2)^{\frac{1}{2}}, \quad \omega_{\vec{k}} = (\vec{k}^2 + \kappa^2)^{\frac{1}{2}}. \qquad (10.5.27)$$

$D_0(\tau-\tau',\vec{x}-\vec{x}')$ and $S_{\underset{\alpha\beta}{0}}(\tau-\tau',\vec{x}-\vec{x}')$ are the "free" temperature Green's functions of the Bose field and the Fermi field, respectively, and are given by

$$D_0(\tau-\tau',\vec{x}-\vec{x}') = \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_{\vec{k}}} \{(f_{\vec{k}}+1)\exp[i\vec{k}(\vec{x}-\vec{x}') - \omega_{\vec{k}}(\tau-\tau')]$$
$$+ f_{\vec{k}} \exp[-i\vec{k}(\vec{x}-\vec{x}') + \omega_{\vec{k}}(\tau-\tau')]\},$$
$$S_{\underset{\alpha\beta}{0}}(\tau-\tau',\vec{x}-\vec{x}') = (i\gamma^\nu\partial_\nu + m)_{\alpha,\beta} \int \frac{d^3\vec{k}}{(2\pi)^3 2\varepsilon_{\vec{k}}}$$
$$\times \begin{cases} \{(N^+_{\vec{k}} - 1)\exp[i\vec{k}(\vec{x}-\vec{x}') - (\varepsilon_{\vec{k}} - \mu)(\tau-\tau')] \\ +N^-_{\vec{k}} \exp[-i\vec{k}(\vec{x}-\vec{x}') + (\varepsilon_{\vec{k}} + \mu)(\tau-\tau')]\}, \\ \qquad\qquad \text{for } \tau > \tau', \\ \{N^+_{\vec{k}} \exp[-i\vec{k}(\vec{x}-\vec{x}') - (\varepsilon_{\vec{k}} - \mu)(\tau-\tau')] \\ +(N^-_{\vec{k}} - 1)\exp[i\vec{k}(\vec{x}-\vec{x}') + (\varepsilon_{\vec{k}} + \mu)(\tau-\tau')]\}, \\ \qquad\qquad \text{for } \tau < \tau', \end{cases}$$
$$\partial_4 \equiv \frac{\partial}{\partial\tau} - \mu, \quad f_{\vec{k}} = \frac{1}{\exp[\beta\omega_{\vec{k}}] - 1}, \quad N^\pm_{\vec{k}} = \frac{1}{\exp[\beta(\varepsilon_{\vec{k}} \mp \mu)] + 1}.$$

The $f_{\vec{k}}$ is the density of the state of the Bose particles at energy $\omega_{\vec{k}}$, and the $N^\pm_{\vec{k}}$ is the density of the state of the (anti-)Fermi particles at energy $\varepsilon_{\vec{k}}$.

We have two ways of expressing $Z_{GC}(\beta; [J,\overline{\eta},\eta])$, Eq. (10.5.25b):

$$Z_{GC}(\beta; [J,\overline{\eta},\eta])$$
$$= Z_0 \exp\left[-\frac{1}{2} \int_0^\beta d\tau \int d^3\vec{x} \int_0^\beta d\tau' \int d^3\vec{x}' \left\{ J(\tau,\vec{x}) - g\gamma_{\beta\alpha} i \frac{\delta}{\delta\eta_\beta(\tau,\vec{x})} \right.\right.$$

$$\times \frac{1}{i}\frac{\delta}{\delta\overline{\eta}_\alpha(\tau,\vec{x})}\Bigg\}$$
$$\times D_0(\tau-\tau',\vec{x}-\vec{x}')\left\{J(\tau',\vec{x}')-g\gamma_{\beta\alpha}i\frac{\delta}{\delta\eta_\beta(\tau',\vec{x}')}\frac{1}{i}\frac{\delta}{\delta\overline{\eta}_\alpha(\tau',\vec{x}')}\right\}\Bigg]$$
$$\times \exp\left[\int_0^\beta d\tau\int d^3\vec{x}\int_0^\beta d\tau'\int d^3\vec{x}'\overline{\eta}_\alpha(\tau,\vec{x})S_{\substack{0\\ \alpha\beta}}(\tau-\tau',\vec{x}-\vec{x}')\eta_\beta(\tau',\vec{x}')\right] \tag{10.5.28}$$

$$= Z_0\left\{\mathrm{Det}(1+gS_0(\tau,\vec{x})\gamma\frac{1}{i}\frac{\delta}{\delta J(\tau,\vec{x})})\right\}^{-1}\exp\Bigg[\int_0^\beta d\tau\int d^3\vec{x}\int_0^\beta d\tau'\int d^3\vec{x}'$$
$$\times\,\overline{\eta}_\alpha(\tau,\vec{x})\left(1+gS_0(\tau,\vec{x})\gamma\frac{1}{i}\frac{\delta}{\delta J(\tau,\vec{x})}\right)^{-1}_{\alpha\varepsilon}S_{\substack{0\\ \varepsilon\beta}}(\tau-\tau',\vec{x}-\vec{x}')\eta_\beta(\tau',\vec{x}')\Bigg]$$
$$\times\exp\left[-\frac{1}{2}\int_0^\beta d\tau\int d^3\vec{x}\int_0^\beta d\tau'\int d^3\vec{x}'J(\tau,\vec{x})D_0(\tau-\tau',\vec{x}-\vec{x}')J(\tau',\vec{x}')\right]. \tag{10.5.29}$$

The thermal expectation value of the τ-ordered function

$$f_{\tau\text{-ordered}}(\hat{\psi},\widehat{\overline{\psi}},\hat{\phi})$$

in the grand canonical ensemble is given by

$$\begin{aligned}\langle f_{\tau\text{-ordered}}(\hat{\psi},\widehat{\overline{\psi}},\hat{\phi})\rangle &\equiv \frac{\mathrm{Tr}\{\hat{\rho}_{GC}(\beta)f_{\tau\text{-ordered}}(\hat{\psi},\widehat{\overline{\psi}},\hat{\phi})\}}{\mathrm{Tr}\hat{\rho}_{GC}(\beta)}\\ &\equiv \frac{1}{Z_{GC}(\beta;[J,\overline{\eta},\eta])}\\ &\quad\times f\left(\frac{1}{i}\frac{\delta}{\delta\overline{\eta}},i\frac{\delta}{\delta\eta},\frac{1}{i}\frac{\delta}{\delta J})Z_{GC}(\beta;[J,\overline{\eta},\eta]\right)\Bigg|_{J=\overline{\eta}=\eta=0}.\end{aligned} \tag{10.5.30}$$

According to this formula, the one body "full" temperature Green's functions of the Bose field and the Fermi field, $D(\tau-\tau',\vec{x}-\vec{x}')$ and $S_{\alpha\beta}(\tau-\tau',\vec{x}-\vec{x}')$, are given, respectively, by

$$\begin{aligned}D(\tau-\tau',\vec{x}-\vec{x}') &= \frac{\mathrm{Tr}\{\hat{\rho}_{GC}(\beta)\mathrm{T}_\tau(\hat{\phi}(\tau,\vec{x})\hat{\phi}(\tau',\vec{x}'))\}}{\mathrm{Tr}\hat{\rho}_{GC}(\beta)}\\ &= -\frac{1}{Z_{GC}(\beta;[J,\overline{\eta},\eta])}\frac{1}{i}\frac{\delta}{\delta J(\tau,\vec{x})}\frac{1}{i}\\ &\quad\times\frac{\delta}{\delta J(\tau',\vec{x}')}Z_{GC}(\beta;[J,\overline{\eta},\eta])\Bigg|_{J=\overline{\eta}=\eta=0},\end{aligned} \tag{10.5.31}$$

$$S_{\alpha\beta}(\tau-\tau',\vec{x}-\vec{x}') = \frac{\mathrm{Tr}\{\hat{\rho}_{\mathrm{GC}}(\beta)\hat{\psi}_\alpha(\tau,\vec{x})\widehat{\overline{\psi}}_\beta(\tau',\vec{x}')\}}{\mathrm{Tr}\hat{\rho}_{\mathrm{GC}}(\beta)}$$
$$= -\frac{1}{Z_{\mathrm{GC}}(\beta;[J,\overline{\eta},\eta])}\frac{1}{i}$$
$$\times \left.\frac{\delta}{\delta\overline{\eta}_\alpha(\tau,\vec{x})} i \frac{\delta}{\delta\eta_\beta(\tau',\vec{x}')} Z_{\mathrm{GC}}(\beta;[J,\overline{\eta},\eta])\right|_{J=\overline{\eta}=\eta=0}. \tag{10.5.32}$$

From the cyclicity of Tr and the (anti-)commutativity of $\hat{\phi}(\tau,\vec{x})$ ($\hat{\psi}_\alpha(\tau,\vec{x})$) under the T_τ-ordering symbol, we have

$$D(\tau-\tau'<0,\vec{x}-\vec{x}') = +D(\tau-\tau'+\beta,\vec{x}-\vec{x}'), \tag{10.5.33}$$

and

$$S_{\alpha\beta}(\tau-\tau'<0,\vec{x}-\vec{x}') = -S_{\alpha\beta}(\tau-\tau'+\beta,\vec{x}-\vec{x}'), \tag{10.5.34}$$

where

$$0 \le \tau,\tau' \le \beta,$$

i.e., the Boson (Fermion) "full" temperature Green's function is (anti-)periodic with the period β. From this, we have the Fourier decompositions as

$$\hat{\phi}(\tau,\vec{x}) = \frac{1}{\beta}\sum_n \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_{\vec{k}}}\Big\{\exp[i(\vec{k}\vec{x}-\omega_n\tau)]a(\omega_n,\vec{k})$$
$$+\exp[-i(\vec{k}\vec{x}-\omega_n\tau)]a^\dagger(\omega_n,\vec{k})\Big\}, \tag{10.5.35}$$
$$\omega_n = \frac{2n\pi}{\beta}, \quad n = \text{integer},$$
$$\hat{\psi}_\alpha(\tau,\vec{x}) = \frac{1}{\beta}\sum_n \int \frac{d^3\vec{k}}{(2\pi)^3 2\varepsilon_{\vec{k}}}\Big\{\exp[i(\vec{k}\vec{x}-\omega_n\tau)]u_{n\alpha}(\vec{k})b(\omega_n,\vec{k})$$
$$+\exp[-i(\vec{k}\vec{x}-\omega_n\tau)]\overline{v}_{n\alpha}(\vec{k})d^\dagger(\omega_n,\vec{k})\Big\}, \tag{10.5.36}$$
$$\omega_n = \frac{(2n+1)\pi}{\beta}, \quad n = \text{integer},$$

where

$$[a(\omega_n,\vec{k}),a^\dagger(\omega_{n'},\vec{k}')] = 2\omega_{\vec{k}}(2\pi)^3\delta^3(\vec{k}-\vec{k}')\delta_{n,n'}, \tag{10.5.37}$$
$$[a(\omega_n,\vec{k}),a(\omega_{n'},\vec{k}')] = [a^\dagger(\omega_n,\vec{k}),a^\dagger(\omega_{n'},\vec{k}')] = 0, \tag{10.5.38}$$
$$\Big\{b(\omega_n,\vec{k}),b^\dagger(\omega_{n'},\vec{k}')\Big\} = \Big\{d(\omega_n,\vec{k}),d^\dagger(\omega_{n'},\vec{k}')\Big\} = 2\varepsilon_{\vec{k}}(2\pi)^3\delta^3(\vec{k}-\vec{k}')\delta_{n,n'}, \tag{10.5.39}$$

the rest of anticommutators $= 0$. (10.5.40)

We shall now address ourselves to the problem of finding the equation of motion of the one-body Boson and Fermion Green's functions. We define the 1-body Boson and Fermion "full" temperature Green's functions, $D^J(x,y)$ and $S^J_{\alpha,\beta}(x,y)$, by for *Boson field Green's function*:

$$\begin{aligned} D^J(x,y) &\equiv -\left.\langle \mathrm{T}_\tau(\hat{\phi}(x)\hat{\phi}(y))\rangle^J\right|_{\overline{\eta}=\eta=0} \\ &\equiv -\left.\frac{\delta^2}{\delta J(x)\delta J(y)}\ln Z_{\mathrm{GC}}(\beta;[J,\overline{\eta},\eta])\right|_{\overline{\eta}=\eta=0} \\ &\equiv -\left.\frac{\delta}{\delta J(x)}\langle\hat{\phi}(y)\rangle^J\right|_{\overline{\eta}=\eta=0}, \end{aligned} \tag{10.5.41}$$

and for *Fermion field Green's function*:

$$\begin{aligned} S^J_{\alpha,\beta}(x,y) &\equiv +\left.\langle \mathrm{T}_\tau(\hat{\psi}_\alpha(x)\widehat{\overline{\psi}}_\beta(y))\rangle^J\right|_{\overline{\eta}=\eta=0} \\ &\equiv -\left.\frac{1}{Z_{\mathrm{GC}}(\beta;[J,\overline{\eta},\eta])}\frac{\delta}{\delta\overline{\eta}_\alpha(x)}\frac{\delta}{\delta\eta_\beta(y)}Z_{\mathrm{GC}}(\beta;[J,\overline{\eta},\eta])\right|_{\overline{\eta}=\eta=0}. \end{aligned} \tag{10.5.42}$$

From Eqs. (10.5.22), (10.5.23), and (10.5.24), we obtain a summary of the Schwinger–Dyson equation satisfied by $D^J(x,y)$ and $S^J_{\alpha,\beta}(x,y)$:

Summary of Schwinger–Dyson equation in configuration space

$$\begin{aligned} &(i\gamma^\nu\partial_\nu - m + g\gamma\langle\hat{\phi}(x)\rangle^J)_{\alpha\varepsilon}S^J_{\varepsilon\beta}(x,y) - \int d^4z\,\boldsymbol{\Sigma}^*_{\alpha\varepsilon}(x,z)S^J_{\varepsilon\beta}(z,y) \\ &\quad = \delta_{\alpha\beta}\delta^4(x-y), \\ &\left(\frac{\partial^2}{\partial x_\nu^2} - \kappa^2\right)\langle\hat{\phi}(x)\rangle^J \\ &\quad = \left.\frac{1}{2}g\gamma_{\beta\alpha}\left\{S^J_{\alpha\beta}(\tau,\vec{x};\tau-\varepsilon,\vec{x}) + S^J_{\alpha\beta}(\tau,\vec{x};\tau+\varepsilon,\vec{x})\right\}\right|_{\varepsilon\to 0^+}, \\ &\left(\frac{\partial^2}{\partial x_\nu^2} - \kappa^2\right)D^J(x,y) - \int d^4z\,\boldsymbol{\Pi}^*(x,z)D^J(z,y) = \delta^4(x-y), \\ &\boldsymbol{\Sigma}^*_{\alpha\beta}(x,y) = g^2\int d^4u\,d^4v\,\gamma_{\alpha\delta}S^J_{\delta\nu}(x,u)\boldsymbol{\Gamma}_{\nu\beta}(u,y;v)D^J(v,x), \\ &\boldsymbol{\Pi}^*(x,y) = g^2\int d^4u\,d^4v\,\gamma_{\alpha\beta}S^J_{\beta\delta}(x,u)\boldsymbol{\Gamma}_{\delta\nu}(u,v;y)S^J_{\nu\alpha}(v,x), \\ &\boldsymbol{\Gamma}_{\alpha\beta}(x,y;z) = \gamma_{\alpha\beta}(z)\delta^4(x-y)\delta^4(x-z) + \frac{1}{g}\frac{\delta\boldsymbol{\Sigma}^*_{\alpha\beta}(x,y)}{\delta\langle\hat{\phi}(z)\rangle^J}. \end{aligned}$$

This system of nonlinear coupled integro-differential equations is exact and closed. Starting from the 0th-order term of $\boldsymbol{\Gamma}_{\alpha\beta}(x,y;z)$, we can develop Feynman–Dyson-type graphical perturbation theory for quantum statistical

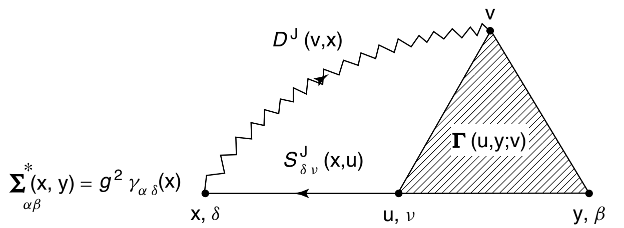

Fig. 10.9 Graphical representation of the proper self-energy part, $\Sigma^*_{\alpha\beta}(x, y)$.

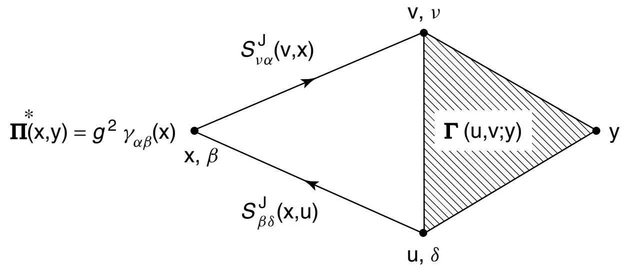

Fig. 10.10 Graphical representation of the proper self-energy part, $\Pi^*(x, y)$.

mechanics in the configuration space by iteration. We here employed the following abbreviation:

$$x \equiv (\tau_x, \vec{x}), \qquad \int d^4x \equiv \int_0^\beta d\tau_x \int d^3\vec{x}.$$

We note that $S^J_{\alpha\beta}(x, y)$ and $D^J(x, y)$ are determined by Eqs. (10.5.3a)–(10.5.3f) only for

$$\tau_x - \tau_y \in [-\beta, \beta],$$

and we assume that they are defined by the periodic boundary condition with the period 2β for other

$$\tau_x - \tau_y \notin [-\beta, \beta].$$

Next we set

$$J \equiv 0,$$

and hence we have

$$\langle \hat{\phi}(x) \rangle^{J \equiv 0} \equiv 0,$$

restoring the translational invariance of the system. We Fourier transform $S_{\alpha\beta}(x)$ and $D(x)$,

$$S_{\alpha\beta}(x) = \frac{1}{\beta}\sum_{p_4}\int \frac{d^3\vec{p}}{(2\pi)^3} S_{\alpha\beta}(p_4, \vec{p}) \exp[i(\vec{p}\vec{x} - p_4\tau_x)], \tag{10.5.43a}$$

$$D(x) = \frac{1}{\beta}\sum_{p_4}\int \frac{d^3\vec{p}}{(2\pi)^3} D(p_4, \vec{p}) \exp[i(\vec{p}\vec{x} - p_4\tau_x)], \tag{10.5.43b}$$

$$\begin{aligned}\mathbf{\Gamma}_{\alpha,\beta}(x, y; z) = \mathbf{\Gamma}_{\alpha,\beta}(x - y, x - z) &= \frac{1}{\beta^2}\sum_{p_4,k_4}\int \frac{d^3\vec{p}d^3\vec{k}}{(2\pi)^6}\mathbf{\Gamma}_{\alpha,\beta}(p, k) \\ &\times \exp[i\{\vec{p}(\vec{x} - \vec{y}) - p_4(\tau_x - \tau_y)\} - i\{\vec{k}(\vec{x} - \vec{z}) - k_4(\tau_x - \tau_z)\}],\end{aligned} \tag{10.5.43c}$$

$$p_4 = \begin{cases} (2n+1)\pi/\beta, & \textit{Fermion}, \quad n = \text{ integer}, \\ 2n\pi/\beta, & \textit{Boson}, \quad n = \text{ integer}. \end{cases} \tag{10.5.43d}$$

We have the Schwinger–Dyson equation in the momentum space:

Summary of Schwinger–Dyson equation in the momentum space

$$\left\{-\vec{\gamma}\vec{p} + \gamma^4(p_4 - i\mu) \ -(m + \mathbf{\Sigma}^*(p))\right\}_{\alpha\varepsilon} S_{\varepsilon\beta}(p) = \delta_{\alpha\beta}\sum_n \delta\left(p_4 - \frac{(2n+1)\pi}{\beta}\right),$$

$$\left\{-k_\nu^2 - (\kappa^2 + \mathbf{\Pi}^*(k))\right\} D(k) = \sum_n \delta\left(k_4 - \frac{2n\pi}{\beta}\right),$$

$$\mathbf{\Sigma}^*_{\alpha,\beta}(p) = g^2\frac{1}{\beta}\sum_{k_4}\int \frac{d^3\vec{k}}{(2\pi)^3}\gamma_{\alpha\delta}S_{\delta\varepsilon}(p+k)\mathbf{\Gamma}_{\varepsilon\beta}(p+k, k)D(k),$$

$$\mathbf{\Pi}^*(k) = g^2\frac{1}{\beta}\sum_{p_4}\int \frac{d^3\vec{p}}{(2\pi)^3}\gamma_{\mu\nu}S_{\nu\lambda}(p+k)\mathbf{\Gamma}_{\lambda\rho}(p+k, k)S_{\rho\mu}(p),$$

$$\begin{aligned}\mathbf{\Gamma}_{\alpha\beta}(p, k) = &\sum_{n,,m}\gamma_{\alpha\beta}\delta\left(p_4 - \frac{(2n+1)\pi}{\beta}\right)\delta\left(k_4 - \frac{(2m+1)\pi}{\beta}\right) \\ &+ \mathbf{\Lambda}_{\alpha\beta}(p, k),\end{aligned}$$

where $\mathbf{\Lambda}_{\alpha\beta}(p, k)$ represents the sum of the vertex diagram except for the first term. This system of nonlinear coupled integral equations is exact and closed. Starting from the 0th-order term of $\mathbf{\Gamma}_{\alpha\beta}(p, k)$, we can develop a Feynman–Dyson-type graphical perturbation theory for quantum statistical mechanics in the momentum space by iteration.

From the Schwinger–Dyson equation, we can derive the Bethe–Goldstone diagram rule of the many-body problems at finite temperature in quantum statistical mechanics, nuclear physics, and condensed matter physics. For details of this diagram rule, we refer the reader to *A.L. Fetter* and *J.D. Walecka*.

10.6 Feynman's Variational Principle

In this section, we shall briefly consider *Feynman'svariational principle in quantum statistical mechanics.*

We consider the canonical ensemble with the Hamiltonian $\hat{H}(\{\vec{q}_j, \vec{p}_j\}_{j=1}^N)$ at finite temperature. The density matrix $\hat{\rho}_C(\beta)$ of this system satisfies the Bloch equation,

$$-\hbar\frac{\partial}{\partial\tau}\hat{\rho}_C(\tau) = \hat{H}(\{\vec{q}_j, \vec{p}_j\}_{j=1}^N)\hat{\rho}_C(\tau), \quad 0 \le \tau \le \beta, \tag{10.6.1}$$

with its formal solution given by

$$\hat{\rho}_C(\tau) = \exp[-\tau\hat{H}(\{\vec{q}_j, \vec{p}_j\}_{j=1}^N/\hbar]\hat{\rho}_C(0). \tag{10.6.2}$$

We compare the Bloch equation and the density matrix, Eqs. (10.6.1) and (10.6.2), with the Schrödinger equation for the state vector $|\psi, t>$,

$$i\hbar\frac{d}{dt}|\psi, t> = \hat{H}(\{\vec{q}_j, \vec{p}_j\}_{j=1}^N)|\psi, t>, \tag{10.6.3}$$

and its formal solution given by

$$|\psi, t> = \exp[-it\hat{H}(\{\vec{q}_j, \vec{p}_j\}_{j=1}^N)/\hbar]|\psi, 0>. \tag{10.6.4}$$

We find that by the analytic continuation,

$$t = -i\tau, \quad 0 \le \tau \le \beta \equiv \hbar/k_B T, \tag{10.6.5}$$
$$k_B = \text{Boltzmann constant}, \quad T = \text{absolute temperature},$$

the (real time) Schrödinger equation and its formal solution, Eqs. (10.6.3) and (10.6.4), are analytically continued into the Bloch equation and the density matrix, Eqs. (10.6.1) and (10.6.2), respectively. By the analytic continuation, Eq. (10.6.5), we divide the interval $[0, \beta]$ into the n equal subinterval, and use the resolution of the identity in both the q-representation and the p-representation. In this way, we obtain the following list of correspondence. Here, we assume the Hamiltonian $\hat{H}(\{\vec{q}_j, \vec{p}_j\}_{j=1}^N)$ of the following form:

$$\hat{H}(\{\vec{q}_j, \vec{p}_j\}_{j=1}^N) = \sum_{j=1}^N \frac{1}{2m}\vec{p}_j^2 + \sum_{j>k} V(\vec{q}_j, \vec{q}_k). \tag{10.6.6}$$

List of Correspondence

Quantum mechanics	**Quantum statistical mechanics**
Schrödinger equation $i\hbar\frac{\partial}{\partial t}\mid\psi,t> = H(\{\vec{q}_j,\vec{p}_j\}_{j=1}^N)\mid\psi,t>.$	Bloch equation $-\hbar\frac{\partial}{\partial\tau}\hat{\rho}_{\mathrm{C}}(\tau) = H(\{\vec{q}_j,\vec{p}_j\}_{j=1}^N)\hat{\rho}_{\mathrm{C}}(\tau).$
Schrödinger state vector $\mid\psi,t> = \exp[-itH(\{\vec{q}_j,\vec{p}_j\}_{j=1}^N)/\hbar]\mid\psi,0>.$	Density matrix $\hat{\rho}_{\mathrm{C}}(\tau) = \exp[-\tau H(\{\vec{q}_j,\vec{p}_j\}_{j=1}^N)/\hbar]\hat{\rho}_{\mathrm{C}}(0).$
Minkowskian Lagrangian $L_{\mathrm{M}}(\{q_j(t),\dot{q}_j(t)\}_{j=1}^N) = \sum_{j=1}^N\frac{1}{2}m\dot{q}_j^2(t) - \sum_{j>k}^N V(\vec{q}_j,\vec{q}_k).$	Euclidean Lagrangian $L_{\mathrm{E}}(\{q_j(\tau),\dot{q}_j(\tau)\}_{j=1}^N) = -\sum_{j=1}^N\frac{1}{2}m\dot{q}_j^2(\tau) - \sum_{j>k}^N V(\vec{q}_j,\vec{q}_k).$
$i\times$ Minkowskian action functional $iI_{\mathrm{M}}[\{\vec{q}_j\}_{j=1}^N;\vec{q}_f,\vec{q}_i] = i\int_{t_i}^{t_f}dtL_{\mathrm{M}}(\{q_j(t),\dot{q}_j(t)\}_{j=1}^N).$	Euclidean action functional $I_{\mathrm{E}}[\{\vec{q}_j\}_{j=1}^N;\vec{q}_f,\vec{q}_i] = \int_0^\beta d\tau\, L_{\mathrm{E}}(\{q_j(\tau),\dot{q}_j(\tau)\}_{j=1}^N).$
Transformation function $\langle\vec{q}_f,t_f\mid\vec{q}_i,t_i\rangle = \int_{\vec{q}(t_i)=\vec{q}_i}^{\vec{q}(t_f)=\vec{q}_f}\mathcal{D}[\vec{q}]\times \times\exp[iI_{\mathrm{M}}[\{\vec{q}_j\}_{j=1}^N;\vec{q}_f,\vec{q}_i]/\hbar].$	Transformation function $Z_{f,i} = \int_{\vec{q}(0)=\vec{q}_i}^{\vec{q}(\beta)=\vec{q}_f}\mathcal{D}[\vec{q}]\times \times\exp[I_{\mathrm{E}}[\{\vec{q}_j\}_{j=1}^N;\vec{q}_f,\vec{q}_i]/\hbar].$
Vacuum-to-vacuum transition amplitude $\langle 0,\mathrm{out}\mid 0,\mathrm{in}\rangle = \int\mathcal{D}[\vec{q}]\exp[iI_{\mathrm{M}}[\{\vec{q}_j\}_{j=1}^N]/\hbar].$	Partition function* $Z_{\mathrm{C}}(\beta) = \mathrm{Tr}\hat{\rho}_{\mathrm{C}}(\beta) = \text{``}\int d\vec{q}_f d\vec{q}_i\delta(\vec{q}_f-\vec{q}_i)Z_{f,i}\text{''}.$
Vacuum expectation value $\langle O(\hat{q})\rangle = \frac{\int\mathcal{D}[\vec{q}]O(\vec{q})\exp[iI_{\mathrm{M}}[\{\vec{q}_j\}_{j=1}^N]/\hbar]}{\int\mathcal{D}[\vec{q}]\exp\{iI_{\mathrm{M}}[\{\vec{q}_j\}_{j=1}^N]/\hbar\}}.$	Thermal expectation value* $\langle O(\hat{q})\rangle = \frac{\mathrm{Tr}\hat{\rho}_{\mathrm{C}}(\beta)O(\hat{q})}{\mathrm{Tr}\hat{\rho}_{\mathrm{C}}(\beta)}.$

In the list above, entries with "*" are given, respectively, by

Partition function:

$$
\begin{aligned}
Z_{\mathrm{C}}(\beta) = \mathrm{Tr}\hat{\rho}_{\mathrm{C}}(\beta) &= \text{``}\int d^3\vec{q}_f d^3\vec{q}_i\delta^3(\vec{q}_f-\vec{q}_i)Z_{f,i}\text{''}\\
&= \frac{1}{N!}\sum_P\delta_P\int d^3\vec{q}_f d^3\vec{q}_i\delta^3(\vec{q}_f-\vec{q}_{Pi})Z_{f,Pi}\\
&= \frac{1}{N!}\sum_P\delta_P\int d^3\vec{q}_f d^3\vec{q}_{Pi}\delta^3(\vec{q}_f-\vec{q}_{Pi})\\
&\quad\times\int_{\vec{q}(0)=\vec{q}_{Pi}}^{\vec{q}(\beta)=\vec{q}_f}\mathcal{D}[\vec{q}]\exp[I_{\mathrm{E}}[\{\vec{q}_j\}_{j=1}^N;\vec{q}_f,\vec{q}_{Pi}]/\hbar],
\end{aligned}
\tag{10.6.7}
$$

Thermal expectation value:

$$\begin{aligned}
\langle \hat{O}(\vec{q}) \rangle &= \frac{\mathrm{Tr}\hat{\rho}_{\mathrm{C}}(\beta)\hat{O}(\hat{q})}{\mathrm{Tr}\hat{\rho}_{\mathrm{C}}(\beta)} \\
&= \frac{\text{“} \int d^3\vec{q}_f d^3\vec{q}_i \delta^3(\vec{q}_f - \vec{q}_i) Z_{f,i} \langle i| \, O(\vec{q}) \, |f\rangle \text{”}}{\int d^3\vec{q}_f d^3\vec{q}_i \delta^3(\vec{q}_f - \vec{q}_i) Z_{f,i}} \\
&= \frac{1}{Z_{\mathrm{C}}(\beta)} \frac{1}{N!} \sum_P \delta_P \int d^3\vec{q}_f d^3\vec{q}_{Pi} \delta^3(\vec{q}_f - \vec{q}_{Pi}) Z_{f,Pi} \langle \vec{q}_{Pi} | \, \hat{O}(\vec{q}) \, |\vec{q}_f\rangle. \quad (10.6.8)
\end{aligned}$$

Here, $\vec{q}_i$ and $\vec{q}_f$ represent the initial position $\{\vec{q}_j(0)\}_{j=1}^N$ and the final position $\{\vec{q}_j(\beta)\}_{j=1}^N$ of N identical particles, P represents the permutation of $\{1, \ldots, N\}$, Pi represents the permutation of the initial position $\{\vec{q}(0)\}_{j=1}^N$ and δ_P represents the signature of the permutation P, respectively.

In this manner, we obtain the path integral representation of the partition function, $Z_{\mathrm{C}}(\beta)$, and the thermal expectation value, $\langle \hat{O}(\vec{q}) \rangle$. Functional $I_{\mathrm{E}}[\{\vec{q}_j\}_{j=1}^N; \vec{q}_f, \vec{q}_i]$ of Eq. (10.6.7) can be obtained from $I_{\mathrm{M}}[\{\vec{q}_j\}_{j=1}^N; \vec{q}_f, \vec{q}_i]$ by replacing t with $-i\tau$. Since $\hat{\rho}_{\mathrm{C}}(\beta)$ is a solution of Eq. (10.6.1), the asymptotic form of $Z_{\mathrm{C}}(\beta)$ for a large τ interval from τ_i to τ_f is

$$Z_{\mathrm{C}}(\beta) \sim \exp[-E_0(\tau_f - \tau_i)/\hbar] \quad \text{as} \quad \tau_f - \tau_i \longrightarrow \infty.$$

Therefore, we must estimate $Z_{\mathrm{C}}(\beta)$ for large $\tau_f - \tau_i$.

We choose any real I_1 which approximates $I_{\mathrm{E}}[\{\vec{q}_j\}_{j=1}^N; \vec{q}_f, \vec{q}_i]$ and write $Z_{\mathrm{C}}(\beta)$ as

$$\begin{aligned}
&\int \mathcal{D}[\vec{q}(\zeta)] \exp[I_{\mathrm{E}}[\{\vec{q}_j\}_{j=1}^N; \vec{q}_f, \vec{q}_i]/\hbar] \\
&= \int \mathcal{D}[\vec{q}(\zeta)] \exp[(I_{\mathrm{E}}[\{\vec{q}_j\}_{j=1}^N; \vec{q}_f, \vec{q}_i] - I_1)/\hbar] \exp[I_1/\hbar]. \quad (10.6.9)
\end{aligned}$$

The expression (10.6.9) can be regarded as the average of $\exp[(I_{\mathrm{E}} - I_1)/\hbar]$ with respect to the positive weight $\exp[I_1/\hbar]$. This observation motivates the variational principle based on Jensen's inequality. Since the exponential function is convex, for any real quantities f, the average of $\exp[f]$ exceeds the exponential of the average $\langle f \rangle$,

$$\langle \exp[f] \rangle \;\geq\; \exp[\langle f \rangle]. \quad (10.6.10)$$

Hence, if in Eq. (10.6.9) we replace $I_{\mathrm{E}}[\{\vec{q}_j\}_{j=1}^N; \vec{q}_f, \vec{q}_i] - I_1$ by its average

$$\langle I_{\mathrm{E}}[\{\vec{q}_j\}_{j=1}^N; \vec{q}_f, \vec{q}_i] - I_1 \rangle \;=\; \frac{\int \mathcal{D}[\vec{q}(\zeta)] \, (I_{\mathrm{E}}[\{\vec{q}_j\}_{j=1}^N; \vec{q}_f, \vec{q}_i] - I_1) \exp[I_1/\hbar]}{\int \mathcal{D}[\vec{q}(\zeta)] \exp[I_1/\hbar]}, \quad (10.6.11)$$

we will underestimate the value of Eq. (10.6.9). If E is computed from

$$\int \mathcal{D}\ [\vec{q}(\zeta)] \exp[\langle I_{\mathrm{E}}[\{\vec{q}_j\}_{j=1}^N; \vec{q}_f, \vec{q}_i] - I_1\rangle / \hbar] \exp[I_1/\hbar] \sim \exp[-E(\tau_f - \tau_i)/\hbar], \tag{10.6.12}$$

we know that E exceeds the true E_0,

$$E \geq E_0. \tag{10.6.13}$$

If there are any free parameters in I_1, we choose as the best values those which minimize E.

Since $\langle I_{\mathrm{E}}[\{\vec{q}_j\}_{j=1}^N; \vec{q}_f, \vec{q}_i] - I_1\rangle$ defined in Eq. (10.6.11) is proportional to $\tau_f - \tau_i$, we write

$$\langle I_{\mathrm{E}}[\{\vec{q}_j\}_{j=1}^N; \vec{q}_f, \vec{q}_i] - I_1\rangle = s(\tau_f - \tau_i). \tag{10.6.14}$$

The factor $\exp[\langle I_{\mathrm{E}}[\{\vec{q}_j\}_{j=1}^N; \vec{q}_f, \vec{q}_i] - I_1\rangle / \hbar]$ in Eq. (10.6.12) is constant and can be taken outside the integral. We suppose the lowest energy E_1 for the action functional I_1 is known,

$$\int \mathcal{D}\ [\vec{q}(\zeta)] \exp[I_1/\hbar] \sim \exp[-E_1(\tau_f - \tau_i)/\hbar] \quad \text{as} \quad \tau_f - \tau_i \longrightarrow \infty. \tag{10.6.15}$$

Then we have

$$E = E_1 - s$$

from Eq. (10.6.12), with s given by Eqs. (10.6.11) and (10.6.14).

If we choose the following *trial action functional*:

$$I_1 = -\frac{1}{2}\int \left(\frac{d\vec{q}}{d\tau}\right)^2 d\tau, \tag{10.6.16}$$

we have what corresponds to the *plane wave Born approximation* in standard perturbation theory. Another choice is

$$I_1 = -\frac{1}{2}\int \left(\frac{d\vec{q}}{d\tau}\right)^2 d\tau + \int V_{\mathrm{trial}}(\vec{q}(\tau)) d\tau, \tag{10.6.17}$$

where $V_{\mathrm{trial}}(\vec{q}(\tau))$ is a trial potential to be chosen. This corresponds to the *distorted wave Born approximation* in standard perturbation theory.

If we choose a *Coulomb potential* as the trial potential,

$$V_{\mathrm{trial}}(R) = Z/R, \tag{10.6.18}$$

we vary the parameter Z. If we choose a *harmonic potential* as the trial potential,

$$V_{\text{trial}}(R) = \frac{1}{2}kR^2, \tag{10.6.19}$$

we vary the parameter k.

A trouble with the trial potential used in Eq. (10.6.17) is that the particle with the coordinate $\vec{q}(\tau)$ is bound to a specific origin. A better choice would be the interparticle potential of the form

$$V_{\text{trial}}(\vec{q}(\tau) - \vec{q}(\sigma)).$$

Polaron Problem: We shall apply Feynman's variational principle to the polaron problem. The polaron problem is the following. An electron in an ionic crystal polarizes the crystal lattice in its neighborhood. When the electron moves in the ionic crystal, the polarized state must move with the electron. An electron moving with its polarized neighborhood is called a *polaron*. The coupling strength of the electron with the crystal lattice varies from weak to strong, depending on the type of the crystal. The major problem is to compute the energy and the effective mass of such an electron. We assume for the sake of mathematical simplicity that

(1) the crystal lattice acts much like a dielectric medium, and
(2) all the important phonon waves have the same frequency.

The trial action functional of the form specified by

$$I_1 = -\frac{1}{2}\int_0^\beta \left(\frac{d\vec{q}}{d\tau}\right)^2 d\tau - \frac{1}{2}C\int_0^\beta d\tau \int_0^\beta d\sigma \left(\vec{q}(\tau) - \vec{q}(\sigma)\right)^2 \exp[-w\,|\tau - \sigma|]$$

was used in the polaron problem by Feynman, after the phonon degrees of freedom were path-integrated out, leaving the electron degrees of freedom for the variational calculation of the ground state energy and the effective mass of the electron in a polar crystal. We shall use the simpler trial action functional with $w = 0$,

$$I_1 = -\frac{1}{2}\int_0^\beta \left(\frac{d\vec{q}}{d\tau}\right)^2 d\tau - \frac{1}{2}C\int_0^\beta d\tau \int_0^\beta d\sigma \left(\vec{q}(\tau) - \vec{q}(\sigma)\right)^2,$$

to simplify the mathematics involved considerably.

For the polaron problem, i.e., for a slow electron in a polar crystal, we usually choose the following action functional:

$$I_1 = -\frac{1}{2}\int \left(\frac{d\vec{q}}{d\tau}\right)^2 d\tau - \frac{1}{2}C\iint \left(\vec{q}(\tau) - \vec{q}(\sigma)\right)^2 \exp[-w\,|\tau - \sigma|]d\tau\, d\sigma, \tag{10.6.20}$$

as the trial action functional with C and w as the adjustable parameters for the variational calculation.

We begin with the Lagrangian for the electron–phonon system in a polar crystal and path-integrate out the phonon degrees of freedom. We then apply the Feynman's variational principle to the electron degrees of freedom.

We write the Lagrangian for the electron–phonon system in a polar crystal as

$$L=\frac{1}{2}m\overrightarrow{\dot{q}}^{2}+\frac{1}{2}\int\frac{d^{3}\vec{k}}{(2\pi)^{3}}\{\dot{q}_{\vec{k}}^{2}-\omega^{2}q_{\vec{k}}^{2}\}-\int\frac{d^{3}\vec{k}}{(2\pi)^{3}}V_{\vec{k}}q_{\vec{k}}\exp[i\vec{k}\cdot\vec{q}], \tag{10.6.21}$$

where $\vec{q}$ is the coordinate of the electron, $q_{\vec{k}}$ is the mode of the phonon with momentum $\hbar\vec{k}$, $V_{\vec{k}}$ is given by

$$V_{\vec{k}}=\frac{\hbar\omega}{\left|\vec{k}\right|}\left(\frac{\hbar}{2m\omega}\right)^{1/4}(4\pi\alpha)^{1/2}, \tag{10.6.22}$$

and α is the coupling constant given by

$$\alpha\equiv\frac{1}{2}\left(\frac{1}{\varepsilon_{\infty}}-\frac{1}{\varepsilon}\right)\frac{e^{2}}{\hbar\omega}\left(\frac{2m\omega}{\hbar}\right)^{1/2}. \tag{10.6.23}$$

We note that ε_∞ and ε are the dielectric constants of the vacuum and the medium, respectively. We now employ the unit system in which $\hbar=m=\omega=1$. In this unit system, we have

$$\left|V_{\vec{k}}\right|^{2}=\frac{4\pi\alpha}{\sqrt{2}\left|\vec{k}\right|^{2}}\quad\text{and}\quad\alpha\equiv\frac{e^{2}}{\sqrt{2}}\left(\frac{1}{\varepsilon_{\infty}}-\frac{1}{\varepsilon}\right). \tag{10.6.24}$$

Path-integrating out the phonon degrees of freedom, after a little algebra, we have the *reduced density matrix* $\rho(\vec{q}_f,\vec{q}_i)$ for the electron degrees of freedom as

$$\rho(\vec{q}_f,\vec{q}_i)=\int\mathcal{D}[\vec{q}]\exp[I_{\mathrm{E}}[\vec{q}]], \tag{10.6.25}$$

where the trial action functional $I_{\mathrm{E}}[\vec{q}]$ is given by

$$I_{\mathrm{E}}[\vec{q}]=-\frac{1}{2}\int_{0}^{\beta}\left(\frac{d\vec{q}}{d\tau}\right)^{2}d\tau+\alpha\int_{0}^{\beta}d\tau\int_{0}^{\beta}d\sigma\frac{\cosh[-|\tau-\sigma|+\beta/2]}{2^{3/2}\sinh[\beta/2]\left|\vec{q}(\tau)-\vec{q}(\sigma)\right|} \tag{10.6.26}$$

which contains the retardation factor $\cosh[-|\tau-\sigma|+\beta/2]$ to account for the effect of the interaction with the phonon degrees of freedom in the original problem.

As a trial action functional, Feynman used the trial action functional, (10.6.20). To simplify the mathematics involved, we apply the following simpler action functional:

$$I_{1}=-\frac{1}{2}\int_{0}^{\beta}\left(\frac{d\vec{q}}{d\tau}\right)^{2}d\tau-\frac{1}{2}C\int_{0}^{\beta}d\tau\int_{0}^{\beta}d\sigma\left(\vec{q}(\tau)-\vec{q}(\sigma)\right)^{2}, \tag{10.6.27}$$

which is the $w = 0$ version of (10.6.20). We recall from the previous section that

$$E = E_1 - s. \tag{10.6.28}$$

Thus we have

$$s = A + B, \tag{10.6.29}$$

where

$$A = \frac{\alpha}{2^{3/2}\beta} \int_0^\beta d\tau \int_0^\beta d\sigma \left\langle \frac{1}{\left|\vec{q}(\tau) - \vec{q}(\sigma)\right|} \right\rangle \exp[-|\tau - \sigma|], \tag{10.6.30}$$

$$B = \frac{C}{2\beta} \int_0^\beta d\tau \int_0^\beta d\sigma \left\langle \left(\vec{q}(\tau) - \vec{q}(\sigma)\right)^2 \right\rangle, \tag{10.6.31}$$

with $\langle F \rangle$ given by

$$\langle F \rangle = \frac{\int F \exp[I_1] \mathcal{D}[\vec{q}]}{\int \exp[I_1] \mathcal{D}[\vec{q}]}. \tag{10.6.32}$$

To compute A, we note

$$\frac{1}{\left|\vec{q}(\tau) - \vec{q}(\sigma)\right|} = \frac{1}{2\pi^2} \int \frac{\exp[i\vec{k} \cdot (\vec{q}(\tau) - \vec{q}(\sigma))]}{\vec{k}^2} d^3\vec{k}. \tag{10.6.33}$$

It is sufficient to compute

$$\left\langle \exp[i\vec{k} \cdot (\vec{q}(\tau) - \vec{q}(\sigma))] \right\rangle = \frac{\int \exp[i\vec{k} \cdot (\vec{q}(\tau) - \vec{q}(\sigma))] \exp[I_1] \mathcal{D}[\vec{q}]}{\int \exp[I_1] \mathcal{D}[\vec{q}]}. \tag{10.6.34}$$

Writing the numerator of the right-hand side of (10.6.34) as J, we have

$$J = \int \exp\left[-\frac{1}{2} \int_0^\beta \left(\frac{d\vec{q}}{d\tau}\right)^2 d\tau - \frac{1}{2} C \int_0^\beta d\tau \int_0^\beta d\sigma \left(\vec{q}(\tau) - \vec{q}(\sigma)\right)^2 \right.$$
$$\left. + \int_0^\beta \vec{f}(\tau) \cdot \vec{q}(\tau) d\tau \right] \mathcal{D}[\vec{q}], \tag{10.6.35}$$

where

$$\vec{f}(\varsigma) = i\vec{k}[\delta(\varsigma - \tau) - \delta(\varsigma - \sigma)]. \tag{10.6.36}$$

We note that J is factorized to the form $J = J_x J_y J_z$, and we are concerned with J_x for the time being. We observe that the exponent of J_x is quadratic in $q_x(\tau)$ which we write as

$$q_x(\tau) = X(\tau) + \delta q_x(\tau), \tag{10.6.37}$$

where $X(\tau)$ extremizes the exponent of (10.6.35). The $\delta q_x(\tau)$ path-integral is Gaussian and we write the result as N_x. We then have

$$J_x = N_x \exp[-\frac{1}{2}\int_0^{\beta}\left(\frac{dX}{d\tau}\right)^2 d\tau - \frac{1}{2}C\int_0^{\beta} d\tau \int_0^{\beta} d\sigma \left(X(\tau)-X(\sigma)\right)^2 + \int_0^{\beta} f_x(\tau)X(\tau)d\tau], \tag{10.6.38}$$

where

$$f_x(\varsigma) = ik_x[\delta(\varsigma-\tau)-\delta(\varsigma-\sigma)]. \tag{10.6.39}$$

Since we have N_x both in the numerator and the denominator of (10.6.34), we can set $N_x = 1$ without loss of generality. As the maximization condition of the exponent of (10.6.35), we obtain the integro-differential equation for $X(\varsigma)$ as

$$\frac{d^2X(\varsigma)}{d\varsigma^2} = 2C\int_0^{\beta}[X(\varsigma)-X(\sigma)]d\sigma - f_x(\varsigma), \tag{10.6.40}$$

with the boundary conditions specified as

$$X(0) = X(\beta) = 0. \tag{10.6.41}$$

With (10.6.40), we have

$$J_x = \exp\left[\frac{1}{2}\int_0^{\beta} f_x(\tau)X(\tau)d\tau\right]. \tag{10.6.42}$$

For the later convenience, we set

$$v^2 \equiv 2C\beta. \tag{10.6.43}$$

Writing

$$F \equiv \int_0^{\beta} X(\sigma)d\sigma, \tag{10.6.44a}$$

and

$$Y(\varsigma) \equiv X(\varsigma) - \frac{2CF}{v^2}, \tag{10.6.44b}$$

we can reduce the given integro-differential equation for $X(\varsigma)$ to the second-order ordinary differential equation for $Y(\varsigma)$ as

$$\frac{d^2}{d\varsigma^2}Y(\varsigma) = v^2 Y(\varsigma) - f(\varsigma). \tag{10.6.45}$$

Observing that

$$\left(\frac{d^2}{d\varsigma^2} - v^2\right)\exp[-v\,|\varsigma - \sigma|] = -2v\delta(\varsigma - \sigma),$$

we obtain the general solution for $Y(\varsigma)$ as

$$Y(\varsigma) = X(\varsigma) - \frac{2CF}{v^2} = P\exp[-v\varsigma] + Q\exp[v\varsigma] + \frac{1}{2v}\int_0^\beta f(\sigma)\exp[-v\,|\varsigma - \sigma|]d\sigma. \tag{10.6.46}$$

We shall determine the integration constants for $X(\varsigma)$ from the boundary conditions of $X(\varsigma)$ in terms of F as

$$P + Q = -\frac{2CF}{v^2} - \frac{ik}{v}\{\exp[-v\tau] - \exp[-v\sigma]\},$$

$$P\exp[-v\beta] + Q\exp[v\beta] = -\frac{2CF}{v^2} - \frac{ik}{v}\{\exp[-v(\beta - \tau)] - \exp[-v(\beta - \sigma)]\}.$$

From this, we obtain

$$Q = -\frac{CF(1 - \exp[-v\beta])}{v^2\sinh v\beta} - \frac{ik\exp[-v\beta](\sinh v\tau - \sinh v\sigma)}{2v\sinh v\beta},$$

$$P = -\frac{2CF}{v^2} - \frac{ik}{v}(\exp[-v\tau] - \exp[-v\sigma]) - Q.$$

In order to determine F from the definition for $\beta \to \infty$, we first observe that $Q = 0$ as $\beta \to \infty$, and hence we have

$$P = -\frac{2CF}{v^2} - \frac{ik}{v}(\exp[-v\tau] - \exp[-v\sigma]) \quad \text{as} \quad \beta \longrightarrow \infty, \tag{10.6.47}$$

so that

$$P + \frac{1}{2}\int_0^\infty d\varsigma \int_0^\infty d\eta ik[\delta(\eta - \tau) - \delta(\eta - \sigma)]\exp[-v\,|\varsigma - \eta|] = 0.$$

Carrying out the integral above and comparing with the expression for P as $\beta \to \infty$ given above, we obtain

$$P = -\frac{ik}{2v}(\exp[-v\sigma] - \exp[-v\tau]) \quad \text{and} \quad \frac{2CF}{v^2} = \frac{ik}{v}(\exp[-v\sigma] - \exp[-v\tau]),$$

so that

$$X(\varsigma) = \frac{ik}{v}(\exp[-v\sigma] - \exp[-v\tau]) - \frac{ik}{2v}(\exp[-v\sigma] - \exp[-v\tau])\exp[-v\varsigma] + \frac{1}{2v}\int_0^\infty f_x(\sigma)\exp[-v\,|\varsigma - \sigma|]d\sigma. \tag{10.6.48}$$

From (10.6.42), we have

$$J_x = \exp\left[\frac{ik_x}{2}(X(\tau) - X(\sigma))\right].$$

From (10.6.48), we have

$$X(\tau) - X(\sigma) = ik_x\left(\frac{(\exp[-v\tau] - \exp[-v\sigma])^2}{2v} + \frac{1 - \exp[-v\,|\tau - \sigma|]}{v}\right).$$

We have the identical results for J_y and J_z with the replacement of k_x with k_y and k_z, respectively. Thus we obtain

$$\begin{aligned}&\left\langle \exp[i\vec{k}\cdot(\vec{q}(\tau) - \vec{q}(\sigma))]\right\rangle \\ &\quad = \exp\left[-\vec{k}^2\left(\frac{(\exp[-v\tau] - \exp[-v\sigma])^2}{4v} + \frac{1 - \exp[-v\,|\tau - \sigma|]}{2v}\right)\right].\end{aligned} \tag{10.6.49}$$

Substituting (10.6.49) into (10.6.33) and carrying out the integral over $\vec{k}$, we obtain

$$\begin{aligned}\left\langle \frac{1}{|\vec{q}(\tau) - \vec{q}(\sigma)|}\right\rangle &= \frac{1}{2\pi^2}\int \frac{\left\langle \exp[i\vec{k}\cdot(\vec{q}(\tau) - \vec{q}(\sigma))]\right\rangle}{\vec{k}^2}d^3\vec{k} \\ &= \frac{1}{\pi^{1/2}}\left[\frac{(\exp[-v\tau] - \exp[-v\sigma])^2}{4v}\right. \\ &\qquad \left. + \frac{1 - \exp[-v\,|\tau - \sigma|]}{2v}\right]^{-1/2}.\end{aligned} \tag{10.6.50}$$

From (10.6.30), we have

$$\begin{aligned}A = {} & \frac{\alpha}{\sqrt{2\pi}\beta}\int_0^\beta d\tau \int_\tau^\beta d\sigma \\ & \times \left[\frac{(\exp[-v\tau] - \exp[-v\sigma])^2}{4v} + \frac{1 - \exp[-v\,|\tau - \sigma|]}{2v}\right]^{-1/2} \\ & \times \exp[-(\sigma - \tau)].\end{aligned} \tag{10.6.51}$$

Setting

$$\sigma = \tau + x,$$

we find that the first term inside the square bracket above is negligible in the limit

$$\beta \longrightarrow \infty \quad (v \longrightarrow \infty)$$

so that we have

$$A = \alpha\sqrt{\frac{v}{\pi}}\int_0^\infty \frac{\exp[-x]dx}{(1 - \exp[-vx])^{1/2}}. \tag{10.6.52}$$

To compute B, we expand both sides of (10.6.49) in power of $\vec{k}^2$ and equate the coefficient of the $\vec{k}^2$ term. Noting

$$\left\langle (X(\tau) - X(\sigma))^2 \right\rangle = \frac{1}{3}\left\langle (\vec{q}(\tau) - \vec{q}(\sigma))^2 \right\rangle,$$

we have

$$\frac{1}{6}\left\langle (\vec{q}(\tau) - \vec{q}(\sigma))^2 \right\rangle = \frac{(\exp[-v\tau] - \exp[-v\sigma])^2}{4v} + \frac{1 - \exp[-v\,|\tau - \sigma|]}{2v},$$

and from (10.6.31), we obtain

$$B = \frac{3v}{4}.$$

To compute E_1, we take the logarithm of the following expression:

$$\int \mathcal{D}\,[\vec{q}]\exp[I_1] \sim \exp[-E_1\beta] \quad \text{as} \quad \beta \longrightarrow \infty, \tag{10.6.53}$$

and take the derivative with respect to C to obtain

$$C\frac{dE_1}{dC} = \frac{C}{2\beta}\int_0^\beta d\tau \int_0^\beta d\sigma \left\langle (\vec{q}(\tau) - \vec{q}(\sigma))^2 \right\rangle = B = \frac{3v}{4}.$$

We integrate this equation with the boundary condition

$$E_1 = 0 \quad \text{for} \quad C = 0,$$

obtaining

$$E_1 = \frac{3v}{2}.$$

Since we have obtained $s = B + A = (3v/4) + A$, we have

$$E = E_1 - s = \frac{3v}{2} - \left(\frac{3v}{4} + A\right) = \frac{3v}{4} - A. \tag{10.6.54}$$

As an application of (10.6.54), we consider the case $v \gg 1$. In this case, we can ignore $\exp[-vx]$ term in (10.6.52), resulting in

$$A \approx \alpha\sqrt{\frac{v}{\pi}}\int_0^\infty \exp[-x]dx = \alpha\sqrt{\frac{v}{\pi}}.$$

With this A, we have E as

$$E = \frac{3v}{4} - \alpha\sqrt{\frac{v}{\pi}}, \tag{10.6.55}$$

attaining the minimum at

$$v = \frac{4\alpha^2}{9\pi} \quad \text{with} \quad E = -\frac{\alpha^2}{3\pi}.$$

We observe that

$$v = 4\alpha^2 \big/ 9\pi \gg 1 \Rightarrow \alpha \gg 1.$$

Hence we have the strong coupling result.

Feynman employed the trial action functional,

$$I_1 = -\frac{1}{2}\int \left(\frac{d\vec{q}}{d\tau}\right)^2 d\tau - \frac{1}{2}C \iint \left(\vec{q}(\tau) - \vec{q}(\sigma)\right)^2 \exp[-w\,|\tau - \sigma\,|]d\tau\, d\sigma.$$

What we discussed so far corresponds to the case $w = 0$. We can perform similar but lengthy calculation to obtain

$$E = \frac{3}{4v}(v - w)^2 - A, \tag{10.6.56}$$

with

$$A = \frac{\alpha v}{\sqrt{\pi}} \int_0^\infty \left[w^2 x + \frac{v^2 - w^2}{v}(1 - \exp[-vx])\right]^{-1/2} \exp[-x]dx. \tag{10.6.57}$$

Here v and w are the variational parameters. If we set $w = 0$ in (10.6.56) and (10.6.57), we recover our results,

$$E = \frac{3v}{4} - A,$$
$$A = \alpha\sqrt{\frac{v}{\pi}} \int_0^\infty \frac{\exp[-x]dx}{(1 - \exp[-vx])^{1/2}}.$$

We can compute the effective mass $m^*(\alpha)$ and the mobility of the polaron as well.

The polaron problem has a long history. It was introduced for the first time by Landau in 1933. Feynman showed the essence of this problem in 1950 by performing similar calculations in *QED*. The polaron problem to this day still attracts considerable attention due to the existence of the large number of different physical problems with the same conceptual origin. Others in this category include problems in electrodynamics, gravitation, quark models of hadrons, and so on. The method of collective coordinates, which is used in quantum statistical mechanics and quantum field theory, was developed in conjunction with the polaron problem.

The general feature of the polaron problem was explained by Adler in 1982, who considered the two fields, a "light field" and a "heavy field." If it is somehow possible to path-integrate out the heavy field, the problem of evaluation of the transformation function is reduced to that of path integral over the light field with a rather complicated effective action functional for the light field. The distinction between the light field and the heavy field refers to the bulk masses of the physical system under consideration.

The treatment of the polaron problem in Feynman's variational principle in quantum statistical mechanics is based on the particle trajectory picture. The particle trajectory picture was once abandoned in 1920s at the time of the birth of quantum mechanics. It was subsequently resurrected in 1942 by Feynman in his formulation of quantum mechanics based on the notion of the sum over all possible histories. It was then applied to the polaron problem in 1955 by Feynman by path-integrating out the phonon degrees of freedom and retaining the electron degrees of freedom for the variational calculation. In relativistic quantum field theory, the particle trajectory technique was originally applied to the pseudoscalar meson theory in 1956, and recently applied for the Monte Carlo simulation of the scalar meson theory in 1996.

Back in 1983, the direct path-integral treatment of the polaron problem was carried out, after the phonon degrees of freedom was path-integrated out, by Fourier transforming the electron coordinate and Laplace transforming the electron time in the evaluation of the transformation function . The result is the standard many-body graphical perturbation theory. The ground state energy and the effective mass of the polaron were obtained perturbatively in the limit of weak coupling.

As for the details of Feynman's variational principle in quantum statistical mechanics and its application to the polaron problem, we refer the reader to the monographs by R.P. Feynman and A.R. Hibbs, *Quantum Mechanics and Path Integrals*, and R.P. Feynman, *Statistical Mechanics*.

10.7 Poincare Transformation and Spin

The operator $\hat{\psi}(x)$ acts in Hilbert space and $|\phi\rangle$ is the state vector in Hilbert space. Under *Poincaré transformation*,

$$x^{\mu} \longrightarrow x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + a^{\mu},$$

the state vector $|\phi\rangle$ transforms as

$$\left|\phi\right\rangle \longrightarrow \left|\phi'\right\rangle = U(\Lambda, a)\left|\phi\right\rangle,$$

where $U(\Lambda, a)$ is a *unitary operator of Poincaré group* in Hilbert space.

Transformation law of the wavefunction is given by, with the use of the *unitary representation matrix* $L(\Lambda)$ of space–time rotation Λ upon $\psi(x)$ as

$$\psi(x) \longrightarrow \psi'(x') = L(\Lambda)\psi(x).$$

Transformation law of the field operator is given by

$$\hat{\psi}(x) \longrightarrow U(\Lambda, a)\hat{\psi}(x)U^{-1}(\Lambda, a) = L^{-1}(\Lambda)\hat{\psi}(x'). \qquad (10.7.1)$$

Rational for the field operator transformation law is that the matrix element of the field operator at the transformed point x', $\hat{\psi}(x')$, evaluated between the transformed states should act like the *wavefunction*. So we demand that the following equation is true:

$$\left\langle \phi_{\alpha'} \left| \hat{\psi}(x') \right| \phi_{\beta'} \right\rangle = L(\Lambda) \left\langle \phi_{\alpha} \left| \hat{\psi}(x) \right| \phi_{\beta} \right\rangle.$$

Thus we have

$$\left\langle \phi_{\alpha} \left| U^{-1}(\Lambda, a)\hat{\psi}(x')U(\Lambda, a) \right| \phi_{\beta} \right\rangle = L(\Lambda) \left\langle \phi_{\alpha} \left| \hat{\psi}(x) \right| \phi_{\beta} \right\rangle,$$

or, as the operator identity, we have

$$U^{-1}(\Lambda, a)\hat{\psi}(x')U(\Lambda, a) = L(\Lambda)\hat{\psi}(x),$$

which is inverted as

$$U(\Lambda, a)\hat{\psi}(x)U^{-1}(\Lambda, a) = L^{-1}(\Lambda)\hat{\psi}(x').$$

Consider the *infinitesimal Poincaré transformation*

$$x^{\mu} \longrightarrow x'\mu = x^{\mu} + \delta x^{\mu}; \quad \delta x^{\mu} = \delta\omega^{\mu\nu}x_{\nu} + \delta\varepsilon^{\mu}.$$

For an infinitesimal Poincaré transformation, we have the unitary operator

$$U(1 + \delta\omega, \delta\varepsilon) \equiv 1 + i\delta\varepsilon^{\mu}P_{\mu} - \frac{i}{2}\delta\omega^{\mu\nu}J_{\mu\nu}, \qquad (10.7.2)$$

where P_{μ} ($J_{\mu\nu}$) is the generator of space–time translation (rotation).
Composition law of two successive Poincaré transformations parameterized by (Λ_1, a_1) and (Λ_2, a_2) is given by

$$\begin{cases} (\Lambda_2, a_2)(\Lambda_1, a_1) & = & (\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2), \\ U(\Lambda_2, a_2)U(\Lambda_1, a_1) & = & U(\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2). \end{cases} \qquad (10.7.3)$$

Consider the following *three successive Poincaré transformations*:

$$U^{-1}(\Lambda, a)U(1 + \delta\omega, \delta\varepsilon)U(\Lambda, a).$$

Since we know

$$U^{-1}(\Lambda, a)U(\Lambda, a) = U(1, 0),$$

we have

$$U^{-1}(\Lambda, a) = U(\Lambda^{-1}, -\Lambda^{-1}a).$$

We can simplify the above Poincaré transformations as

$$\begin{aligned} &U^{-1}(\Lambda, a)\, U(1+\delta\omega, \delta\varepsilon)\, U(\Lambda, a) \\ &\quad = U(\Lambda^{-1}, -\Lambda^{-1}a)\, U(1+\delta\omega, \delta\varepsilon)\, U(\Lambda, a) \\ &\quad = U(\Lambda^{-1}, -\Lambda^{-1}a)\, U((1+\delta\omega)\Lambda, (1+\delta\omega)a + \delta\varepsilon) \\ &\quad = U(\Lambda^{-1}(1+\delta\omega)\Lambda, \Lambda^{-1}[(1+\delta\omega)a + \delta\varepsilon] - \Lambda^{-1}a) \\ &\quad = U(1 + \Lambda^{-1}\delta\omega\Lambda, \Lambda^{-1}\delta\omega a + \Lambda^{-1}\delta\varepsilon). \end{aligned}$$

Thus we have

$$\begin{aligned} &U^{-1}(\Lambda, a)(1 + i\delta\varepsilon^{\mu} P_{\mu} - \frac{i}{2}\delta\omega^{\mu\nu} J_{\mu\nu})\, U(\Lambda, a) \\ &\quad = 1 + i(\Lambda^{-1}\delta\omega a + \Lambda^{-1}\delta\varepsilon)^{\lambda} P_{\lambda} - \frac{i}{2}(\Lambda^{-1}\delta\omega\Lambda)^{\lambda\kappa} J_{\lambda\kappa}. \end{aligned}$$

Recalling that $(\Lambda^{-1})^{\mu}{}_{\nu} = \Lambda_{\nu}{}^{\mu}$, we have

$$\begin{aligned} (\Lambda^{-1}\delta\omega\Lambda)^{\lambda\kappa} &= (\Lambda^{-1})^{\lambda}{}_{\mu}\delta\omega^{\mu\nu}\Lambda_{\nu}{}^{\kappa} = \delta\omega^{\mu\nu}\Lambda_{\mu}{}^{\lambda}\Lambda_{\nu}{}^{\kappa}, \\ (\Lambda^{-1}\delta\omega a + \Lambda^{-1}\delta\varepsilon)^{\lambda} &= (\Lambda^{-1})^{\lambda}{}_{\mu}(\delta\omega^{\mu\nu} a_{\nu} + \delta\varepsilon^{\mu}) \\ &= \Lambda_{\mu}{}^{\lambda}\delta\omega^{\mu\nu} a_{\nu} + \Lambda_{\mu}{}^{\lambda}\delta\varepsilon^{\mu} \\ &= \frac{1}{2}\delta\omega^{\mu\nu}(\Lambda_{\mu}{}^{\lambda} a_{\nu} - \Lambda_{\nu}{}^{\lambda} a_{\mu}) + \delta\varepsilon^{\mu}\Lambda_{\mu}{}^{\lambda}. \end{aligned}$$

We thus have

$$\begin{aligned} &U^{-1}(\Lambda, a)(1 + i\delta\varepsilon^{\mu} P_{\mu} - \frac{i}{2}\delta\omega^{\mu\nu} J_{\mu\nu})\, U(\Lambda, a) \\ &\quad = 1 + i\delta\varepsilon^{\mu}\Lambda_{\mu}{}^{\lambda} P_{\lambda} - \frac{i}{2}\delta\omega^{\mu\nu}\{\Lambda_{\mu}{}^{\lambda}\Lambda_{\nu}{}^{\kappa} J_{\lambda\kappa} + (a_{\mu}\Lambda_{\nu}{}^{\lambda} - a_{\nu}\Lambda_{\mu}{}^{\lambda}) P_{\lambda}\}. \end{aligned}$$

Hence, for arbitrary $\delta\varepsilon$ and $\delta\omega$, we have

$$\begin{cases} U^{-1}(\Lambda, a) P_{\mu} U(\Lambda, a) = \Lambda_{\mu}{}^{\lambda} P_{\lambda}, \\ U^{-1}(\Lambda, a) J_{\mu\nu} U(\Lambda, a) = \Lambda_{\mu}{}^{\lambda}\Lambda_{\nu}{}^{\kappa} J_{\lambda\kappa} + (a_{\mu}\Lambda_{\nu}{}^{\lambda} - a_{\nu}\Lambda_{\mu}{}^{\lambda}) P_{\lambda}. \end{cases}$$

We further specialize to the following case:

$$\Lambda^{\mu}{}_{\nu} = \eta^{\mu}{}_{\nu} + \delta\omega^{\mu}{}_{\nu}, \quad a^{\mu} = \delta\varepsilon^{\mu}.$$

$\mathbf{P}_{\mu}$ transformation:

$$U^{-1}(1+\delta\omega, \delta\varepsilon) P_{\mu} U(1+\delta\omega, \delta\varepsilon) = P_{\mu} + \delta\omega_{\mu\nu} P^{\nu}.$$

We recall

$$\begin{cases} U(1+\delta\omega,\delta\varepsilon) & = & 1+i\delta\varepsilon^{\nu}P_{\nu}-\frac{i}{2}\delta\omega^{\lambda\kappa}J_{\lambda\kappa}, \\ U^{-1}(1+\delta\omega,\delta\varepsilon) & = & 1-i\delta\varepsilon^{\nu}P_{\nu}+\frac{i}{2}\delta\omega^{\lambda\kappa}J_{\lambda\kappa}. \end{cases}$$

To the first order in the infinitesimals, $\delta\varepsilon$ and $\delta\omega$, we have

$$P_{\mu}-i\delta\varepsilon^{\nu}[P_{\nu},P_{\mu}]+\frac{i}{2}\delta\omega^{\lambda\kappa}[J_{\lambda\kappa},P_{\mu}]=P_{\mu}+\delta\omega_{\mu\nu}P^{\nu}.$$

From $\delta\varepsilon$ term, we obtain

$$[P_{\nu},P_{\mu}]=0. \tag{10.7.4}$$

From $\delta\omega$ term, we observe

$$\delta\omega_{\mu\nu}P^{\nu}=\delta\omega^{\lambda\kappa}\eta_{\lambda\mu}P_{\kappa}=\frac{1}{2}\delta\omega^{\lambda\kappa}(\eta_{\lambda\mu}P_{\kappa}-\eta_{\kappa\mu}P_{\lambda}),$$

and obtain

$$i[P_{\mu},J_{\lambda\kappa}]=\eta_{\kappa\mu}P_{\lambda}-\eta_{\lambda\mu}P_{\kappa}. \tag{10.7.5}$$

$J_{\mu\nu}$ transformation:

$$U^{-1}(\Lambda,a)J_{\mu\nu}U(\Lambda,a)=J_{\mu\nu}-\delta\omega_{\mu}{}^{\lambda}J_{\lambda\nu}-\delta\omega_{\nu}{}^{\kappa}J_{\mu\kappa}+\delta\varepsilon_{\mu}P_{\nu}-\delta\varepsilon_{\nu}P_{\mu}.$$

Thus we have

$$J_{\mu\nu}-i\delta\varepsilon^{\sigma}[P_{\sigma},J_{\mu\nu}]+\frac{i}{2}\delta\omega^{\lambda\kappa}[J_{\lambda\kappa},J_{\mu\nu}]=J_{\mu\nu}+\delta\varepsilon_{\mu}P_{\nu}-\delta\varepsilon_{\nu}P_{\mu}-\delta\omega_{\mu}{}^{\lambda}J_{\lambda\nu}-\delta\omega_{\nu}{}^{\kappa}J_{\mu\kappa}.$$

We observe that

$$\begin{aligned}\delta\omega_{\mu}{}^{\lambda}J_{\lambda\nu}+\delta\omega_{\nu}{}^{\kappa}J_{\mu\kappa}&=\eta_{\mu\kappa}\delta\omega^{\kappa\lambda}J_{\lambda\nu}+\eta_{\nu\lambda}\delta\omega^{\lambda\kappa}J_{\mu\kappa}\\&=\frac{1}{2}\delta\omega^{\lambda\kappa}(\eta_{\mu\kappa}J_{\nu\lambda}-\eta_{\mu\lambda}J_{\nu\kappa}+\eta_{\nu\lambda}J_{\mu\kappa}-\eta_{\nu\kappa}J_{\mu\lambda}).\end{aligned}$$

We thus obtain

$$i[J_{\mu\nu},J_{\lambda\kappa}]=\eta_{\mu\lambda}J_{\nu\kappa}-\eta_{\mu\kappa}J_{\nu\lambda}+\eta_{\nu\kappa}J_{\mu\lambda}-\eta_{\nu\lambda}J_{\mu\kappa}. \tag{10.7.6}$$

We finally consider the *transformation property* of the field operator with the allowance for the case of the *multicomponent field operator*. Transformation law of the field operator is given by

$$\hat{\psi}_{\rho}(x)\longrightarrow U(\Lambda,a)\hat{\psi}_{\rho}(x)U^{-1}(\Lambda,a)=(L^{-1}(\Lambda))_{\rho\sigma}\hat{\psi}_{\sigma}(x').$$

Infinitesimal translation:

$$\begin{cases} U(1,\delta\varepsilon) &= 1 + i\delta\varepsilon^{\mu}P_{\mu}, \\ x'^{\mu} &= x^{\mu} + \delta\varepsilon^{\mu}, \end{cases} \qquad (L(\Lambda = 1))_{\rho\sigma} = \delta_{\rho\sigma}.$$

We observe that

$$\hat{\psi}_{\rho}(x) + i\delta\varepsilon^{\mu}[P_{\mu}, \hat{\psi}_{\rho}(x)] = \hat{\psi}_{\rho}(x + \delta\varepsilon) = \hat{\psi}_{\rho}(x) + \delta\varepsilon^{\mu}\partial_{\mu}\hat{\psi}_{\rho}(x).$$

We thus obtain

$$i[P_{\mu}, \hat{\psi}_{\rho}(x)] = \partial_{\mu}\hat{\psi}_{\rho}(x). \tag{10.7.7}$$

Infinitesimal rotation:

$$\begin{cases} U(1+\delta\omega, 0) &= 1 - \frac{i}{2}\delta\omega^{\mu\nu}J_{\mu\nu}, \\ x'^{\mu} &= x^{\mu} + \delta\omega^{\mu\nu}x_{\nu}, \end{cases} \qquad (L(1+\delta\omega))_{\rho\sigma} = (1 + \frac{i}{2}\delta\omega^{\mu\nu}S_{\mu\nu})_{\rho\sigma}.$$

We observe that

$$\begin{aligned} \hat{\psi}_{\rho}(x) - \frac{i}{2}\delta\omega^{\mu\nu}[J_{\mu\nu}, \hat{\psi}_{\rho}(x)] &= (1 - \frac{i}{2}\delta\omega^{\mu\nu}S_{\mu\nu})_{\rho\sigma}\hat{\psi}_{\sigma}(x + \delta\omega\cdot x) \\ &= (1 - \frac{i}{2}\delta\omega^{\mu\nu}S_{\mu\nu})_{\rho\sigma}(\hat{\psi}_{\sigma}(x) + \delta\omega^{\mu\nu}x_{\nu}\partial_{\mu}\hat{\psi}_{\sigma}(x)) \\ &= \hat{\psi}_{\rho}(x) - \frac{i}{2}\delta\omega^{\mu\nu}(S_{\mu\nu})_{\rho\sigma}\hat{\psi}_{\sigma}(x) + \delta\omega^{\mu\nu}x_{\nu}\partial_{\mu}\hat{\psi}_{\rho}(x) \\ &= \hat{\psi}_{\rho}(x) - \frac{i}{2}\delta\omega^{\mu\nu}(S_{\mu\nu})_{\rho\sigma}\hat{\psi}_{\sigma}(x) + \frac{1}{2}\delta\omega^{\mu\nu}(x_{\nu}\partial_{\mu} - x_{\mu}\partial_{\nu})\hat{\psi}_{\rho}(x). \end{aligned}$$

We thus obtain

$$\begin{aligned} [J_{\mu\nu}, \hat{\psi}_{\rho}(x)] &= (S_{\mu\nu})_{\rho\sigma}\hat{\psi}_{\sigma}(x) + i(x_{\nu}\partial_{\mu} - x_{\mu}\partial_{\nu})\hat{\psi}_{\rho}(x) \\ &= (S_{\mu\nu})_{\rho\sigma}\hat{\psi}_{\sigma}(x) + (x_{\mu}\frac{1}{i}\partial_{\nu} - x_{\nu}\frac{1}{i}\partial_{\mu})\hat{\psi}_{\rho}(x). \end{aligned} \tag{10.7.8}$$

The mixing matrix $S_{\mu\nu}$ of the multicomponent field is thus identified as the spin of the field operator.

10.8 Conservation Laws and Noether's Theorem

There exists the close relationship between the continuous symmetries of the Lagrangian $L(q_r(t), \dot{q}_r(t), t)$ (the Lagrangian density $\mathcal{L}(\psi_a(x), \partial_{\mu}\psi_a(x))$) and the conservation laws, commonly known as Noether's theorem.

We shall first consider classical mechanics, described by the Lagrangian $L(q_r(t), \dot{q}_r(t), t)$. Suppose that when we perform the variation,

$$q_r(t) \longrightarrow q_r(t) + \delta q_r(t),$$

we have

$$\delta L(q_r(t), \dot{q}_r(t), t) = \frac{d}{dt}\delta\Lambda, \tag{10.8.1}$$

where $\delta\Lambda$ is the function of $q_r(t)$'s and $\dot{q}_r(t)$'s. Then, without the use of Lagrange equation of motion, we have

$$\sum_{r=1}^{f}\left[\frac{\partial L}{\partial q_r}\delta q_r + \frac{\partial L}{\partial \dot{q}_r}\delta\dot{q}_r\right] - \frac{d}{dt}\delta\Lambda = 0.$$

Now, with the use of Lagrange equation of motion, we have

$$\sum_{r=1}^{f}\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right)\delta q_r + \frac{\partial L}{\partial \dot{q}_r}\frac{d}{dt}\delta q_r\right] - \frac{d}{dt}\delta\Lambda = \frac{d}{dt}\left[\sum_{r=1}^{f}\frac{\partial L}{\partial \dot{q}_r}\delta q_r - \delta\Lambda\right] = 0. \tag{10.8.2}$$

We have the *conserved Noether charge* δQ as

$$\delta Q = \sum_{r=1}^{f}\frac{\partial L}{\partial \dot{q}_r}\delta q_r - \delta\Lambda, \tag{10.8.3}$$

which is independent of t.

For the ease of presentation, we consider the n particle system with the three spatial dimension,

$$\vec{x}_a = (q_{3(a-1)+1}, q_{3(a-1)+2}, q_{3(a-1)+3}), \quad a = 1, \ldots, n, \quad f = 3n.$$

(1) $\delta\mathbf{x}_a = \boldsymbol{\delta}$, a fixed displacement with $L(q_r, \dot{q}_r, t)$ invariant,

$$\delta Q = \sum_{a=1}^{n}\mathbf{p}_a \cdot \boldsymbol{\delta} = \boldsymbol{\delta}\cdot\sum_{a=1}^{n}\mathbf{p}_a = \boldsymbol{\delta}\cdot\sum_{a=1}^{n}\nabla_{\dot{\mathbf{x}}_a}L,$$

$$\nabla_{\dot{\mathbf{x}}_a} = \hat{\mathbf{e}}_x\frac{\partial}{\partial \dot{x}_a} + \hat{\mathbf{e}}_y\frac{\partial}{\partial \dot{y}_a} + \hat{\mathbf{e}}_z\frac{\partial}{\partial \dot{z}_a}.$$

The following quantity is conserved:

$$\mathbf{P} = \sum_{a=1}^{n}\nabla_{\dot{\mathbf{x}}_a}L. \tag{10.8.4}$$

Total momentum conservation results from the translational invariance of L.
(2) $\delta\mathbf{x}_a = \delta\boldsymbol{\theta} \times \mathbf{x}_a$, a fixed rotation about the origin with $L(q_r, \dot{q}_r, t)$ invariant,

$$\delta Q = \sum_{a=1}^{n} \nabla_{\dot{\mathbf{x}}_a} L \cdot (\delta\boldsymbol{\theta} \times \mathbf{x}_a) = \delta\boldsymbol{\theta} \cdot \sum_{a=1}^{n} \mathbf{x}_a \times \nabla_{\dot{\mathbf{x}}_a} L.$$

The following quantity is conserved:

$$\mathbf{L} = \sum_{a=1}^{n} \mathbf{x}_a \times \nabla_{\dot{\mathbf{x}}_a} L. \tag{10.8.5}$$

Total angular momentum conservation results from the rotational invariance of L.
(3) The Lagrangian $L(q_r, \dot{q}_r, t)$ does not depend on t explicitly. Then we have

$$\delta q_r = \dot{q}_r \delta t \quad \text{and} \quad \delta\dot{q}_r(t) = \ddot{q}_r \delta t \quad \text{under} \quad t \longrightarrow t + \delta t,$$

so that, with the use of Lagrange equation of motion, we have

$$\begin{aligned} \delta L &= \sum_{r=1}^{f} \left[\frac{\partial L}{\partial q_r} \dot{q}_r + \frac{\partial L}{\partial \dot{q}_r} \ddot{q}_r \right] \delta t = \sum_{r=1}^{f} \left[\left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_r} \right) \dot{q}_r + \frac{\partial L}{\partial \dot{q}_r} \frac{d}{dt} \dot{q}_r \right] \delta t \\ &= \frac{d}{dt} \sum_{r=1}^{f} \left[\left(\frac{\partial L}{\partial \dot{q}_r} \right) \dot{q}_r \right] \delta t = \frac{dL}{dt} \delta t. \end{aligned}$$

The following quantity is conserved:

$$E = \sum_{r=1}^{f} \frac{\partial L}{\partial \dot{q}_r} \dot{q}_r - L. \tag{10.8.6}$$

Total energy conservation results from the explicit t independence of L.

We shall now consider classical field theory, described by the Lagrangian density $\mathcal{L}(\psi_a(x), \partial_\mu \psi_a(x))$. Suppose that when we perform the variation,

$$\psi_a(x) \to \psi_a(x) + \delta\psi_a(x),$$

we have $\delta\mathcal{L}(\psi_a(x), \partial_\mu \psi_a(x))$ as

$$\delta\mathcal{L}(\psi_a(x), \partial_\mu \psi_a(x)) = \partial_\mu \delta\Lambda^\mu, \tag{10.8.7}$$

where $\delta\Lambda^\mu$ is the function of $\psi_a(x)$'s and $\partial_\mu \psi_a(x)$'s. Then, without the use of Euler–Lagrange equation of motion, we have

$$\sum_a \left[\frac{\partial\mathcal{L}}{\partial\psi_a} \delta\psi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu \psi_a)} \delta(\partial_\mu \psi_a) \right] - \partial_\mu \delta\Lambda^\mu = 0.$$

Now, with the use of Euler–Lagrange equation of motion, we have

$$\sum_a \left[\left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_a)} \right) \delta\psi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_a)} \partial_\mu(\delta\psi_a) \right] - \partial_\mu \delta\Lambda^\mu$$
$$= \partial_\mu \left[\sum_a \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_a)} \delta\psi_a - \delta\Lambda^\mu \right] = 0. \tag{10.8.8}$$

We have the *conserved Noether current* δC^μ as

$$\delta C^\mu = \sum_a \pi_a^\mu \delta\psi_a - \delta\Lambda^\mu, \quad \text{with} \quad \pi_a^\mu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_a)}. \tag{10.8.9}$$

(4.) $\delta x^\lambda = \varepsilon^\lambda$, a space–time translation with ε^λ constant

$$\delta\psi_a = \varepsilon^\lambda \partial_\lambda \psi_a, \quad \text{and} \quad \delta\Lambda^\mu = \varepsilon^\mu \mathcal{L},$$

where we assume that the Lagrangian density $\mathcal{L}$ has no explicit space–time dependence.

The conserved "current" δC^μ is given by

$$\delta C^\mu = \sum_a \pi_a^\mu \varepsilon^\lambda \partial_\lambda \psi_a - \varepsilon^\mu \mathcal{L} = \varepsilon_\lambda \left(\sum_a \pi_a^\mu \partial^\lambda \psi_a - \eta^{\mu\lambda} \mathcal{L} \right).$$

We call the object in the bracket as the canonical energy–momentum tensor $\Theta^{\mu\lambda}$,

$$\Theta^{\mu\lambda} = \sum_a \pi_a^\mu \partial^\lambda \psi_a - \eta^{\mu\lambda} \mathcal{L}, \quad \text{with} \quad \partial_\mu \Theta^{\mu\lambda} = 0. \tag{10.8.10}$$

We can define the conserved four-vector P^λ, from the canonical energy–momentum tensor $\Theta^{\mu\lambda}$ given above, by

$$P^\lambda = \int d^3\vec{x}\,\Theta^{0\lambda} \quad \text{such that} \quad \frac{d}{dx^0} P^\lambda = 0, \tag{10.8.11}$$

which is the energy–momentum conservation law. The space–time translation charge P^λ generates the space–time translation as it should.

(5) $\delta x^\mu = \varepsilon_{\alpha\beta}(\eta^{\alpha\mu} x^\beta - \eta^{\beta\mu} x^\alpha)$, Lorentz transformation with $\varepsilon_{\alpha\beta}$ infinitesimal and antisymmetric with respect to α and β,

$$\delta\psi_a = \varepsilon_{\alpha\beta} \{ \delta_{ab}(x^\alpha \partial^\beta - x^\beta \partial^\alpha) + S^{\alpha\beta}_{(a,b)} \} \psi_b.$$

Here, $S^{\mu\lambda}_{(a,b)}$ is the spin matrix of the multicomponent field ψ_a, given by

$$S^{\mu\lambda}_{(a,b)} = \begin{cases} 0 & \text{for scalar field,} \\ (i/2)[\gamma^\mu, \gamma^\lambda]_{ab} & \text{for spinor field,} \\ \delta^\mu_a \delta^\lambda_b - \delta^\lambda_a \delta^\mu_b & \text{for vector field.} \end{cases} \tag{10.8.12}$$

Since the Lorentz transformation law is coordinate dependent, the derivative of the field variable, $\partial_\mu \psi_a$, transforms as

$$\delta\partial_\mu \psi_a = \varepsilon_{\alpha\beta}\Big[\Big\{\delta_{ab}(x^\alpha\partial^\beta - x^\beta\partial^\alpha) + S^{\alpha\beta}_{(a,b)}\Big\}\partial_\mu + \delta_{ab}(\eta^\alpha_\mu\partial^\beta - \eta^\beta_\mu\partial^\alpha)\Big]\psi_b,$$

resulting in the Lorentz covariance condition

$$\pi_{\mu a}S^{\alpha\beta}_{(a,b)}\partial^\mu\psi_b + \frac{\partial\mathcal{L}}{\partial\psi_a}S^{\alpha\beta}_{(a,b)}\psi_b = \pi^\beta_a\partial^\alpha\psi_a - \pi^\alpha_a\partial^\beta\psi_a,$$
$$\delta\Lambda^\mu = \varepsilon_{\alpha\beta}(\eta^{\mu\beta}x^\alpha - \eta^{\mu\alpha}x^\beta)\mathcal{L}.$$

The conserved "current" δC^μ is given by

$$\delta C^\mu = (\pi^\mu_a\delta\psi_a - \delta\Lambda^\mu) = \varepsilon_{\alpha\beta}\Big(x^\alpha\Theta^{\mu\beta} - x^\beta\Theta^{\mu\alpha} + \pi^\mu_a S^{\alpha\beta}_{(a,b)}\psi_b\Big).$$

We write the third-rank tensor inside the bracket as

$$M^{\mu\alpha\beta} = x^\alpha\Theta^{\mu\beta} - x^\beta\Theta^{\mu\alpha} + \pi^\mu_a S^{\alpha\beta}_{(a,b)}\psi_b, \quad \text{with} \quad \partial_\mu M^{\mu\alpha\beta} = 0. \tag{10.8.13}$$

We can define the conserved second-rank tensor $L^{\alpha\beta}$, from the third-rank tensor $M^{\mu\alpha\beta}$ given above, by

$$L^{\alpha\beta} = \int d^3\vec{x}M^{0\alpha\beta}, \quad \text{such that} \quad \frac{d}{dx^0}L^{\alpha\beta} = 0, \tag{10.8.14}$$

which is the angular momentum conservation law. To be more precise, the spatial (i, j)-components provide the angular momentum conservation law and the time-spatial $(0, k)$-components provide the conservation law of the Lorentz boost charge. The conserved Lorentz charge $L^{\alpha\beta}$ generates the Lorentz transformation as it should.

The energy–momentum tensor $\Theta^{\mu\lambda}$ is not symmetric with respect to μ and λ in general. We can construct the symmetric energy–momentum tensor $T^{\mu\lambda}$ from the explicit inclusion of the spin degrees of freedom. We define the symmetric energy–momentum tensor $T^{\mu\lambda}$ by

$$T^{\mu\lambda} = \Theta^{\mu\lambda} + \partial_\sigma\Phi^{\sigma\mu\lambda}, \tag{10.8.15}$$

with

$$\Phi^{\sigma\mu\lambda} = \frac{1}{2}\sum_{a,b}(\pi^\sigma_a S^{\mu\lambda}_{(a,b)} - \pi^\mu_a S^{\sigma\lambda}_{(a,b)} - \pi^\lambda_a S^{\sigma\mu}_{(a,b)})\psi_b. \tag{10.8.16}$$

The new tensor $T^{\mu\lambda}$ differs from the old tensor $\Theta^{\mu\lambda}$ by the divergence of $\Phi^{\sigma\mu\lambda}$ with respect to x^σ. The construction of the symmetric energy–momentum tensor $T^{\mu\lambda}$ is properly accomplished with the inclusion of the spin degrees of freedom. We can define the conserved four-vector $\tilde{P}^{\lambda}$, from the symmetric energy–momentum tensor $T^{\mu\lambda}$ given above, by

$$\tilde{P}^{\lambda} = \int d^3\vec{x}T^{0\lambda} \quad \text{such that} \quad \frac{d}{dx^0}\tilde{P}^{\lambda} = 0, \tag{10.8.17}$$

with the requisite identity, $\tilde{P}^{\lambda} = P^{\lambda}$.

We can also define the third-rank tensor $\tilde{M}^{\mu\alpha\beta}$ in terms of the symmetric energy–momentum tensor $T^{\mu\beta}$ given above by

$$\tilde{M}^{\mu\alpha\beta} = x^{\alpha}T^{\mu\beta} - x^{\beta}T^{\mu\alpha}, \quad \text{with} \quad \partial_{\mu}\tilde{M}^{\mu\alpha\beta} = 0, \tag{10.8.18}$$

also leading to the angular momentum conservation law

$$\tilde{L}^{\alpha\beta} = \int d^3\vec{x}\tilde{M}^{0\alpha\beta} \quad \text{such that} \quad \frac{d}{dx^0}\tilde{L}^{\alpha\beta} = 0, \tag{10.8.19}$$

with the requisite identity $\tilde{L}^{\alpha\beta} = L^{\alpha\beta}$.

The space–time translation and the Lorentz transformation are unified to the 10-parameter Poincaré transformation. Beyond the 10-parameter Poincaré transformation, we have the 15-parameter conformal group, which constitutes the space–time translation, the Lorentz transformation, the scale transformation (the dilatation) and the conformal transformation, where the last one consists of the inversion, the translation followed by the inversion.

We consider the scale transformation and the conformal transformation.

(6.) $\delta x^{\mu} = x^{\mu}\delta\rho$, the scale transformation with $\delta\rho$ infinitesimal,

$$\delta\psi_a = \delta\rho(d + x\cdot\partial)\psi_a,$$

and

$$\delta\Lambda^{\mu} = \delta\rho x^{\mu}\mathcal{L},$$

with the scale invariance condition

$$-4\mathcal{L} + \pi_{\mu a}(d+1)\partial^{\mu}\psi_a + \frac{\partial\mathcal{L}}{\partial\psi_a}\cdot d\cdot\psi_a = 0,$$

where d is the scale dimension of the field ψ_a, given by

$$d = \begin{cases} 1, & \text{for} \quad \text{boson field,} \\ 3/2, & \text{for} \quad \text{fermion field.} \end{cases} \tag{10.8.20}$$

The above-stated scale invariance condition demands that the scale dimension of the Lagrangian density $\mathcal{L}$ must be 4. Namely, any dimensional parameter in the Lagrangian density $\mathcal{L}$ is forbidden for the scale invariance . The mass term in the Lagrangian density violates the scale invariance condition.

The conserved "current" δC^{μ} is given by

$$\delta C^{\mu} = \pi_a^{\mu}\delta\psi_a - \delta\Lambda^{\mu} = \delta\rho(x_{\alpha}\Theta^{\mu\alpha} + \pi_a^{\mu}\cdot d\cdot\psi_a).$$

We write the four-vector inside the bracket as

$$D^{\mu} = x_{\alpha}\Theta^{\mu\alpha} + \pi_a^{\mu}\cdot d\cdot\psi_a, \quad \text{with} \quad \partial_{\mu}D^{\mu} = 0. \tag{10.8.21}$$

The dilatation charge D is given by

$$D = \int d^3\vec{x}D^0 \quad \text{such that} \quad \frac{d}{dx^0}D = 0. \tag{10.8.22}$$

The dilatation charge D generates the scale transformation as it should.
(7.) $\delta x^{\mu} = \delta c_{\alpha}(2x^{\alpha}x^{\mu} - \eta^{\alpha\mu}x^2)$, the conformal transformation with the four-vector δc_{α} infinitesimal,

$$\delta\psi_a = \delta c_{\alpha}\left[(2x^{\alpha}x^{\lambda} - \eta^{\alpha\lambda}x^2)\partial_{\lambda}\psi_a + 2x_{\lambda}(\delta_{ab}\eta^{\lambda\alpha}d - S^{\lambda\alpha}_{(a,b)})\psi_b\right],$$

and

$$\delta\Lambda^{\mu} = \delta c_{\alpha}(2x^{\alpha}x^{\mu} - \eta^{\alpha\mu}x^2)\mathcal{L},$$

with the scale invariance condition

$$-4\mathcal{L} + \pi_{\mu a}(d+1)\partial^{\mu}\psi_a + \frac{\partial\mathcal{L}}{\partial\psi_a}\cdot d\cdot\psi_a = 0,$$

plus the second condition that the field virial V^{μ} defined below must be a total divergence,

$$V^{\mu} \equiv \pi_{\alpha a}(\delta_{ab}\eta^{\alpha\mu}d - S^{\alpha\mu}_{(a,b)})\psi_b = \partial_{\lambda}\sigma^{\lambda\mu}$$

for some $\sigma^{\lambda\mu}$.
The conserved "current" δC^{μ} is given by

$$\delta C^{\mu} = \delta c_{\alpha}((2x^{\alpha}x_{\lambda} - \eta^{\alpha}_{\lambda}x^2)\Theta^{\mu\lambda} + 2x_{\lambda}\pi_a^{\mu}(\delta_{ab}\eta^{\lambda\alpha}d - S^{\lambda\alpha}_{(a,b)})\psi_b - 2\sigma^{\alpha\mu}).$$

We write the second-rank tensor inside the bracket as

$$K^{\alpha\mu} = (2x^{\alpha}x_{\lambda} - \eta^{\alpha}_{\lambda}x^2)\Theta^{\mu\lambda} + 2x_{\lambda}\pi_a^{\mu}(\delta_{ab}\eta^{\lambda\alpha}d - S^{\lambda\alpha}_{(a,b)})\psi_b - 2\sigma^{\alpha\mu},$$
$$\text{with } \partial_{\mu}K^{\alpha\mu} = 0. \tag{10.8.23}$$

The conformal charge K^{α} is given by

$$K^{\alpha} = \int d^3\vec{x}K^{\alpha 0} \quad \text{such that} \quad \frac{d}{dx^0}K^{\alpha} = 0. \tag{10.8.24}$$

The conformal charge K^α generates the conformal transformation as it should.

In contradistinction to the infinitesimal transformation forms discussed above, the geometrical meaning of the scale transformation and the conformal transformation becomes clear in the finite transformation forms

$$\begin{cases} \text{scale transformation} & x'^\mu = \exp[\rho]x^\mu, \\ \text{conformal transformation} & x'^\mu = (x^\mu - c^\mu x^2)(1 - 2cx + c^2x^2)^{-1}. \end{cases}$$

The conformal group as a whole is not the invariance group of the nature. The mass term in the Lagrangian density violates the scale invariance. If the scale transformation is the symmetry transformation of the theory, the theory must possess the continuous mass spectrum, which is not the case in the nature. Still for the conformal field theory, the conformal group is important.

All of the transformations discussed above, specifically, the space–time translation, the Lorentz transformation , the scale transformation and the conformal transformation are space–time coordinate dependent. Generally, for the *space–time coordinate transformation*, we have $\delta\Lambda^\mu \neq 0$. On the other hand, for the *internal transformation*, we have $\delta\Lambda^\mu = 0$.

The global internal symmetry, which is space–time independent, results in the internal current conservation. Extension to the local internal symmetry which is space–time dependent is accomplished with the introduction of the appropriate gauge fieldsby invoking Weyl's gauge principle. This has been discussed in the next section.

10.9 Weyl's Gauge Principle

In *electrodynamics*, we have a property known as the *gauge invariance*. In the *theory of gravitational field*, we have a property known as the *scale invariance*. Before the birth of quantum mechanics, H. Weyl attempted to construct the unified theory of classical electrodynamics and gravitational field. But he failed to accomplish his goal. After the birth of quantum mechanics, he realized that the gauge invariance of electrodynamics is not related to the scale invariance of the gravitational field, but is related to the invariance of the matter field $\phi(x)$ under the local phase transformation. The matter field in interaction with the electromagnetic field has a property known as the *charge conservation law* or the *current conservation law*. In this section, we discuss Weyl's gauge principle for the $U(1)$ gauge field and the non-Abelian gauge field, and Kibble's gauge principle for the gravitational field.

Weyl's gauge principle: Electrodynamics is described by the total Lagrangian density with the use of the four-vector potential $A_\mu(x)$ by

$$\mathcal{L}_{\text{tot}} = \mathcal{L}_{\text{matter}}(\phi(x), \partial_\mu\phi(x)) + \mathcal{L}_{\text{int}}(\phi(x), A_\mu(x)) + \mathcal{L}_{\text{gauge}}(A_\mu(x), \partial_\nu A_\mu(x)). \tag{10.9.1}$$

This system is invariant under the local $U(1)$ transformations

$$A_\mu(x) \to A'_\mu(x) \equiv A_\mu(x) - \partial_\mu \varepsilon(x), \tag{10.9.2}$$

$$\phi(x) \to \phi'(x) \equiv \exp[iq\varepsilon(x)]\phi(x). \tag{10.9.3}$$

The interaction Lagrangian density $\mathcal{L}_{\text{int}}(\phi(x), A_\mu(x))$ is generated by the substitution

$$\partial_\mu \phi(x) \to D_\mu \phi(x) \equiv (\partial_\mu + iqA_\mu(x))\phi(x), \tag{10.9.4}$$

in the original matter field Lagrangian density $\mathcal{L}_{\text{matter}}(\phi(x), \partial_\mu \phi(x))$. The derivative $D_\mu \phi(x)$ is called the *covariant derivative* of $\phi(x)$ and transforms exactly like $\phi(x)$,

$$D_\mu \phi(x) \to (D_\mu \phi(x))' = \exp[iq\varepsilon(x)] D_\mu \phi(x), \tag{10.9.5}$$

under the local $U(1)$ transformations, Eqs. (10.9.2) and (10.9.3).

The physical meaning of this local $U(1)$ invariance lies in its weaker version, the global $U(1)$ invariance, namely,

$$\varepsilon(x) = \varepsilon, \quad \text{space–time independent constant.}$$

The global $U(1)$ invariance of the matter field Lagrangian density,

$$\mathcal{L}_{\text{matter}}(\phi(x), \partial_\mu \phi(x)),$$

under the global $U(1)$ transformation of $\phi(x)$,

$$\phi(x) \to \phi''(x) = \exp[iq\varepsilon]\phi(x), \quad \varepsilon = \text{constant}, \tag{10.9.6}$$

in its infinitesimal version,

$$\delta\phi(x) = iq\varepsilon\phi(x), \quad \varepsilon = \text{infinitesimal constant}, \tag{10.9.7}$$

results in

$$\frac{\partial \mathcal{L}_{\text{matter}}(\phi(x), \partial_\mu \phi(x))}{\partial \phi(x)} \delta\phi(x) + \frac{\partial \mathcal{L}_{\text{matter}}(\phi(x), \partial_\mu \phi(x))}{\partial(\partial_\mu \phi(x))} \delta(\partial_\mu \phi(x)) = 0. \tag{10.9.8}$$

With the use of the Euler–Lagrange equation of motion for $\phi(x)$,

$$\frac{\partial \mathcal{L}_{\text{matter}}(\phi(x), \partial_\mu \phi(x))}{\partial \phi(x)} - \partial_\mu \frac{\partial \mathcal{L}_{\text{matter}}(\phi(x), \partial_\mu \phi(x))}{\partial(\partial_\mu \phi(x))} = 0,$$

we obtain the current conservation law

$$\partial_\mu J^\mu_{\text{matter}}(x) = 0, \tag{10.9.9}$$

$$\varepsilon J^{\mu}_{\text{matter}}(x) = \frac{\partial \mathcal{L}_{\text{matter}}(\phi(x), \partial_\mu \phi(x))}{\partial(\partial_\mu \phi(x))} \delta\phi(x). \tag{10.9.10}$$

This in its integrated form becomes the charge conservation law,

$$\frac{d}{dt} Q_{\text{matter}}(t) = 0, \tag{10.9.11}$$

$$Q_{\text{matter}}(t) = \int d^3\vec{x} J^{0}_{\text{matter}}(t, \vec{x}). \tag{10.9.12}$$

Weyl's gauge principle considers the analysis backward. The extension of the "current conserving" global $U(1)$ invariance, Eqs. (10.9.7) and (10.9.8), of the matter field Lagrangian density $\mathcal{L}_{\text{matter}}(\phi(x), \partial_\mu \phi(x))$ to the local $U(1)$ invariance necessitates

(1) the introduction of the $U(1)$ gauge field $A_\mu(x)$, and the replacement of the derivative $\partial_\mu \phi(x)$ in the matter field Lagrangian density with the covariant derivative $D_\mu \phi(x)$,

$$\partial_\mu \phi(x) \rightarrow D_\mu \phi(x) \equiv (\partial_\mu + iqA_\mu(x))\phi(x), \tag{10.9.4}$$

and

(2) the requirement that the covariant derivative $D_\mu \phi(x)$ transforms exactly like the matter field $\phi(x)$ under the local $U(1)$ phase transformation of $\phi(x)$,

$$D_\mu \phi(x) \rightarrow (D_\mu \phi(x))' = \exp[iq\varepsilon(x)] D_\mu \phi(x), \tag{10.9.5}$$

under

$$\phi(x) \rightarrow \phi'(x) \equiv \exp[iq\varepsilon(x)]\phi(x). \tag{10.9.3}$$

From requirement (2), we obtain the transformation law of the $U(1)$ gauge field $A_\mu(x)$ immediately,

$$A_\mu(x) \rightarrow A'_\mu(x) \equiv A_\mu(x) - \partial_\mu \varepsilon(x). \tag{10.9.2}$$

From requirement (2), the local $U(1)$ invariance of $\mathcal{L}_{\text{matter}}(\phi(x), D_\mu \phi(x))$ is also self-evident. In order to give dynamical content to the $U(1)$ gauge field, we introduce the field strength tensor $F_{\mu\nu}(x)$ to the gauge field Lagrangian density $\mathcal{L}_{\text{gauge}}(A_\mu(x), \partial_\nu A_\mu(x))$ by the trick,

$$[D_\mu, D_\nu]\phi(x) = iq(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x))\phi(x) \equiv iqF_{\mu\nu}(x)\phi(x). \tag{10.9.13}$$

From the transformation law of the $U(1)$ gauge field $A_\mu(x)$, Eq. (10.9.2), we observe that the field strength tensor $F_{\mu\nu}(x)$ is the locally invariant quantity

$$F_{\mu\nu}(x) \to F'_{\mu\nu}(x) = \partial_\mu A'_\nu(x) - \partial_\nu A'_\mu(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) = F_{\mu\nu}(x). \tag{10.9.14}$$

As the gauge field Lagrangian density $\mathcal{L}_{\text{gauge}}(A_\mu(x), \partial_\nu A_\mu(x))$, we choose

$$\mathcal{L}_{\text{gauge}}(A_\mu(x), \partial_\nu A_\mu(x)) = -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x). \tag{10.9.15}$$

In this manner, we obtain the total Lagrangian density of the matter-gauge system which is locally $U(1)$ invariant as

$$\mathcal{L}_{\text{tot}} = \mathcal{L}_{\text{matter}}(\phi(x), D_\mu\phi(x)) + \mathcal{L}_{\text{gauge}}(F_{\mu\nu}(x)). \tag{10.9.16}$$

We obtain the interaction Lagrangian density $\mathcal{L}_{\text{int}}(\phi(x), A_\mu(x))$ as

$$\mathcal{L}_{\text{int}}(\phi(x), A_\mu(x)) = \mathcal{L}_{\text{matter}}(\phi(x), D_\mu\phi(x)) - \mathcal{L}_{\text{matter}}(\phi(x), \partial_\mu\phi(x)), \tag{10.9.17}$$

which is the universal coupling generated by Weyl's gauge principle. As a result of the local extension of the global $U(1)$ invariance, we derived the electrodynamics from the current conservation law, Eq. (10.9.9), or the charge conservation law, Eq. (10.9.11).

We shall now consider the extension of the present discussion to the non-Abelian gauge field. We let the semisimple Lie group G be the gauge group. We let the representation of G in the Hilbert space be $U(g)$, and its matrix representation on the field operator $\hat{\psi}_n(x)$ in the internal space be $D(g)$,

$$U(g)\hat{\psi}_n(x)U^{-1}(g) = D_{n,m}(g)\hat{\psi}_m(x), \quad g \in G. \tag{10.9.18}$$

For the element $g_\varepsilon \in G$ continuously connected to the identity of G by the parameter $\{\varepsilon_\alpha\}_{\alpha=1}^N$, we have

$$U(g_\varepsilon) = \exp[i\varepsilon_\alpha T_\alpha] = 1 + i\varepsilon_\alpha T_\alpha + \cdots, \quad T_\alpha : \text{generator of Lie group } G, \tag{10.9.19}$$

$$D(g_\varepsilon) = \exp[i\varepsilon_\alpha t_\alpha] = 1 + i\varepsilon_\alpha t_\alpha + \cdots, \quad t_\alpha : \text{realization of } T_\alpha \text{ on } \hat{\psi}_n(x), \tag{10.9.20}$$

$$[T_\alpha, T_\beta] = iC_{\alpha\beta\gamma} T_\gamma, \tag{10.9.21}$$

$$[t_\alpha, t_\beta] = iC_{\alpha\beta\gamma} t_\gamma. \tag{10.9.22}$$

We shall assume that the action functional $I_{\text{matter}}[\psi_n]$ of the matter field Lagrangian density $\mathcal{L}_{\text{matter}}(\psi_n(x), \partial_\mu \psi_n(x))$, given by

$$I_{\text{matter}}[\psi_n] \equiv \int d^4x \mathcal{L}_{\text{matter}}(\psi_n(x), \partial_\mu \psi_n(x)), \tag{10.9.23}$$

is invariant under the global G transformation,

$$\delta\psi_n(x) = i\varepsilon_\alpha(t_\alpha)_{n,m}\psi_m(x), \quad \varepsilon_\alpha = \text{infinitesimal constant.} \tag{10.9.24}$$

Namely, we have

$$\frac{\partial\mathcal{L}_{\text{matter}}(\psi_n(x), \partial_\mu\psi_n(x))}{\partial\psi_n(x)}\delta\psi_n(x) + \frac{\partial\mathcal{L}_{\text{matter}}(\psi_n(x), \partial_\mu\psi_n(x))}{\partial(\partial_\mu\psi_n(x))}\delta(\partial_\mu\psi_n(x)) = 0. \tag{10.9.25}$$

With the use of the Euler–Lagrange equation of motion,

$$\frac{\partial\mathcal{L}_{\text{matter}}(\psi(x), \partial_\mu\psi(x))}{\partial\psi_n(x)} - \partial_\mu(\frac{\partial\mathcal{L}_{\text{matter}}(\psi(x), \partial_\mu\psi(x))}{\partial(\partial_\mu\psi_n(x))}) = 0, \tag{10.9.26}$$

we have the current conservation law and the charge conservation law,

$$\partial_\mu J^\mu_{\alpha,\text{matter}}(x) = 0, \quad \alpha = 1, \ldots, N, \tag{10.9.27a}$$

where the conserved matter current $J^\mu_{\alpha,\text{matter}}(x)$ is given by

$$\varepsilon_\alpha J^\mu_{\alpha,\text{matter}}(x) = \frac{\partial\mathcal{L}_{\text{matter}}(\psi(x), \partial_\mu\psi(x))}{\partial(\partial_\mu\psi_n(x))}\delta\psi_n(x), \tag{10.9.27b}$$

and

$$\frac{d}{dt}Q^{\text{matter}}_\alpha(t) = 0, \quad \alpha = 1, \ldots, N, \tag{10.9.28a}$$

where the conserved matter charge $Q^{\text{matter}}_\alpha(t)$ is given by

$$Q^{\text{matter}}_\alpha(t) = \int d^3\vec{x} J^0_{\alpha,\text{matter}}(t, \vec{x}), \quad \alpha = 1, \ldots, N. \tag{10.9.28b}$$

Invoking Weyl's gauge principle, we extend the global G invariance of the matter system to the local G invariance of the matter-gauge system under the local G phase transformation,

$$\delta\psi_n(x) = i\varepsilon_\alpha(x)(t_\alpha)_{n,m}\psi_m(x). \tag{10.9.29}$$

Weyl's gauge principle requires the following:

(1) the introduction of the non-Abelian gauge field $A_{\alpha\mu}(x)$ and the replacement of the derivative $\partial_\mu \psi_n(x)$ in the matter field Lagrangian density with the covariant derivative $(D_\mu \psi(x))_n$,

$$\partial_\mu \psi_n(x) \to (D_\mu \psi(x))_n \equiv (\partial_\mu \delta_{n,m} + i(t_\gamma)_{n,m} A_{\gamma\mu}(x))\psi_m(x), \tag{10.9.30}$$

and
(2) the requirement that the covariant derivative $(D_\mu \psi(x))_n$ transforms exactly like the matter field $\psi_n(x)$ under the local G phase transformation of $\psi_n(x)$, Eq. (10.9.29),

$$\delta(D_\mu \psi(x))_n = i\varepsilon_\alpha(x)(t_\alpha)_{n,m}(D_\mu \psi(x))_m, \tag{10.9.31}$$

where t_γ is the realization of the generator T_γ upon the multiplet $\psi_n(x)$.

From Eqs. (10.9.29) and (10.9.31), the infinitesimal transformation law of the non-Abelian gauge field $A_{\alpha\mu}(x)$ follows,

$$\delta A_{\alpha\mu}(x) = -\partial_\mu \varepsilon_\alpha(x) + i\varepsilon_\beta(x)(t_\beta^{\text{adj}})_{\alpha\gamma} A_{\gamma\mu}(x) \tag{10.9.32a}$$

$$= -\partial_\mu \varepsilon_\alpha(x) + \varepsilon_\beta(x) C_{\beta\alpha\gamma} A_{\gamma\mu}(x). \tag{10.9.32b}$$

Then the local G invariance of the gauged matter field Lagrangian density $\mathcal{L}_{\text{matter}}(\psi(x), D_\mu \psi(x))$ becomes self-evident as long as the ungauged matter field Lagrangian density $\mathcal{L}_{\text{matter}}(\psi(x), \partial_\mu \psi(x))$ is globally G invariant.

In order to provide dynamical content to the non-Abelian gauge field $A_{\alpha\mu}(x)$, we introduce the field strength tensor $F_{\gamma\mu\nu}(x)$ by the following trick:

$$[D_\mu, D_\nu]\psi(x) \equiv i(t_\gamma) F_{\gamma\mu\nu}(x)\psi(x), \tag{10.9.33}$$

$$F_{\gamma\mu\nu}(x) = \partial_\mu A_{\gamma\nu}(x) - \partial_\nu A_{\gamma\mu}(x) - C_{\alpha\beta\gamma} A_{\alpha\mu}(x) A_{\beta\nu}(x). \tag{10.9.34}$$

We can easily show that the field strength tensor $F_{\gamma\mu\nu}(x)$ undergoes the local G rotation under the local G transformations, Eqs. (10.9.29) and (10.9.32a), under the adjoint representation

$$\delta F_{\gamma\mu\nu}(x) = i\varepsilon_\alpha(x)(t_\alpha^{\text{adj}})_{\gamma\beta} F_{\beta\mu\nu}(x) \tag{10.9.35a}$$

$$= \varepsilon_\alpha(x) C_{\alpha\gamma\beta} F_{\beta\mu\nu}(x). \tag{10.9.35b}$$

As the Lagrangian density of the non-Abelian gauge field $A_{\alpha\mu}(x)$, we choose

$$\mathcal{L}_{\text{gauge}}(A_{\gamma\mu}(x), \partial_\nu A_{\gamma\mu}(x)) \equiv -\frac{1}{4} F_{\gamma\mu\nu}(x) F_\gamma^{\mu\nu}(x). \tag{10.9.36}$$

The total Lagrangian density $\mathcal{L}_{\text{total}}$ of the matter-gauge system is given by

$$\mathcal{L}_{\text{total}} = \mathcal{L}_{\text{matter}}(\psi(x), D_\mu\psi(x)) + \mathcal{L}_{\text{gauge}}(F_{\gamma\mu\nu}(x)). \tag{10.9.37}$$

The interaction Lagrangian density $\mathcal{L}_{\text{int}}$ consists of two parts due to the nonlinearity of the field strength tensor $F_{\gamma\mu\nu}(x)$ with respect to $A_{\gamma\mu}(x)$,

$$\begin{aligned}\mathcal{L}_{\text{int}} = {} & \mathcal{L}_{\text{matter}}(\psi(x), D_\mu\psi(x)) - \mathcal{L}_{\text{matter}}(\psi(x), \partial_\mu\psi(x)) \\ & + \mathcal{L}_{\text{gauge}}(F_{\gamma\mu\nu}(x)) - \mathcal{L}^{\text{quad}}_{\text{gauge}}(F_{\gamma\mu\nu}(x)),\end{aligned} \tag{10.9.38}$$

which provides the universal coupling just like the $U(1)$ gauge field theory. The conserved current $J^\mu_{\alpha,\text{total}}(x)$ and the conserved charge $\{Q^{\text{total}}_\alpha(t)\}^N_{\alpha=1}$ after the extension to the local G invariance also consist of two parts,

$$J^\mu_{\alpha,\text{total}}(x) \equiv J^{\mu\ \text{gauged}}_{\alpha,\text{matter}}(x) + J^\mu_{\alpha,\text{gauge}}(x) \equiv \frac{\delta I_{\text{total}}[\psi, A_{\alpha\mu}]}{\delta A_{\alpha\mu}(x)}, \tag{10.9.39a}$$

$$I_{\text{total}}[\psi, A_{\alpha\mu}] = \int d^4x \{\mathcal{L}_{\text{matter}}(\psi(x), D_\mu\psi(x)) + \mathcal{L}_{\text{gauge}}(F_{\gamma\mu\nu}(x))\}, \tag{10.9.39b}$$

$$Q^{\text{total}}_\alpha(t) = Q^{\text{matter}}_\alpha(t) + Q^{\text{gauge}}_\alpha(t) = \int d^3\vec{x}\{J^{0\ \text{gauged}}_{\alpha,\text{matter}}(t,\vec{x}) + J^0_{\alpha,\text{gauge}}(t,\vec{x})\}. \tag{10.9.40}$$

We note that the gauged matter current $J^{\mu\ \text{gauged}}_{\alpha,\text{matter}}(x)$ of Eq. (10.9.39a) is not identical to the ungauged matter current $J^\mu_{\alpha,\text{matter}}(x)$ of Eq. (10.9.27b):

$$\varepsilon_\alpha J^\mu_{\alpha,\text{matter}}(x) \text{ of (10.9.27b)} = \frac{\partial\mathcal{L}_{\text{matter}}(\psi(x), \partial_\mu\psi(x))}{\partial(\partial_\mu\psi_n(x))}\delta\psi_n(x),$$

whereas after the local G extension,

$$\begin{aligned}\varepsilon_\alpha J^{\mu\ \text{gauged}}_{\alpha,\text{matter}}(x) \text{ of (10.9.39a)} &= \frac{\partial\mathcal{L}_{\text{matter}}(\psi(x), D_\mu\psi(x))}{\partial(D_\mu\psi_n(x))}\delta\psi_n(x) \\ &= \frac{\partial\mathcal{L}_{\text{matter}}(\psi(x), D_\mu\psi(x))}{\partial(D_\mu\psi(x))_n} i\varepsilon_\alpha(t_\alpha)_{n,m}\psi_m(x) \\ &= \varepsilon_\alpha\frac{\partial\mathcal{L}_{\text{matter}}(\psi(x), D_\mu\psi(x))}{\partial(D_\nu\psi(x))_n}\frac{\partial(D_\nu\psi(x))_n}{\partial A_{\alpha\mu}(x)} \\ &= \varepsilon_\alpha\frac{\partial\mathcal{L}_{\text{matter}}(\psi(x), D_\mu\psi(x))}{\partial A_{\alpha\mu}(x)} \\ &= \varepsilon_\alpha\frac{\delta}{\delta A_{\alpha\mu}(x)} I^{\text{gauged}}_{\text{matter}}[\psi, D_\mu\psi].\end{aligned} \tag{10.9.41}$$

Here we note that $I^{\text{gauged}}_{\text{matter}}[\psi, D_\mu\psi]$ is not identical to the ungauged matter action functional $I_{\text{matter}}[\psi_n]$ given by Eq. (10.9.23), but is the gauged matter action functional defined by

$$I^{\text{gauged}}_{\text{matter}}[\psi, D_\mu\psi] \equiv \int d^4x\mathcal{L}_{\text{matter}}(\psi(x), D_\mu\psi(x)). \tag{10.9.42}$$

We emphasize here that the conserved Noether current after the extension of the global G invariance to the local G invariance is not the gauged matter current $J^{\mu\ \text{gauged}}_{\alpha,\text{matter}}(x)$ but the total current $J^{\mu}_{\alpha,\text{total}}(x)$, Eq. (10.9.39a). At the same time, we note that the strict conservation law of the total current $J^{\mu}_{\alpha,\text{total}}(x)$ is enforced at the expense of loss of covariance, which we will see further . The origin of this problem is the self-interaction of the non-Abelian gauge field $A_{\alpha\mu}(x)$ and the nonlinearity of the Euler–Lagrange equation of motion for the non-Abelian gauge field $A_{\alpha\mu}(x)$.

We shall make a table of the global $U(1)$ transformation law and the global G transformation law.

Global $U(1)$ transformation law		Global G transformation law	
$\delta\psi_n(x) = i\varepsilon q_n \psi_n(x),$	charged	$\delta\psi_n(x) = i\varepsilon_\alpha (t_\alpha)_{n,m}\psi_m(x),$	charged
$\delta A_\mu(x) = 0,$	neutral	$\delta A_{\alpha\mu}(x) = i\varepsilon_\beta (t^{\text{adj}}_\beta)_{\alpha\gamma} A_{\gamma\mu}(x),$	charged.

(10.9.43)

In the global transformation law of internal symmetry, Eq. (10.9.43), the matter fields $\psi_n(x)$ which have the group charge undergo global ($U(1)$ or G) rotation. As for the gauge fields, $A_\mu(x)$ and $A_{\alpha\mu}(x)$, the Abelian gauge field $A_\mu(x)$ remains unchanged under global $U(1)$ transformation while the non-Abelian gauge field $A_{\alpha\mu}(x)$ undergoes global G rotation under global G transformation. Hence the Abelian gauge field $A_\mu(x)$ is $U(1)$-neutral while the non-Abelian gauge field $A_{\alpha\mu}(x)$ is G-charged. The field strength tensors, $F_{\mu\nu}(x)$ and $F_{\alpha\mu\nu}(x)$, behave like $A_\mu(x)$ and $A_{\alpha\mu}(x)$, under global $U(1)$ and G transformations. The field strength tensor $F_{\mu\nu}(x)$ is $U(1)$-neutral, while the field strength tensor $F_{\alpha\mu\nu}(x)$ is G-charged, which originates from their linearity and nonlinearity in $A_\mu(x)$ and $A_{\alpha\mu}(x)$, respectively.

Global $U(1)$ transformation law		Global G transformation law	
$\delta F_{\mu\nu}(x) = 0,$	neutral	$\delta F_{\alpha\mu\nu}(x) = i\varepsilon_\beta (t^{\text{adj}}_\beta)_{\alpha\gamma} F_{\gamma\mu\nu}(x),$	charged.

(10.9.44)

When we write the Euler–Lagrange equation of motion for each case, the linearity and the nonlinearity with respect to the gauge fields become clear.

Abelian $U(1)$ gauge field		non-Abelian G gauge field.	
$\partial_\nu F^{\nu\mu}(x) = j^\mu_{\text{matter}}(x),$	linear	$D^{\text{adj}}_\nu F^{\nu\mu}_\alpha(x) = j^\mu_{\alpha,\text{matter}}(x),$	nonlinear.

(10.9.45)

From the antisymmetry of the field strength tensorwith respect to the Lorentz indices, μ and ν, we have the following current conservation as an identity:

$$\begin{array}{ll} \text{Abelian } U(1) \text{ gauge field} & \text{non-Abelian } G \text{ gauge field} \\ \partial_\mu j^\mu_{\text{matter}}(x) = 0. & D^{\text{adj}}_\mu j^\mu_{\alpha,\text{matter}}(x) = 0. \end{array} \tag{10.9.46}$$

Here we have

$$D^{\text{adj}}_\mu = \partial_\mu + i t^{\text{adj}}_\gamma A_{\gamma\mu}(x). \tag{10.9.47}$$

As a result of the extension to the local ($U(1)$ or G) invariance, in the case of the Abelian $U(1)$ gauge field, due to the neutrality of $A_\mu(x)$, the matter current $j^\mu_{\text{matter}}(x)$ alone that originates from the global $U(1)$ invariance is conserved, while in the case of the non-Abelian G gauge field, due to the G-charge of $A_{\alpha\mu}(x)$, the gauged matter current $j^\mu_{\alpha,\text{matter}}(x)$ alone that originates from the local G invariance is not conserved, but the sum with the gauge current $j^\mu_{\alpha,\text{gauge}}(x)$ which originates from the self-interaction of the non-Abelian gauge field $A_{\alpha\mu}(x)$ is conserved at the expense of the loss of covariance. A similar situation exists for the charge conservation law.

$$\begin{array}{ll} \text{Abelian } U(1) \text{ gauge field} & \text{non-Abelian } G \text{ gauge field} \\ \frac{d}{dt} Q^{\text{matter}}(t) = 0. & \frac{d}{dt} Q^{\text{tot}}_\alpha(t) = 0. \\ Q^{\text{matter}}(t) = \int d^3\vec{x} \cdot j^0_{\text{matter}}(t,\vec{x}). & Q^{\text{tot}}_\alpha(t) = \int d^3\vec{x} \cdot j^0_{\alpha,\text{tot}}(t,\vec{x}). \end{array} \tag{10.9.48}$$

We discuss the finite gauge transformation property of the non-Abelian gauge field $A_{\alpha\mu}(x)$. Under the finite local G-phase transformation of $\psi_n(x)$,

$$\psi_n(x) \to \psi'_n(x) = (\exp[i\varepsilon_\alpha(x)t_\alpha])_{n,m}\psi_m(x), \tag{10.9.49}$$

we demand that the covariant derivative $D_\mu\psi(x)$ defined by Eq. (10.9.30) transforms exactly like $\psi(x)$,

$$\begin{aligned} D_\mu\psi(x) \to (D_\mu\psi(x))' &= (\partial_\mu + i t_\gamma A'_{\gamma\mu}(x))\psi'(x) \\ &= \exp[i\varepsilon_\alpha(x)t_\alpha] D_\mu\psi(x). \end{aligned} \tag{10.9.50}$$

From Eq. (10.9.50), we obtain the following equation:

$$\exp[-i\varepsilon_\alpha(x)t_\alpha](\partial_\mu + i t_\gamma A'_{\gamma\mu}(x))\exp[i\varepsilon_\alpha(x)t_\alpha]\psi(x) = (\partial_\mu + i t_\gamma A_{\gamma\mu}(x))\psi(x).$$

Canceling $\partial_\mu\psi(x)$ term from both sides of the above equation, we obtain

$$\begin{aligned} &\exp[-i\varepsilon_\alpha(x)t_\alpha](\partial_\mu \exp[i\varepsilon_\alpha(x)t_\alpha]) + \exp[-i\varepsilon_\alpha(x)t_\alpha](i t_\gamma A'_{\gamma\mu}(x))\exp[i\varepsilon_\alpha(x)t_\alpha] \\ &= i t_\gamma A_{\gamma\mu}(x). \end{aligned}$$

Solving the above equation for $t_\gamma A'_{\gamma\mu}(x)$, we finally obtain the finite gauge transformation law of $A'_{\gamma\mu}(x)$,

$$t_\gamma A'_{\gamma\mu}(x) \\ = \exp[i\varepsilon_\alpha(x)t_\alpha]\{t_\gamma A_{\gamma\mu}(x) + \exp[-i\varepsilon_\beta(x)t_\beta](i\partial_\mu \exp[i\varepsilon_\beta(x)t_\beta])\}\exp[-i\varepsilon_\alpha(x)t_\alpha]. \tag{10.9.51}$$

At first sight, we get the impression that the finite gauge transformation law of $A'_{\gamma\mu}(x)$, (10.9.51), may depend on the specific realization $\{t_\gamma\}_{\gamma=1}^N$ of the generator $\{T_\gamma\}_{\gamma=1}^N$ upon the multiplet $\psi(x)$. Actually, $A'_{\gamma\mu}(x)$ transforms under the adjoint representation $\{t_\gamma^{\text{adj}}\}_{\gamma=1}^N$. The infinitesimal version of the finite gauge transformation, (10.9.51), does reduce to the infinitesimal gauge transformation, (10.9.32a) and (10.9.32b), under the adjoint representation.

Unity of All Forces: Electro-weak unification of Glashow–Weinberg–Salam is based on the gauge group

$$SU(2)_{\text{weak isospin}} \times U(1)_{\text{weak hypercharge}}.$$

It suffers from the problem of the nonrenormalizability due to the triangular anomaly in the lepton sector. In the early 1970s, it is discovered that non-Abelian gauge field theory is asymptotically free at short distance, i.e., it behaves like a free field at short distance. Then the relativistic quantum field theory of the strong interaction based on the gauge group $SU(3)_{\text{color}}$ is invented and is called quantum chromodynamics.

Standard model with the gauge group

$$SU(3)_{\text{color}} \times SU(2)_{\text{weak isospin}} \times U(1)_{\text{weak hypercharge,}}$$

which describes the weal interaction, the electromagnetic interaction and the strong interaction is free from the triangular anomaly. It suffers, however, from a serious defect; the existence of the classical instanton solution to the field equation in the Euclidean metric for the $SU(2)$ gauge field theory. In the $SU(2)$ gauge field theory, we have the Belavin–Polyakov–Schwartz–Tyupkin instanton solution which is a classical solution to the field equation in the Euclidean metric. A proper account for the instanton solution requires the addition of the strong CP-violating term to the QCD Lagrangian density in the path integral formalism. The Peccei–Quinn axion and the invisible axion scenario resolve this strong CP-violation problem. In the grand unified theories, we assume that the subgroup of the grand unifying gauge group is the gauge group $SU(3)_{\text{color}} \times SU(2)_{\text{weak isospin}} \times U(1)_{\text{weak hypercharge}}$. We now attempt to unify the weak interaction, the electromagnetic interaction and the strong interaction by starting from the much larger gauge group G which is reduced to $SU(3)_{\text{color}} \times SU(2)_{\text{weak isospin}} \times U(1)_{\text{weak hypercharge}}$ and further down to $SU(3)_{\text{color}} \times U(1)_{\text{E.M.}}$.

Gravitational field: We can extend Weyl's gauge principle to Utiyama's gauge principle and Kibble's gauge principle to obtain the Lagrangian density for the gravitational field. We note that Weyl's gauge principle, Utiyama's gauge principle and Kibble's gauge principle belong to the category of the invariant variational principle.

R. Utiyama derived the theory of the gravitational field from his version of the gauge principle, based on the requirement of the invariance of the action functional $I[\phi]$ under the *local 6-parameter Lorentz transformation*. T.W.B. Kibble derived the theory of the gravitational field from his version of the gauge principle, based on the requirement of the invariance of the action functional $I[\phi]$ under the *local 10-parameter Poincaré transformation*, extending the treatment of Utiyama.

We let ϕ represent the set of generic matter field variables $\phi_a(x)$, which we regard as the elements of a column vector $\phi(x)$, and define the matter action functional $I_{\text{matter}}[\phi]$ in terms of the matter Lagrangian density $\mathcal{L}_{\text{matter}}(\phi, \partial_\mu \phi)$ as

$$I_{\text{matter}}[\phi] = \int d^4x \mathcal{L}_{\text{matter}}(\phi, \partial_\mu \phi). \tag{10.9.52}$$

We first discuss the infinitesimal transformationof both the coordinates x^μ and the matter field variables $\phi(x)$,

$$x^\mu \to x'^\mu = x^\mu + \delta x^\mu, \quad \phi(x) \to \phi'(x') = \phi(x) + \delta\phi(x), \tag{10.9.53}$$

where the invariance group G is not specified. It is convenient to allow the possibility that the matter Lagrangian density $\mathcal{L}_{\text{matter}}$ explicitly depends on the coordinates x^μ. Then, under the infinitesimal transformation, (10.9.53), we have

$$\delta\mathcal{L}_{\text{matter}} \equiv \frac{\partial \mathcal{L}_{\text{matter}}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}_{\text{matter}}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu \phi) + \left.\frac{\partial \mathcal{L}_{\text{matter}}}{\partial x^\mu}\right|_{\phi \text{ fixed}} \delta x^\mu.$$

It is also useful to consider the variation of $\phi(x)$ at a fixed value of x^μ,

$$\delta_0\phi \equiv \phi'(x) - \phi(x) = \delta\phi - \delta x^\mu \partial_\mu \phi. \tag{10.9.54}$$

It is obvious that δ_0 commutes with ∂_μ, so we have

$$\delta(\partial_\mu \phi) = \partial_\mu(\delta\phi) - (\partial_\mu \delta x^\nu)\partial_\nu \phi. \tag{10.9.55}$$

The matter action functional, (10.9.52), over a space–time region Ω is transformed under the transformations,(10.9.53), into

$$I'_{\text{matter}}[\Omega] \equiv \int_\Omega \mathcal{L}'_{\text{matter}}(x') \det(\partial_\nu x'^\mu) d^4x.$$

Thus the matter action functional $I_{\text{matter}}[\Omega]$ over an arbitrary region Ω is invariant if

$$\delta\mathcal{L}_{\text{matter}} + (\partial_\mu \delta x^\mu)\mathcal{L}_{\text{matter}} \equiv \delta_0\mathcal{L}_{\text{matter}} + \partial_\mu(\delta x^\mu \mathcal{L}_{\text{matter}}) \equiv 0. \tag{10.9.56}$$

We now consider the specific case of the Poincaré transformation,

$$\delta x^\mu = i\varepsilon^\mu_\nu x^\nu + \varepsilon^\mu, \quad \delta\phi = \frac{1}{2} i\varepsilon^{\mu\nu} S_{\mu\nu}\phi, \tag{10.9.57}$$

where $\{\varepsilon^\mu\}$ and $\{\varepsilon^{\mu\nu}\}$ with $\varepsilon^{\mu\nu} = -\varepsilon^{\nu\mu}$ are the 10 infinitesimal constant parameters of the Poincaré group, and $S_{\mu\nu}$ with $S_{\mu\nu} = -S_{\nu\mu}$, are the mixing matrices of the components of a column vector $\phi(x)$ satisfying

$$[S_{\mu\nu}, S_{\rho\sigma}] = i(\eta_{\nu\rho} S_{\mu\sigma} + \eta_{\mu\sigma} S_{\nu\rho} - \eta_{\nu\sigma} S_{\mu\rho} - \eta_{\mu\rho} S_{\nu\sigma}).$$

$\{S_{\mu\nu}\}$ will be identified as the spin matrices of matter field ϕ later. From Eq. (10.9.55), we have

$$\delta(\partial_\mu\phi) = \frac{1}{2} i\varepsilon^{\rho\sigma} S_{\rho\sigma}\partial_\mu\phi - i\varepsilon^\rho_\mu \partial_\rho\phi. \tag{10.9.58}$$

Since we have $\partial_\mu(\delta x^\mu) = \varepsilon^\mu_\mu = 0$, the conditions, (10.9.56), for the invariance of the matter action functional $I_{\text{matter}}[\phi]$ under the infinitesimal Poincaré transformations, (10.9.57), reduce to

$$\delta\mathcal{L}_{\text{matter}} \equiv 0,$$

and result in 10 identities,

$$\frac{\partial\mathcal{L}_{\text{matter}}}{\partial x^\rho} \equiv \partial_\rho\mathcal{L}_{\text{matter}} - \frac{\partial\mathcal{L}_{\text{matter}}}{\partial\phi}\partial_\rho\phi - \frac{\partial\mathcal{L}_{\text{matter}}}{\partial(\partial_\mu\phi)}\partial_\rho\partial_\mu\phi \equiv 0, \tag{10.9.59}$$

$$\frac{\partial\mathcal{L}_{\text{matter}}}{\partial\phi} iS_{\rho\sigma}\phi + \frac{\partial\mathcal{L}_{\text{matter}}}{\partial(\partial_\mu\phi)}(iS_{\rho\sigma}\partial_\mu\phi + \eta_{\mu\rho}\partial_\sigma\phi - \eta_{\mu\sigma}\partial_\rho\phi) \equiv 0. \tag{10.9.60}$$

The conditions (10.9.59) express the translational invariance of the system and are equivalent to the requirement that $\mathcal{L}_{\text{matter}}$ is explicitly independent of x^μ. We use the Euler–Lagrange equations of motion in Eqs. (10.9.59) and (10.9.60), obtaining the ten conservation laws, which we write as

$$\partial_\mu T^\mu_\rho = 0, \quad \partial_\mu(S^\mu_{\rho\sigma} - x_\rho T^\mu_\sigma + x_\sigma T^\mu_\rho) = 0, \tag{10.9.61}$$

$$T^\mu_\rho \equiv \frac{\partial\mathcal{L}_{\text{matter}}}{\partial(\partial_\mu\phi)}\partial_\rho\phi - \delta^\mu_\rho\mathcal{L}_{\text{matter}}, \quad S^\mu_{\rho\sigma} \equiv -i\frac{\partial\mathcal{L}_{\text{matter}}}{\partial(\partial_\mu\phi)} S_{\rho\sigma}\phi. \tag{10.9.62}$$

The ten conservation laws, Eqs. (10.9.59) and (10.9.60), are the conservation laws of energy, momentum, and angular momentum. Thus $\{S_{\mu\nu}\}$ are the spin matrices of matter field $\phi(x)$.

We shall also examine the transformations in terms of the variation $\delta_0\phi$, which in this case is

$$\delta_0\phi = -i\varepsilon^{\rho}\partial_\rho\phi + \frac{1}{2}i\varepsilon^{\rho\sigma}(S_{\rho\sigma} + x_\rho\frac{1}{i}\partial_\sigma - x_\sigma\frac{1}{i}\partial_\rho)\phi. \tag{10.9.63}$$

On comparing with Weyl's gauge principle, the role of the realizations $\{t_\alpha\}$ of the generators $\{T_\alpha\}$ upon the multiplet ϕ is played by the differential operators,

$$\frac{1}{i}\partial_\rho \quad \text{and} \quad S_{\rho\sigma} + x_\rho\frac{1}{i}\partial_\sigma - x_\sigma\frac{1}{i}\partial_\rho.$$

Then, by the definition of the currents, we expect the currents corresponding to ε^ρ and $\varepsilon^{\rho\sigma}$ to be given, respectively, by

$$J_\rho^\mu \equiv \frac{\partial\mathcal{L}_{\text{matter}}}{\partial(\partial_\mu\phi)}\partial_\rho\phi \quad \text{and} \quad J_{\rho\sigma}^\mu \equiv S_{\rho\sigma}^\mu - x_\rho\frac{1}{i}J_\sigma^\mu + x_\sigma\frac{1}{i}J_\rho^\mu. \tag{10.9.64}$$

In terms of δ_0, however, the invariance condition (10.9.56) is not simply $\delta_0\mathcal{L}_{\text{matter}} \equiv 0$, and the additional term $\delta x^\rho\partial_\rho\mathcal{L}_{\text{matter}}$ results in the appearance of the term $\partial_\rho\mathcal{L}_{\text{matter}}$ in the identities (10.9.59) and thus for the term $\delta_\rho^\mu\mathcal{L}_{\text{matter}}$ in T_ρ^μ.

We shall now consider the local 10-parameter Poincaré transformation in which the 10 arbitrary infinitesimal constants, $\{\varepsilon^\mu\}$ and $\{\varepsilon^{\mu\nu}\}$, in Eq. (10.9.57) become the 10 arbitrary infinitesimal functions, $\{\varepsilon^\mu(x)\}$ and $\{\varepsilon^{\mu\nu}(x)\}$. It is convenient to regard

$$\varepsilon^{\mu\nu}(x) \quad \text{and} \quad \xi^\mu(x) \equiv i\varepsilon^\mu_\nu(x)x^\nu + \varepsilon^\mu(x),$$

as the 10 independent infinitesimal functions. Such a choice avoids the explicit appearance of x^μ. Furthermore, we can always choose $\varepsilon^\mu(x)$ such that

$$\xi^\mu(x) = 0 \quad \text{and} \quad \varepsilon^{\mu\nu}(x) \neq 0,$$

so that the coordinate and field transformations are completely separated.

Based on this fact, we use the Latin indices for $\varepsilon^{ij}(x)$ and the Greek indices for ξ^μ and x^μ. The Latin indices, $i, j, k, \ldots$, also assume the values 0, 1, 2, and 3. Then the transformations under consideration are

$$\delta x^\mu = \xi^\mu(x) \quad \text{and} \quad \delta\phi(x) = \frac{1}{2}i\varepsilon^{ij}(x)S_{ij}\phi(x), \tag{10.9.65}$$

or

$$\delta_0\phi(x) = -\xi^\mu(x)\partial_\mu\phi(x) + \frac{1}{2}i\varepsilon^{ij}(x)S_{ij}\phi(x). \tag{10.9.66}$$

This notation emphasizes the similarity of the $\varepsilon^{ij}(x)$ transformations to the linear transformations of Weyl's gauge principle. Actually, in Utiyama's gauge principle, the $\varepsilon^{ij}(x)$ transformations alone are considered in the local 6-parameter Lorentz transformation . The $\xi^{\mu}(x)$ transformations correspond to the general coordinate transformation.

According to the convention we just employed, the differential operator ∂_{μ} must have a Greek index. In the matter Lagrangian density $\mathcal{L}_{\text{matter}}$, we then have the two kinds of indices, and we shall regard $\mathcal{L}_{\text{matter}}$ as a given function of $\phi(x)$ and $\tilde{\partial}_k\phi(x)$, satisfying the identities, (10.9.59) and (10.9.60). The original matter Lagrangian density $\mathcal{L}_{\text{matter}}$ is obtained by setting

$$\tilde{\partial}_k\phi(x) = \delta_k^{\mu}\partial_{\mu}\phi(x).$$

The matter Lagrangian density $\mathcal{L}_{\text{matter}}$ is not invariant under the local 10-parameter transformations, (10.9.65) or (10.9.66), but we shall later obtain an invariant expression by replacing $\tilde{\partial}_k\phi(x)$ with a suitable covariant derivative $D_k\phi(x)$ in the matter Lagrangian density $\mathcal{L}_{\text{matter}}$.

The transformation of $\partial_{\mu}\phi(x)$ is given by

$$\delta\partial_{\mu}\phi = \frac{1}{2}i\varepsilon^{ij}S_{ij}\partial_{\mu}\phi + \frac{1}{2}i(\partial_{\mu}\varepsilon^{ij})S_{ij}\phi - (\partial_{\mu}\xi^{\nu})(\partial_{\nu}\phi), \tag{10.9.67}$$

and the original matter Lagrangian density $\mathcal{L}_{\text{matter}}$ transforms according to

$$\delta\mathcal{L}_{\text{matter}} \equiv -(\partial_{\mu}\xi^{\rho})J_{\rho}^{\mu} - \frac{1}{2}i(\partial_{\mu}\varepsilon^{ij})S_{ij}^{\mu}.$$

We note that it is J_{ρ}^{μ} instead of T_{ρ}^{μ} that appears here. The reason for this is that we have not included the extra term $(\partial_{\mu}\delta x^{\mu})\mathcal{L}_{\text{matter}}$ in Eq,(10.9.56). The left-hand side of Eq. (10.9.56) actually has the value

$$\delta\mathcal{L}_{\text{matter}} + (\partial_{\mu}\delta x^{\mu})\,\mathcal{L}_{\text{matter}} \equiv -(\partial_{\mu}\xi^{\rho})T_{\rho}^{\mu} - \frac{1}{2}i(\partial_{\mu}\varepsilon^{ij})S_{ij}^{\mu}.$$

We shall now look for the modified matter Lagrangian density $\mathcal{L}'_{\text{matter}}$ which makes the matter action functional $I_{\text{matter}}[\phi]$ invariant under (10.9.65) or (10.9.66). The extra term just mentioned is of a different kind in that it involves $\mathcal{L}_{\text{matter}}$ and not $\partial\mathcal{L}_{\text{matter}}/\partial(\tilde{\partial}_k\phi)$. In particular, the extra term includes the contributions from terms in $\mathcal{L}_{\text{matter}}$ which do not contain the derivatives. Thus it is clear that we cannot remove the extra term by replacing the derivative $\tilde{\partial}_{\mu}$ with a suitable covariant derivative D_{μ}. For this reason, we shall consider the problem in two stages. First, we eliminate the noninvariance arising from the fact that $\partial_{\mu}\phi(x)$ is not a covariant quantity, and second, we obtain an expression $\mathcal{L}'_{\text{matter}}$ satisfying

$$\delta\mathcal{L}'_{\text{matter}} \equiv 0. \tag{10.9.68}$$

Because the invariance condition (10.9.56) for the matter action functional I_{matter} requires the matter Lagrangian density $\mathcal{L}'_{\text{matter}}$ to be an invariant scalar density rather than an invariant scalar, we shall make a further modification, replacing $\mathcal{L}'_{\text{matter}}$ with $\mathcal{L}''_{\text{matter}}$, which satisfies

$$\delta\mathcal{L}''_{\text{matter}} + (\partial_\mu \xi^\mu)\mathcal{L}''_{\text{matter}} \equiv 0. \qquad (10.9.69)$$

The first part of this program can be accomplished by replacing $\tilde{\partial}_k\phi$ in $\mathcal{L}_{\text{matter}}$ with a covariant derivative $D_k\phi$ which transforms according to

$$\delta(D_k\phi) = \frac{1}{2}i\varepsilon^{ij}S_{ij}(D_k\phi) - i\varepsilon_k^{\,i}(D_i\phi). \qquad (10.9.70)$$

The condition (10.9.68) follows from the identities, (10.9.59) and (10.9.60). To do this, it is necessary to introduce 40 new field variables toward the end. We first consider the ε^{ij} transformations, and eliminate the $\partial_\mu\varepsilon^{ij}$ term in (10.9.67) by setting

$$D_{|\mu}\,\phi \equiv \partial_\mu\phi + \frac{1}{2}A_\mu^{ij}S_{ij}\phi, \qquad (10.9.71)$$

where A_μ^{ij} with

$$A_\mu^{ij} = -A_\mu^{ji}$$

are 24 new field variables.

We can then impose the condition

$$\delta(D_{|\mu}\,\phi) = \frac{1}{2}i\varepsilon^{ij}S_{ij}(D_{|\mu}\,\phi) - (\partial_\mu\xi^\nu)(D_{|\nu}\,\phi), \qquad (10.9.72)$$

which determines the transformation properties of A_μ^{ij} uniquely. They are

$$\delta A_\mu^{ij} = -\partial_\mu\varepsilon^{ij} + \varepsilon_k^{\,i}A_\mu^{kj} + \varepsilon_k^{\,j}A_\mu^{ik} - (\partial_\mu\xi^\nu)A_\nu^{ij}. \qquad (10.9.73)$$

The position of the last term in Eq. (10.9.67) is rather different. The term involving $\partial_\mu\varepsilon^{ij}$ is inhomogeneous in the sense that it contains ϕ rather than $\partial_\mu\phi$, but this is not true for the last term. Correspondingly, the transformation law for $D_{|\mu}\,\phi$, (10.9.72), is already homogeneous. This means that to force the covariant derivative $D_k\phi$ to transform according to Eq. (10.9.70), we shall add to $D_{|\mu}\,\phi$ not a term in ϕ but rather a term in $D_{|\mu}\,\phi$ itself. In other words, we merely multiply by a new field,

$$D_k\phi \equiv e_k^{\mu}\,D_{|\mu}\,\phi. \qquad (10.9.74)$$

Here, the e_k^{μ} are 16 new field variables with the transformation properties determined by Eq. (10.9.70) to be

$$\delta e_k^{\mu} = (\partial_\nu\xi^\mu)e_k^{\nu} - i\varepsilon_k^{\,i}e_i^{\mu}. \qquad (10.9.75)$$

We note that the fields e_k^{μ} and A_{μ}^{ij} are independent and unrelated at this stage, though they will be related by the Euler–Lagrange equations of motion.

We find the invariant matter Lagrangian density $\mathcal{L}'_{\text{matter}}$ defined by

$$\mathcal{L}'_{\text{matter}} \equiv \mathcal{L}_{\text{matter}}(\phi, D_k\phi),$$

which is an invariant scalar. We can obtain the invariant matter Lagrangian density $\mathcal{L}''_{\text{matter}}$, which is an invariant scalar density by multiplying $\mathcal{L}'_{\text{matter}}$ by a suitable function of the new field variables,

$$\mathcal{L}''_{\text{matter}} \equiv \mathcal{E}\mathcal{L}'_{\text{matter}} = \mathcal{E}\mathcal{L}_{\text{matter}}(\phi, D_k\phi).$$

The invariance condition (10.9.69) for $\mathcal{L}''_{\text{matter}}$ is satisfied if a factor $\mathcal{E}$ itself is an invariant scalar density,

$$\delta\mathcal{E} + (\partial_{\mu}\xi^{\mu})\mathcal{E} \equiv 0.$$

The only function of the new field variable e_k^{μ} which obeys this transformation law and does not involve the derivatives is

$$\mathcal{E} = [\det(e_k^{\mu})]^{-1}, \tag{10.9.76}$$

where the arbitrary constant factor has been chosen such that $\mathcal{E} = 1$ when e_k^{μ} is set equal to δ_k^{μ}. The final form of the modified matter Lagrangian density $\mathcal{L}''_{\text{matter}}$, which is an invariant scalar density is given by

$$\mathcal{L}''_{\text{matter}}(\phi, \partial_{\mu}\phi, e_k^{\mu}, A_{\mu}^{ij}) \equiv \mathcal{E}\mathcal{L}_{\text{matter}}(\phi, D_k\phi). \tag{10.9.77}$$

As in the case of Weyl's gauge principle, we can define the modified current densities in terms of $\mathcal{L}_{\text{matter}}(\phi, D_k\phi)$ by

$$\mathcal{T}^k_{\ \mu} \equiv \frac{\partial\mathcal{L}''_{\text{matter}}}{\partial e_k^{\mu}} \equiv \mathcal{E}e_{\mu}^{i}\left[\frac{\partial\mathcal{L}_{\text{matter}}}{\partial(D_k\phi)}D_i\phi - \delta_i^k\mathcal{L}_{\text{matter}}\right], \tag{10.9.78}$$

$$\mathcal{S}^{\mu}_{\ ij} \equiv -2\frac{\partial\mathcal{L}''_{\text{matter}}}{\partial A_{\mu}^{ij}} \equiv i\mathcal{E}e_k^{\mu}\frac{\partial\mathcal{L}_{\text{matter}}}{\partial(D_k\phi)}S_{ij}\phi, \tag{10.9.79}$$

where e_{μ}^{i} is the inverse of e_i^{μ}, satisfying

$$e_{\mu}^{i}e_i^{\nu} = \delta_{\mu}^{\nu}, \quad e_{\mu}^{i}e_j^{\mu} = \delta_j^i. \tag{10.9.80}$$

In order to express the conservation laws of these currents in a simple form, we extend the definition of $D_{|\mu}\phi$. Originally, it was defined for $\phi(x)$, and is to be defined for any other quantity which is invariant under the ξ^{μ} transformations and transforms linearly under the ε^{ij} transformations . We extend $D_{|\mu}$ to any quantity which transforms linearly under the ε^{ij} transformations by ignoring the ξ^{μ} transformations altogether. Thus we have

$$D_{|\nu}\, e_i^{\mu} \equiv \partial_\nu e_i^{\mu} - A^k_{i\nu} e_k^{\mu}, \tag{10.9.81}$$

according to the ε^{ij} transformation law of e_i^{μ}. We call this the ε covariant derivative. We calculate the commutator of the ε covariant derivatives as,

$$[D_{|\mu}\,, D_{|\nu}\,]\phi = \frac{1}{2} i R^{ij}_{\ \ \mu\nu} S_{ij}\phi, \tag{10.9.82}$$

where $R^i_{\ j\mu\nu}$ is defined by the following equation:

$$R^i_{\ j\mu\nu} \equiv \partial_\nu A^i_{\ j\mu} - \partial_\mu A^i_{\ j\nu} - A^i_{\ k\mu} A^k_{\ j\nu} + A^i_{\ k\nu} A^k_{\ j\mu}. \tag{10.9.83}$$

This quantity is covariant under the ε^{ij} transformations. $R^i_{\ j\mu\nu}$ is closely analogous to the field strength tensor $F_{\alpha\mu\nu}$ of the non-Abelian gauge field. $R^{ij}_{\ \ \mu\nu}$ is antisymmetric in both pairs of the indices.

In terms of the ε covariant derivative, the ten conservation laws of the currents, (10.9.78) and (10.9.79), are expressed as

$$D_{|\mu}\,(\mathcal{T}^k_{\ \nu} e_k^{\mu}) + \mathcal{T}^k_{\ \mu}(D_{|\nu}\, e_k^{\mu}) = \mathcal{S}^{\mu}_{\ ij} R^{ij}_{\ \ \mu\nu}, \tag{10.9.84}$$

$$D_{|\mu}\, \mathcal{S}^{\mu}_{\ ij} = \mathcal{T}_{i\mu} e_j^{\mu} - \mathcal{T}_{j\mu} e_i^{\mu}. \tag{10.9.85}$$

Now we examine our ultimate goal, the Lagrangian density $\mathcal{L}_{\mathrm{G}}$ of the "free" self-interacting gravitational field. We examine the commutator of D_k and D_l acting on $\phi(x)$. After some algebra, we obtain

$$[D_k, D_l]\phi = \frac{1}{2} i R^{ij}_{\ \ kl} S_{ij}\phi - C^i_{\ kl} D_i\phi, \tag{10.9.86}$$

where

$$R^{ij}_{\ \ kl} \equiv e_k^{\mu} e_l^{\nu} R^{ij}_{\ \ \mu\nu}, \quad C^i_{\ kl} \equiv (e_k^{\mu} e_l^{\nu} - e_l^{\mu} e_k^{\nu}) D_{|\nu}\, e^i_{\ \mu}. \tag{10.9.87}$$

We note that the right-hand side of Eq. (10.9.86) is not simply proportional to ϕ but also involves $D_i\phi$.

The Lagrangian density $\mathcal{L}_{\mathrm{G}}$ for the "free" self-interacting gravitational field must be an invariant scalar density. If we set $\mathcal{L}_{\mathrm{G}} = \mathcal{E}\mathcal{L}_0$, then $\mathcal{L}_0$ must be an invariant scalar and a function only of the covariant quantities $R^{ij}_{\ \ kl}$ and $C^i_{\ kl}$. All the indices of these expressions are of the same kind, unlike the case of the non-Abelian gauge field, so that we can take the contractions of the upper indices with the lower indices.

The requirement that $\mathcal{L}_0$ is an invariant scalar in two separate spaces is reduced to the requirement that it is an invariant scalar in one space. We have a linear invariant scalar which has no analog in the case of the non-Abelian gauge field, namely, $R \equiv R^{ij}_{\ ij}$. There exists a few quadratic invariants, but we choose the lowest order invariant. Thus we are led to the Lagrangian density $\mathcal{L}_G$ for the "free" self-interacting gravitational field,

$$\mathcal{L}_G = \frac{1}{2\kappa^2}\mathcal{E}R, \tag{10.9.88}$$

which is linear in the derivatives. In Eq. (10.9.88), κ is Newton's gravitational constant.

So far, we have given neither any geometrical interpretation of the local ten-parameter Poincaré transformation , (10.9.65), nor any interpretation of the 40 new fields, $e^{\mu}_{k}(x)$ and $A^{ij}_{\mu}(x)$. We shall now establish the connection of the present theory with the standard metric theory of the gravitational field.

Under the ξ^{μ} transformation which is a general coordinate transformation, $e^{\mu}_{k}(x)$ transforms like a contravariant vector, while $e^{k}_{\mu}(x)$ and $A^{ij}_{\mu}(x)$ transform like covariant vectors. Then the quantity

$$g_{\mu\nu}(x) \equiv e^{k}_{\mu}(x)e_{k\nu}(x) \tag{10.9.89}$$

is a symmetric covariant tensor, and therefore may be interpreted as the metric tensor of a Riemannian space. It remains invariant under the ε^{ij} transformations. We shall abandon the convention that all the indices are to be lowered or raised by the flat-space metric $\eta_{\mu\nu}$, and we use $g_{\mu\nu}(x)$ instead as the metric tensor. We can easily show that

$$\mathcal{E} = \sqrt{-g(x)} \quad \text{with} \quad g(x) \equiv \det(g_{\mu\nu}(x)). \tag{10.9.90}$$

From Eq. (10.9.89), we realize that $e^{\mu}_{k}(x)$ and $e^{k}_{\mu}(x)$ are the contravariant and covariant components of a tetrad system in Riemannian space. The ε^{ij} transformations are the tetrad rotations. The Greek indices are the world tensor indices and the Latin indices are the local tensor indices of this system. The original generic matter field $\phi(x)$ may be decomposed into local tensors and local spinors. From the local tensors, we can form the corresponding world tensors by multiplying by $e^{\mu}_{k}(x)$ or $e^{k}_{\mu}(x)$.

For example, from a local vector $v^{i}(x)$, we can form the world vector as

$$v^{\mu}(x) = e^{\mu}_{i}(x)v^{i}(x) \quad \text{and} \quad v_{\mu}(x) = e^{i}_{\mu}(x)v_{i}(x). \tag{10.9.91}$$

We note that

$$v_{\mu}(x) = g_{\mu\nu}(x)v^{\nu}(x),$$

so that Eq. (10.9.91) is consistent with the definition of the metric $g_{\mu\nu}(x)$, Eq. (10.9.89).

The field $A^i_{\ j\mu}(x)$ is regarded as a local affine connection with respect to the tetrad system since it specifies the covariant derivatives of local tensors or local spinors. For a local vector, we have

$$\begin{cases} D_{|\nu}\, v^i &= \partial_\nu v^i + A^i_{\ j\nu} v^j, \\ D_{|\nu}\, v_j &= \partial_\nu v_j - A^i_{\ j\nu} v_i. \end{cases} \tag{10.9.92}$$

We notice that the relationship between $D_{|\mu}\,\phi$ and $D_k\phi$, (10.9.74), could be written simply as

$$D_\mu\phi = D_{|\mu}\,\phi, \tag{10.9.93}$$

according to the convention (10.9.91). We shall, however, make a distinction between D_μ and $D_{|\mu}$ for a later purpose. We define the covariant derivative of a world tensor in terms of the covariant derivative of the associated local tensor. Thus, we have

$$\begin{cases} D_\nu v^\lambda &\equiv e^\lambda_i\, D_{|\nu}\, v^i = \partial_\nu v^\lambda + \Gamma^\lambda_{\ \mu\nu} v^\mu, \\ D_\nu v_\mu &\equiv e^i_\mu\, D_{|\nu}\, v_i = \partial_\nu v_\mu - \Gamma^\lambda_{\ \mu\nu} v_\lambda, \end{cases} \tag{10.9.94}$$

$$\Gamma^\lambda_{\ \mu\nu} \equiv e^\lambda_i\, D_{|\nu}\, e^i_\mu \equiv -e^i_\mu\, D_{|\nu}\, e^\lambda_i. \tag{10.9.95}$$

We note that this definition of $\Gamma^\lambda_{\ \mu\nu}$ is equivalent to the requirement that the covariant derivative of the tetrad components shall vanish,

$$\begin{cases} D_\nu e^\lambda_i &\equiv 0, \\ D_\nu e^i_\mu &\equiv 0. \end{cases} \tag{10.9.96}$$

For a generic quantity α transforming according to

$$\delta\alpha = \frac{1}{2} i\varepsilon^{ij} S_{ij}\alpha + (\partial_\mu \xi^\lambda)\Sigma_\lambda^{\ \mu}\alpha, \tag{10.9.97}$$

the covariant derivative of α is defined by

$$D_\nu\alpha \equiv \partial_\nu\alpha + \frac{1}{2} iA^{ij}_{\ \ \nu} S_{ij}\alpha + \Gamma^\lambda_{\ \mu\nu}\Sigma_\lambda^{\ \mu}\alpha. \tag{10.9.98}$$

The ε covariant derivative of α, defined by Eq. (10.9.81), is obtained by simply dropping the last term in (10.9.98). We calculate the commutator of the covariant derivative of α with the result,

$$[D_\mu, D_\nu]\alpha = \frac{1}{2} iR^{ij}_{\ \ \mu\nu} S_{ij}\alpha + R^\rho_{\ \sigma\mu\nu}\Sigma_\rho^{\ \sigma}\alpha - C^\lambda_{\ \mu\nu} D_\lambda\alpha,$$

where $R^{\rho}_{\ \sigma\mu\nu}$ and $C^{\lambda}_{\ \mu\nu}$ are defined in terms of $R^{i}_{\ j\mu\nu}$ and $C^{i}_{\ kl}$ in the usual way. These quantities are the world tensors and can be expressed in terms of $\Gamma^{\lambda}_{\ \mu\nu}$ in the form

$$R^{\rho}_{\ \sigma\mu\nu} = \partial_\nu \Gamma^{\rho}_{\ \sigma\mu} - \partial_\mu \Gamma^{\rho}_{\ \sigma\nu} - \Gamma^{\rho}_{\ \lambda\mu}\Gamma^{\lambda}_{\ \sigma\nu} + \Gamma^{\rho}_{\ \lambda\nu}\Gamma^{\lambda}_{\ \sigma\mu}, \quad C^{\lambda}_{\ \mu\nu} = \Gamma^{\lambda}_{\ \mu\nu} - \Gamma^{\lambda}_{\ \nu\mu}. \tag{10.9.99}$$

We see that $R^{\rho}_{\ \sigma\mu\nu}$ is the Riemann tensor formed from the affine connection $\Gamma^{\lambda}_{\ \mu\nu}$. From Eq. (10.9.96), we have $D_\rho g_{\mu\nu}(x) \equiv 0$. It is consistent to interpret $\Gamma^{\lambda}_{\ \mu\nu}$ as an affine connection in a Riemannian space. The definition of $\Gamma^{\lambda}_{\ \mu\nu}$, Eq. (10.9.95), does not guarantee that it is symmetric so that it is not the Christoffel symbol in general. In the absence of the matter field, however, $\Gamma^{\lambda}_{\ \mu\nu}$ is symmetric so that it is the Christoffel symbol. The curvature scalar has the usual form, $R \equiv R^{\mu}_{\ \mu}$, where $R_{\mu\nu} \equiv R^{\lambda}_{\ \mu\lambda\nu}$. The Lagrangian density $\mathcal{L}_{\mathrm{G}}$ for the "free" self-interacting gravitational field , Eq. (10.9.88), is the usual one,

$$\mathcal{L}_{\mathrm{G}}(g^{\mu\nu}(x), \Gamma^{\lambda}_{\ \mu\nu}(x)) = \frac{1}{2\kappa^2}\sqrt{-g}g^{\mu\nu}(\partial_\nu \Gamma^{\lambda}_{\ \mu\lambda} - \partial_\lambda \Gamma^{\lambda}_{\ \mu\nu} + \Gamma^{\rho}_{\ \mu\lambda}\Gamma^{\lambda}_{\ \nu\rho} - \Gamma^{\lambda}_{\ \mu\nu}\Gamma^{\rho}_{\ \lambda\rho}). \tag{10.9.100}$$

10.10
Path Integral Quantization of Gauge Field I

In this section, we discuss the *path integral quantization of the gauge field*. When we apply the path integral representation of $\langle 0, \mathrm{out} | 0, \mathrm{in} \rangle$,

$$\left\langle 0, \mathrm{out} \middle| 0, \mathrm{in} \right\rangle = \int \mathcal{D}[\phi_i] \exp\left[i \int d^4x \mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x)) \right],$$

naively to the path integral quantization of the gauge field, we obtain the results that violate the unitarity. The origin of this difficulty lies in the *gauge invariance* of the gauge field action functional $I_{\mathrm{gauge}}[A_{\alpha\mu}]$, i.e., the *four-dimensional transversality* of the kernel of the quadratic part of the gauge field action functional $I_{\mathrm{gauge}}[A_{\alpha\mu}]$. In another word, when we path-integrate along the gauge equivalent class, the functional integrand remains constant and we merely calculate the orbit volume V_G. We introduce the hypersurface (the *gauge-fixing condition*),

$$F_\alpha(A_{\beta\mu}(x)) = 0, \quad \alpha = 1, \ldots, N,$$

in the manifold of the gauge field which intersects with each gauge equivalent class only once and perform the path integration and the group integration on this hypersurface. We shall complete the *path integral quantization* of the gauge field and the *separation of the orbit volume* V_G simultaneously. This method is called the *Faddeev–Popov method*. Next we generalize the gauge-fixing condition to the form

$$F_\alpha(A_{\beta\mu}(x)) = a_\alpha(x), \quad \alpha = 1, \ldots, N,$$

with $a_\alpha(x)$ arbitrary function independent of $A_{\alpha\mu}(x)$. By this consideration, we obtain the second formula of Faddeev–Popov. We arrive at the *covariant gauge*. We introduce the *Faddeev–Popov ghost* (fictitious scalar fermion) only in the internal loop, which restores the unitarity. We discuss the various gauge-fixing conditions, and the *advantage of the axial gauge*.

The First Formula of Faddeev–Popov: In Section 10.2, we derived the path integral formula for the vacuum-to-vacuum transition amplitude $\langle 0, \text{out} | 0, \text{in} \rangle$ for the nonsingular Lagrangian density $\mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x))$ as

$$\left\langle 0, \text{out} \middle| 0, \text{in} \right\rangle = \int \mathcal{D}[\phi_i] \exp\left[i \int d^4x \mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x)) \right]. \qquad (10.10.1)$$

We cannot apply this path integral formula naively to the non-Abelian gauge field theory. The kernel $K_{\alpha\mu,\beta\nu}(x-y)$ of the quadratic part of the action functional of the gauge field $A_{\gamma\mu}(x)$,

$$I_{\text{gauge}}[A_{\gamma\mu}] = \int d^4x \mathcal{L}_{\text{gauge}}(F_{\gamma\mu\nu}(x)) = -\frac{1}{4} \int d^4x F_{\gamma\mu\nu}(x) F_\gamma^{\mu\nu}(x), \qquad (10.10.2a)$$

$$F_{\gamma\mu\nu}(x) = \partial_\mu A_{\gamma\nu}(x) - \partial_\nu A_{\gamma\mu}(x) - C_{\alpha\beta\gamma} A_{\alpha\mu}(x) A_{\beta\nu}(x), \qquad (10.10.2b)$$

is given by

$$I_{\text{gauge}}^{\text{quad}}[A_{\gamma\mu}] = -\frac{1}{2} \iint d^4x d^4y A_\alpha^\mu(x) K_{\alpha\mu,\beta\nu}(x-y) A_\beta^\nu(y), \qquad (10.10.3a)$$

$$K_{\alpha\mu,\beta\nu}(x-y) = \delta_{\alpha\beta} \left(-\eta_{\mu\nu} \partial^2 + \partial_\mu \partial_\nu \right) \delta^4(x-y), \quad \mu, \nu = 0, 1, 2, 3. \qquad (10.10.3b)$$

$K_{\alpha\mu,\beta\nu}(x-y)$ is singular and noninvertible. We cannot define the "free" Green's function of $A_{\alpha\mu}(x)$ as it stands now, and cannot apply the Feynman path integral formula, (10.10.1), for the path integral quantization of the gauge field. As a matter of fact, this $K_{\alpha\mu,\beta\nu}(x-y)$ is the four-dimensional transverse projection operator for arbitrary $A_{\alpha\mu}(x)$. The origin of such calamity lies in the gauge invariance of the gauge field action functional $I_{\text{gauge}}[A_{\gamma\mu}]$ under the gauge transformation,

$$A_{\gamma\mu}(x) \longrightarrow A_{\gamma\mu}^g(x). \qquad (10.10.4a)$$

$A_{\gamma\mu}^g(x)$ is given by

$$\begin{cases} t_\gamma A_{\gamma\mu}^g(x) = U(g(x)) \left\{ t_\gamma A_{\gamma\mu}(x) + U^{-1}(g(x)) \left[i\partial_\mu U(g(x)) \right] \right\} U^{-1}(g(x)), \\ U(g(x)) = \exp\left[ig_\gamma(x) t_\gamma \right]. \end{cases} \qquad (10.10.4b)$$

In another words, it originates from the fact that the gauge field action functional $I_{\text{gauge}}[A_{\gamma\mu}]$ is constant on the orbit of the gauge group G. By the orbit here, we mean the gauge equivalent class

$$\{A^{g}_{\gamma\mu}(x) : A_{\gamma\mu}(x) \text{ fixed}, \forall g(x) \in G\}. \tag{10.10.5}$$

Here $A^{g}_{\gamma\mu}(x)$ is defined by (10.10.4b). The naive expression for the vacuum-to-vacuum transition amplitude $\langle 0, \text{out} | 0, \text{in} \rangle$ of the gauge field $A_{\gamma\mu}(x)$,

$$\left\langle 0, \text{out} \middle| 0, \text{in} \right\rangle \;''=''\; \int \mathcal{D}[A_{\gamma\mu}] \exp\left[i \int d^4x \mathcal{L}_{\text{gauge}}(F_{\gamma\mu\nu}(x)) \right], \tag{10.10.6}$$

is hence proportional to the orbit volume V_G,

$$V_G = \int dg(x), \tag{10.10.7}$$

which is infinite and independent of $A_{\gamma\mu}(x)$.

In order to accomplish the path integral quantization of the gauge field $A_{\gamma\mu}(x)$ correctly, we must extract the orbit volume V_G from the vacuum-to-vacuum transition amplitude, $\langle 0, \text{out} | 0, \text{in} \rangle$. In the path integral, $\int \mathcal{D}[A_{\gamma\mu}]$, in (10.10.6), we should not path-integrate over all possible configurations of the gauge field, but we should path-integrate over only distinct orbit of the gauge field $A_{\gamma\mu}(x)$. We shall introduce the hypersurface,

$$F_\alpha(A_{\gamma\mu}(x)) = 0, \quad \alpha = 1, \ldots, N, \tag{10.10.8}$$

in the manifold of the gauge field $A_{\gamma\mu}(x)$, which intersects with each orbit only once. This implies that

$$F_\alpha(A^{g}_{\gamma\mu}(x)) = 0, \quad \alpha = 1, \ldots, N, \tag{10.10.9}$$

has the unique solution $g(x) \in G$ for arbitrary $A_{\gamma\mu}(x)$. In this sense, the hypersurface, (10.10.8), is called the gauge-fixing condition.

Here we recall the group theory.

(1)

$$g(x), g'(x) \in G \quad \Longrightarrow \quad (gg')(x) \in G. \tag{10.10.10a}$$

(2)

$$U(g(x))\, U(g'(x)) = U((gg')(x)). \tag{10.10.10b}$$

(3)

$$dg'(x) = d(gg')(x), \quad \text{Invariant Hurwitz measure.} \tag{10.10.10c}$$

We parametrize $U(g(x))$ in the neighborhood of the identity element of G as

$$U(g(x)) = 1 + it_{\gamma}\varepsilon_{\gamma}(x) + O(\varepsilon^2), \tag{10.10.11a}$$

$$\prod_{x} dg(x) = \prod_{\alpha,x} d\varepsilon_{\alpha}(x) \quad \text{for } g(x) \approx \text{Identity}. \tag{10.10.11b}$$

We define *Faddeev–Popov determinant* $\Delta_F[A_{\gamma\mu}]$ by

$$\Delta_F[A_{\gamma\mu}] \int \prod_{x} dg(x) \prod_{\alpha,x} \delta(F_{\alpha}(A^{g}_{\gamma\mu}(x))) = 1. \tag{10.10.12}$$

We show that $\Delta_F[A_{\gamma\mu}]$ is gauge invariant under the gauge transformation

$$A_{\gamma\mu}(x) \longrightarrow A^{g}_{\gamma\mu}(x). \tag{10.10.4a}$$

Under the gauge transformation, (10.10.4a), we have

$$\begin{aligned}
\left(\Delta_F[A^{g}_{\gamma\mu}]\right)^{-1} &= \int \prod_{x} dg'(x) \prod_{\alpha,x} \delta\left(F_{\alpha}\left(A^{gg'}_{\gamma\mu}\right)\right) \\
&= \int \prod_{x} d(gg')(x) \prod_{\alpha,x} \delta\left(F_{\alpha}\left(A^{gg'}_{\gamma\mu}\right)\right) \\
&= \int \prod_{x} dg''(x) \prod_{\alpha,x} \delta\left(F_{\alpha}\left(A^{g''}_{\gamma\mu}\right)\right) = \left(\Delta_F[A_{\gamma\mu}]\right)^{-1}.
\end{aligned}$$

Namely, we have

$$\Delta_F[A^{g}_{\gamma\mu}] = \Delta_F[A_{\gamma\mu}]. \tag{10.10.13}$$

We substitute the defining equation of the Faddeev–Popov determinant into the functional integrand on the right-hand side of the naive expression, (10.10.6), for the vacuum-to-vacuum transition amplitude $\langle 0, \text{out} | 0, \text{in} \rangle$.

$$\begin{aligned}
\left\langle 0, \text{out} \middle| 0, \text{in} \right\rangle \text{"} = \text{"} &\int \prod_{x} dg(x) \mathcal{D}[A_{\gamma\mu}] \Delta_F[A_{\gamma\mu}] \prod_{\alpha,x} \delta\left(F_{\alpha}\left(A^{g}_{\gamma\mu}(x)\right)\right) \\
&\times \exp\left[i \int d^4x \mathcal{L}_{\text{gauge}}\left(F_{\gamma\mu\nu}(x)\right)\right].
\end{aligned} \tag{10.10.14}$$

In (10.10.14), we perform the following gauge transformation:

$$A_{\gamma\mu}(x) \longrightarrow A^{g^{-1}}_{\gamma\mu}(x). \tag{10.10.15}$$

Under this gauge transformation, we have

$$\mathcal{D}[A_{\gamma\mu}] \longrightarrow \mathcal{D}[A_{\gamma\mu}],\ \Delta_F[A_{\gamma\mu}] \longrightarrow \Delta_F[A_{\gamma\mu}],\ \delta\left(F_\alpha\left(A^g_{\gamma\mu}(x)\right)\right) \rightarrow \delta\left(F_\alpha\left(A_{\gamma\mu}(x)\right)\right),$$

$$\int d^4x\mathcal{L}_{\text{gauge}}\left(F_{\gamma\mu\nu}(x)\right) \longrightarrow \int d^4x\mathcal{L}_{\text{gauge}}\left(F_{\gamma\mu\nu}(x)\right). \qquad (10.10.16)$$

Since the value of the functional integral remains unchanged under the change of the function variable, from (10.10.14) and (10.10.16), we have

$$\left\langle 0,\text{out}\middle|\,0,\text{in}\right\rangle '' = '' \int \prod_x dg(x) \int \mathcal{D}[A_{\gamma\mu}]\Delta_F[A_{\gamma\mu}] \prod_{\alpha,x} \delta\left(F_\alpha\left(A_{\gamma\mu}(x)\right)\right)$$
$$\times \exp\left[i\int d^4x\mathcal{L}_{\text{gauge}}\left(F_{\gamma\mu\nu}(x)\right)\right]$$

$$= V_G \int \mathcal{D}[A_{\gamma\mu}]\Delta_F[A_{\gamma\mu}] \prod_{\alpha,x} \delta\left(F_\alpha\left(A_{\gamma\mu}(x)\right)\right)$$
$$\times \exp\left[i\int d^4x\mathcal{L}_{\text{gauge}}\left(F_{\gamma\mu\nu}(x)\right)\right]. \qquad (10.10.17)$$

In this manner, we extract the orbit volume V_G, which is infinite and is independent of $A_{\gamma\mu}(x)$, and at the expense of the introduction of $\Delta_F[A_{\gamma\mu}]$, we perform the path integral of the gauge field $\int \mathcal{D}[A_{\gamma\mu}]$ on the hypersurface,

$$F_\alpha(A_{\gamma\mu}(x)) = 0, \quad \alpha = 1, \ldots, N, \qquad (10.10.8)$$

which intersects with each orbit of the gauge field only once. From now on, we drop V_G and we write the vacuum-to-vacuum transition amplitude under the gauge-fixing condition, (10.10.8), as

$$\left\langle 0,\text{out}\middle|\,0,\text{in}\right\rangle_F = \int \mathcal{D}[A_{\gamma\mu}]\Delta_F[A_{\gamma\mu}] \prod_{\alpha,x} \delta\left(F_\alpha\left(A_{\gamma\mu}(x)\right)\right)$$
$$\times \exp\left[i\int d^4x\mathcal{L}_{\text{gauge}}\left(F_{\gamma\mu\nu}(x)\right)\right]. \qquad (10.10.18)$$

In (10.10.18), the Faddeev–Popov determinant $\Delta_F[A_{\gamma\mu}]$ gets multiplied by $\prod_{\alpha,x}\delta(F_\alpha(A_{\gamma\mu}(x)))$. We calculate $\Delta_F[A_{\gamma\mu}]$ for $A_{\gamma\mu}(x)$ which satisfies the gauge-fixing condition, (10.10.8). Mimicking the parameterization of $U(g(x))$, we parameterize $F_\alpha(A^g_{\gamma\mu}(x))$ as

$$F_\alpha\left(A^g_{\gamma\mu}(x)\right) = F_\alpha\left(A_{\gamma\mu}(x)\right) + \sum_{\beta=1}^{N}\int d^4y M^F_{\alpha x,\beta y}(A_{\gamma\mu})\varepsilon_\beta(y) + O\left(\varepsilon^2\right). \qquad (10.10.19)$$

The matrix $M^F_{\alpha x,\beta y}(A_{\gamma\mu})$ is the kernel of a linear response with respect to $\varepsilon_\beta(y)$ of the gauge-fixing condition under the infinitesimal gauge transformation,

$$\delta A^g_{\gamma\mu}(x) = -\left(D^{\text{adj}}_\mu \varepsilon(x)\right)_\gamma . \tag{10.10.20}$$

Thus, from (10.10.19) and (10.10.20), we have

$$M^F_{\alpha x,\beta y}\left(A_{\gamma\mu}\right) = \left.\frac{\delta F_\alpha\left(A^g_{\gamma\mu}(x)\right)}{\delta\varepsilon_\beta(y)}\right|_{g=1} = \frac{\delta F_\alpha\left(A_{\gamma\mu}(x)\right)}{\delta A_{\gamma\mu}(x)}\left(-D^{\text{adj}}_\mu\right)_{\gamma\beta}\delta^4(x-y)$$

$$\equiv M^F_{\alpha,\beta}\left(A_{\gamma\mu}(x)\right)\delta^4(x-y). \tag{10.10.21}$$

Here, we have $\left(D^{\text{adj}}_\mu\right)_{\alpha\beta} = \partial_\mu\delta_{\alpha\beta} + C_{\alpha\beta\gamma}A_{\gamma\mu}(x)$. For those $A_{\gamma\mu}(x)$ which satisfy the gauge-fixing condition, (10.10.8), we have

$$\left(\Delta_F[A_{\gamma\mu}]\right)^{-1} = \int\prod_{\alpha,x} d\varepsilon_\alpha(x)\prod_{\alpha,x}\delta\left(M^F_{\alpha,\beta}\left(A_{\gamma\mu}(x)\right)\varepsilon_\beta(x)\right) = \left(\text{Det}M^F\left(A_{\gamma\mu}\right)\right)^{-1}.$$

Namely, we have

$$\Delta_F[A_{\gamma\mu}] = \text{Det}M^F\left(A_{\gamma\mu}\right). \tag{10.10.22}$$

We introduce the Faddeev–Popov ghost (fictitious scalar fermion), $\overline{c}_\alpha(x)$ and $c_\beta(x)$, in order to exponentiate the Faddeev–Popov determinant $\Delta_F[A_{\gamma\mu}]$. Then the determinant $\Delta_F[A_{\gamma\mu}]$ can be written as

$$\Delta_F[A_{\gamma\mu}] = \text{Det}M^F\left(A_{\gamma\mu}\right)$$

$$= \int\mathcal{D}[\overline{c}_\alpha]\mathcal{D}[c_\beta]\exp\left[i\int d^4x\,\overline{c}_\alpha(x)M^F_{\alpha,\beta}\left(A_{\gamma\mu}(x)\right)c_\beta(x)\right]. \tag{10.10.23}$$

Thus, we obtain the first formula of Faddeev–Popov:

$$\left\langle 0,\text{out}\middle|\,0,\text{in}\right\rangle_F = \int\mathcal{D}[A_{\gamma\mu}]\Delta_F[A_{\gamma\mu}]\prod_{\alpha,x}\delta\left(F_\alpha\left(A_{\gamma\mu}(x)\right)\right)$$
$$\times\exp\left[i\int d^4x\mathcal{L}_{\text{gauge}}\left(F_{\gamma\mu\nu}(x)\right)\right] \tag{10.10.24a}$$

$$= \int\mathcal{D}[A_{\gamma\mu}]\mathcal{D}[\overline{c}_\alpha]\mathcal{D}[c_\beta]\prod_{\alpha,x}\delta\left(F_\alpha\left(A_{\gamma\mu}(x)\right)\right)$$
$$\times\exp\left[i\int d^4x\left\{\mathcal{L}_{\text{gauge}}\left(F_{\gamma\mu\nu}(x)\right)\right.\right.$$
$$\left.\left.+\,\overline{c}_\alpha(x)M^F_{\alpha,\beta}\left(A_{\gamma\mu}(x)\right)c_\beta(x)\right\}\right]. \tag{10.10.24b}$$

Faddeev–Popov determinant $\Delta_F[A_{\gamma\mu}]$ plays the role of restoring the unitarity.

The Second Formula of Faddeev–Popov: We now generalize the gauge-fixing condition, (10.10.8):

$$F_\alpha\left(A_{\gamma\mu}(x)\right) = a_\alpha(x), \quad \alpha = 1, \ldots, N; \quad a_\alpha(x) \text{ independent of } A_{\alpha\mu}(x). \tag{10.10.25}$$

According to the first formula of Faddeev–Popov, (10.10.24a), we have

$$\mathcal{Z}_F(a(x)) \equiv \left\langle 0, \text{out} \middle| 0, \text{in} \right\rangle_{F,a} = \int \mathcal{D}[A_{\gamma\mu}]\Delta_F[A_{\gamma\mu}] \prod_{\alpha,x} \delta\left(F_\alpha\left(A_{\gamma\mu}(x)\right) - a_\alpha(x)\right) \times \exp\left[i \int d^4x \mathcal{L}_{\text{gauge}}\left(F_{\gamma\mu\nu}(x)\right)\right]. \tag{10.10.26}$$

We perform the nonlinear infinitesimal gauge transformation g_0, in the path integrand of (10.10.26), parametrized by $\lambda_\beta(y)$,

$$\varepsilon_\alpha\left(x; M^F\left(A_{\gamma\mu}\right)\right) = \left(M^F\left(A_{\gamma\mu}\right)\right)^{-1}_{\alpha x, \beta y} \lambda_\beta(y), \tag{10.10.27}$$

where $\lambda_\beta(y)$ is an arbitrary infinitesimal function, independent of $A_{\alpha\mu}(x)$. Under this g_0, we have the following three statements:

(1) The gauge invariance of the gauge field action functional,

$$\int d^4x \mathcal{L}_{\text{gauge}}\left(F_{\gamma\mu\nu}(x)\right) \text{ is invariant under } g_0.$$

(2) The gauge invariance of the integration measure,

$$\mathcal{D}[A_{\gamma\mu}]\Delta_F[A_{\gamma\mu}] = \text{ invariant measure under } g_0. \tag{10.10.28}$$

(3) The gauge-fixing condition $F_\alpha(A_{\gamma\mu}(x))$ is transformed into the following form:

$$F_\alpha\left(A^{g_0}_{\gamma\mu}(x)\right) = F_\alpha\left(A_{\gamma\mu}(x)\right) + \lambda_\alpha(x) + O\left(\lambda^2_\alpha(x)\right). \tag{10.10.29}$$

Since the value of the functional integral remains unchanged under the change of the function variable, the value of $\mathcal{Z}_F(a(x))$ remains unchanged under the nonlinear gauge transformation g_0. When we choose the infinitesimal parameter $\lambda_\alpha(x)$ as

$$\lambda_\alpha(x) = \delta a_\alpha(x), \quad \alpha = 1, \ldots, N, \tag{10.10.30}$$

we have from the above stated (1), (2), and (3),

$$\mathcal{Z}_F(a(x)) = \mathcal{Z}_F\left(a(x) + \delta a(x)\right) \quad \text{or} \quad \frac{d}{da(x)}\mathcal{Z}_F(a(x)) = 0. \tag{10.10.31}$$

Namely, $\mathcal{Z}_F(a(x))$ is independent of $a(x)$. Introducing an arbitrary weighting functional $H[a_\alpha(x)]$ for $\mathcal{Z}_F(a(x))$, and path-integrating with respect to $a_\alpha(x)$, we obtain the weighted $\mathcal{Z}_F(a(x))$,

$$\mathcal{Z}_F \equiv \int \prod_{\alpha,x} da_\alpha(x) H[a_\alpha(x)] \mathcal{Z}_F(a(x))$$

$$= \int \mathcal{D}\left[A_{\gamma\mu}\right] \Delta_F \left[A_{\gamma\mu}\right] H\left[F_\alpha(A_{\gamma\mu}(x))\right] \exp\left[i \int d^4x \mathcal{L}_{\text{gauge}}\left(F_{\gamma\mu\nu}(x)\right)\right]. \tag{10.10.32}$$

As the weighting functional, we choose the quasi-Gaussian functional,

$$H\left[a_\alpha(x)\right] = \exp\left[-\frac{i}{2}\int d^4x a_\alpha^2(x)\right]. \tag{10.10.33}$$

Then we obtain as $\mathcal{Z}_F$,

$$\mathcal{Z}_F = \int \mathcal{D}[A_{\gamma\mu}]\Delta_F[A_{\gamma\mu}] \exp\left[i \int d^4x \left\{\mathcal{L}_{\text{gauge}}\left(F_{\gamma\mu\nu}(x)\right) - \frac{1}{2}F_\alpha^2\left(A_{\gamma\mu}(x)\right)\right\}\right]$$

$$= \int \mathcal{D}[A_{\gamma\mu}]\mathcal{D}[\bar{c}_\alpha]\mathcal{D}[c_\beta] \exp\left[i \int d^4x \left\{\mathcal{L}_{\text{gauge}}\left(F_{\gamma\mu\nu}(x)\right) - \frac{1}{2}F_\alpha^2\left(A_{\gamma\mu}(x)\right)\right.\right.$$

$$\left.\left. + \bar{c}_\alpha(x) M^F_{\alpha,\beta}\left(A_{\gamma\mu}(x)\right) c_\beta(x)\right\}\right]. \tag{10.10.34}$$

Thus we obtain the second formula of Faddeev–Popov:

$$\left\langle 0, \text{out} \middle| 0, \text{in} \right\rangle_F = \int \mathcal{D}[A_{\gamma\mu}]\mathcal{D}[\bar{c}_\alpha]\mathcal{D}[c_\beta] \exp\left[iI_{\text{eff}}[A_{\gamma\mu}, \bar{c}_\alpha, c_\beta]\right], \tag{10.10.35a}$$

with the effective action functional $I_{\text{eff}}[A_{\gamma\mu}, \bar{c}_\alpha, c_\beta]$,

$$I_{\text{eff}}[A_{\gamma\mu}, \bar{c}_\alpha, c_\beta] = \int d^4x \mathcal{L}_{\text{eff}}, \tag{10.10.35b}$$

where the effective Lagrangian density $\mathcal{L}_{\text{eff}}$ is given by

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{gauge}}\left(F_{\gamma\mu\nu}(x)\right) - \frac{1}{2}F_\alpha^2\left(A_{\gamma\mu}(x)\right) + \bar{c}_\alpha(x) M^F_{\alpha,\beta}\left(A_{\gamma\mu}(x)\right) c_\beta(x). \tag{10.10.35c}$$

There are the other methods to arrive at this second formula, due to Vassiliev.

Choice of Gauge Fixing Condition: We consider the following gauge-fixing conditions:

(1) *Axial gauge*:

$$F_\alpha\left(A_{\gamma\mu}(x)\right) = n^\mu A_{\alpha\mu}(x) = 0, \quad \alpha = 1, \ldots, N, \tag{10.10.36a}$$

$$n^\mu = (0; 0, 0, 1), \quad n^2 = n^\mu n_\mu = -1, \quad nk = n^\mu k_\mu = -k_3. \tag{10.10.36b}$$

(2) *Landau gauge*:

$$F_\alpha\left(A_{\gamma\mu}(x)\right) = \partial^\mu A_{\alpha\mu}(x) = 0, \quad \alpha = 1, \ldots, N. \tag{10.10.37}$$

(3) *Covariant gauge*:

$$F_\alpha\left(A_{\gamma\mu}(x)\right) = \sqrt{\xi}\partial^\mu A_{\alpha\mu}(x), \quad \alpha = 1, \ldots, N, \quad 0 < \xi < \infty. \tag{10.10.38}$$

We obtain Green's functions $D^{(A)}_{\alpha\mu,\beta\nu}(x-y)$ of the gauge field $A_{\gamma\mu}(x)$ and $D^{(C)}_{\alpha,\beta}(x-y)$ of the Faddeev–Popov ghost fields, $\bar{c}_\alpha(x)$ and $c_\beta(x)$, and the effective interaction Lagrangian density $\mathcal{L}^{\text{int}}_{\text{eff}}$.

(1) *Axial gauge*: We use the first formula of Faddeev–Popov, (10.10.24a). Kernel $K^{(\text{axial})}_{\alpha\mu,\beta\nu}(x-y)$ of the quadratic part of the gauge field action functional $I_{\text{gauge}}[A_{\gamma\mu}]$ in the axial gauge, (10.10.36a) and (10.10.36b), is given by

$$\begin{aligned} I^{\text{quad}}_{\text{gauge}}[A_{\gamma\mu}] &= -\frac{1}{2}\int d^4x d^4y A^\mu_\alpha(x) K^{(\text{axial})}_{\alpha\mu,\beta\nu}(x-y) A^\nu_\beta(y) \\ K^{(\text{axial})}_{\alpha\mu,\beta\nu}(x-y) &= \delta_{\alpha\beta}\left(-\eta_{\mu\nu}\partial^2 + \partial_\mu\partial_\nu\right)\delta^4(x-y) \\ &\qquad \mu, \nu = 0, 1, 2; \quad \alpha, \beta = 1, \ldots, N. \end{aligned} \tag{10.10.39}$$

At first sight, this axial gauge kernel $K^{(\text{axial})}_{\alpha\mu,\beta\nu}(x-y)$ appears to be the noninvertible kernel. Since the Lorentz indices, μ and ν, run through 0,1,2 only, this axial gauge kernel $K^{(\text{axial})}_{\alpha\mu,\beta\nu}(x-y)$ is invertible. This has to do with the fact that the third spatial component of the gauge field $A_{\gamma\mu}(x)$ gets killed by the gauge-fixing condition, (10.10.36a) and (10.10.36b).

The "free" Green's function $D^{(\text{axial})}_{\alpha\mu,\beta\nu}(x-y)$ of the gauge field in the axial gauge satisfies the following equation:

$$\delta_{\alpha\beta}\left(\eta^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\right) D^{(\text{axial})}_{\beta\nu,\gamma\lambda}(x-y) = \delta_{\alpha\gamma}\eta^\mu_\lambda\delta^4(x-y). \tag{10.10.40}$$

Upon Fourier transforming the "free" Green's function $D^{(\text{axial})}_{\alpha\mu,\beta\nu}(x-y)$,

$$\begin{aligned} D^{(\text{axial})}_{\alpha\mu,\beta\nu}(x-y) &= \delta_{\alpha\beta} D^{(\text{axial})}_{\mu,\nu}(x-y) \\ &= \delta_{\alpha\beta}\int \frac{d^4k}{(2\pi)^4}\exp\left[-ik(x-y)\right] D^{(\text{axial})}_{\mu,\nu}(k), \end{aligned} \tag{10.10.41}$$

we have the momentum space Green's function $D^{(\text{axial})}_{\mu,\nu}(k)$ satisfying

$$\left(-\eta^{\mu\nu}k^2 + k^\mu k^\nu\right) D^{(\text{axial})}_{\nu\lambda}(k) = \eta^\mu_\lambda. \tag{10.10.42a}$$

We introduce the transverse projection operator $\Lambda_{\mu\nu}(n)$,

$$\begin{cases} \Lambda_{\mu\nu}(n) & = & \eta_{\mu\nu} - n_\mu n_\nu/n^2 & = & \eta_{\mu\nu} + n_\mu n_\nu, \\ n\Lambda(n) & = & \Lambda(n)n & = & 0, \\ \Lambda^2(n) & = & \Lambda(n), \end{cases} \tag{10.10.43}$$

and write the momentum space Green's function $D^{(\text{axial})}_{\nu\lambda}(k)$ as

$$D^{(\text{axial})}_{\nu\lambda}(k) = \Lambda_{\nu\sigma}(n)\left(\eta^{\sigma\rho}A(k^2) + k^\sigma k^\rho B(k^2)\right)\Lambda_{\rho\lambda}(n). \tag{10.10.44}$$

On multiplying by $\Lambda_{\tau\mu}(n)$ on the left-hand side of (10.10.42a), we have

$$\Lambda_{\tau\mu}(n)\left(-\eta^{\mu\nu}k^2 + k^\mu k^\nu\right) D^{(\text{axial})}_{\nu\lambda}(k) = \Lambda_{\tau\lambda}(n). \tag{10.10.42b}$$

Substituting (10.10.44) into (10.10.42b) and making use of the identities,

$$\left(k\Lambda(n)\right)_\mu = \left(\Lambda(n)k\right)_\mu = k_\mu - \frac{(nk)}{n^2}n_\mu, \quad k\Lambda(n)k = k^2 - \frac{(nk)^2}{n^2}, \tag{10.10.45}$$

we have

$$\left(-k^2A(k^2)\right)\Lambda_{\tau\lambda}(n) + \left(A(k^2) - \frac{(nk)^2}{n^2}B(k^2)\right)\left(\Lambda(n)k\right)_\tau \left(k\Lambda(n)\right)_\lambda = \Lambda_{\tau\lambda}(n). \tag{10.10.46}$$

We obtain $A(k^2)$, $B(k^2)$ and $D^{(\text{axial})}_{\nu\lambda}(k)$ as

$$A(k^2) = -\frac{1}{k^2}, \quad B(k^2) = -\frac{1}{k^2}\frac{n^2}{(nk)^2}, \tag{10.10.47}$$

$$D^{(\text{axial})}_{\nu\lambda}(k) = -\frac{1}{k^2}\left\{\eta_{\nu\lambda} - \frac{n_\nu n_\lambda}{n^2} + \frac{n^2}{(nk)^2}\left(k_\nu - \frac{(nk)}{n^2}n_\nu\right)\left(k_\lambda - \frac{(nk)}{n^2}n_\lambda\right)\right\} \tag{10.10.48a}$$

$$= -\frac{1}{k^2}\left\{\eta_{\nu\lambda} + \frac{n^2}{(nk)^2}k_\nu k_\lambda - \frac{1}{(nk)}n_\nu k_\lambda - \frac{1}{(nk)}k_\nu n_\lambda\right\} \tag{10.10.48b}$$

$$= -\frac{1}{k^2}\left\{\eta_{\nu\lambda} - \frac{k_\nu k_\lambda}{k_3^2} + \frac{n_\nu k_\lambda}{k_3} + \frac{k_\nu n_\lambda}{k_3}\right\}. \tag{10.10.48c}$$

Here in (10.10.48c) for the first time in this derivation, the following explicit representations for the axial gauge are used,

$$n^2 = -1, \quad nk = -k_3. \tag{10.10.36b}$$

Next we demonstrate that the Faddeev–Popov ghost does not show up in the axial gauge. The kernel of the Faddeev–Popov ghost Lagrangian density is given by

$$M^{(\text{axial})}_{\alpha x,\beta y}\left(A_{\gamma\mu}\right) = \frac{\delta F_\alpha\left(A_{\gamma\mu}(x)\right)}{\delta A_{\gamma\mu}(x)}\left(-D^{\text{adj}}_\mu\right)_{\gamma\beta}\delta^4(x-y)$$

$$= -n^\mu\left(\delta_{\alpha\beta}\partial_\mu + C_{\alpha\beta\gamma}A_{\gamma\mu}(x)\right)\delta^4(x-y) = \delta_{\alpha\beta}\partial_3\delta^4(x-y). \tag{10.10.49}$$

So the Faddeev–Popov determinant $\Delta_F[A_{\gamma\mu}]$ is an infinite number independent of the gauge field $A_{\gamma\mu}(x)$, and the Faddeev–Popov ghost does not show up.

In this way, we note that we can carry out the canonical quantization of the non-Abelian gauge field theory in the axial gauge, starting out with the separation of the dynamical variable and the constraint variable. Thus we can apply the phase space path integral formula with the minor change of the insertion of the gauge-fixing condition $\prod_{\alpha,x}\delta(n^\mu A_{\alpha\mu}(x))$ in the functional integrand. In the phase space path integral formula, with the use of the constraint equation, we perform the momentum integration and obtain

$$\left\langle 0,\text{out}\middle|\,0,\text{in}\right\rangle = \int \mathcal{D}[A_{\gamma\mu}]\prod_{\alpha,x}\delta\left(n^\mu A_{\alpha\mu}(x)\right)\exp\left[i\int d^4x\mathcal{L}_{\text{gauge}}\left(F_{\gamma\mu\nu}(x)\right)\right]. \tag{10.10.50}$$

In (10.10.50), the manifest covariance is miserably destroyed. This point will be overcome by the gauge transformation from the axial gauge to some covariant gauge, say Landau gauge, and the Faddeev–Popov determinant $\Delta_F[A_{\gamma\mu}]$ will show up again as the Jacobian of the change of the function variable (the gauge transformation) in the delta function,

$$F^{(\text{axial})}_\alpha\left(A_{\gamma\mu}(x)\right) \longrightarrow F^{(\text{covariant})}_\alpha\left(A_{\gamma\mu}(x)\right). \tag{10.10.51}$$

The effective interaction Lagrangian density $\mathcal{L}^{\text{int}}_{\text{eff}}$ is given by

$$\mathcal{L}^{\text{int}}_{\text{eff}} = \mathcal{L}_{\text{eff}} - \mathcal{L}^{\text{quad}}_{\text{eff}}. \tag{10.10.52}$$

(2) *Landau gauge*: Kernel $K^{(\text{Landau})}_{\alpha\mu,\beta\nu}(x-y)$ of the quadratic part of the gauge field action functional $I_{\text{gauge}}[A_{\gamma\mu}]$ in the Landau gauge, (10.10.37),

$$I^{\text{quad}}_{\text{gauge}}[A_{\gamma\mu}] = -\frac{1}{2}\int d^4xd^4yA^{\mu}_{\alpha}(x)K^{(\text{Landau})}_{\alpha\mu,\beta\nu}(x-y)A^{\nu}_{\beta}(y)$$

is given by

$$K^{(\text{Landau})}_{\alpha\mu,\beta\nu}(x-y) = \delta_{\alpha\beta}\left(-\eta_{\mu\nu}\partial^2 + \partial_{\mu}\partial_{\nu}\right)\delta^4(x-y), \tag{10.10.53a}$$

$$\mu,\nu = 0,1,2,3; \quad \alpha,\beta = 1,\ldots,N, \tag{10.10.53b}$$

which is the four-dimensional transverse projection operator and hence is not invertible. The "free" Green's function $D^{(\text{Landau})}_{\alpha\mu,\beta\nu}(x-y)$ of the gauge field in the Landau gauge satisfies the following equation:

$$\delta_{\alpha\beta}\left(\eta^{\mu\nu}\partial^2 - \partial^{\mu}\partial^{\nu}\right)D^{(\text{Landau})}_{\beta\nu,\gamma\lambda}(x-y) = \delta_{\alpha\gamma}\eta^{\mu}_{\lambda}\delta^4(x-y). \tag{10.10.54}$$

We Fourier-transform the "free" Green's function $D^{(\text{Landau})}_{\alpha\mu,\beta\nu}(x-y)$ as

$$\begin{aligned} D^{(\text{Landau})}_{\alpha\mu,\beta\nu}(x-y) &= \delta_{\alpha\beta}D^{(\text{Landau})}_{\mu,\nu}(x-y) \\ &= \delta_{\alpha\beta}\int\frac{d^4k}{(2\pi)^4}\exp\left[-ik(x-y)\right]D^{(\text{Landau})}_{\mu\nu}(k). \end{aligned} \tag{10.10.55}$$

We have the momentum space Green's function $D^{(\text{Landau})}_{\mu\nu}(k)$ satisfying

$$\left(-\eta^{\mu\nu}k^2 + k^{\mu}k^{\nu}\right)D^{(\text{Landau})}_{\nu\lambda}(k) = \eta^{\mu}_{\lambda}. \tag{10.10.56a}$$

We introduce the four-dimensional transverse projection operator $\Lambda_{\text{T}}(k)$ by

$$\Lambda_{\text{T}\,\mu\nu}(k) = \eta_{\mu\nu} - \frac{k_{\mu}k_{\nu}}{k^2},\ k\Lambda_{\text{T}}(k) = \Lambda_{\text{T}}(k)k = 0,\ \Lambda^2_{\text{T}}(k) = \Lambda_{\text{T}}(k), \tag{10.10.57}$$

and write $D^{(\text{Landau})}_{\nu\lambda}(k)$ as

$$D^{(\text{Landau})}_{\nu\lambda}(k) = \Lambda_{\text{T}\,\nu\lambda}(k)A(k^2). \tag{10.10.58}$$

Multiplying by $\Lambda_{\text{T}\,\tau\mu}(k)$ on the left-hand side of (10.10.56a), we have

$$\Lambda_{\text{T}\,\tau\mu}(k)(-k^2)\Lambda^{\mu\nu}_{\text{T}}(k)D^{(\text{Landau})}_{\nu\lambda}(k) = \Lambda_{\text{T}\,\tau\lambda}(k). \tag{10.10.56b}$$

Substituting (10.10.58) into the left-hand side of (10.10.56b), and noting the idempotence of $\Lambda_{\mathrm{T}}(k)$, we have

$$\left(-k^2 A(k^2)\right) \Lambda_{\mathrm{T}\,\tau\lambda}(k) = \Lambda_{\mathrm{T}\,\tau\lambda}(k). \tag{10.10.59}$$

From (10.10.59), we obtain

$$A(k^2) = -\frac{1}{k^2}, \tag{10.10.60}$$

$$D^{(\mathrm{Landau})}_{\nu\lambda}(k) = -\frac{1}{k^2}\left(\eta_{\nu\lambda} - \frac{k_\nu k_\lambda}{k^2}\right). \tag{10.10.61}$$

We now calculate the kernel $M^{(\mathrm{Landau})}_{\alpha x,\beta y}(A_{\gamma\mu})$ of the Faddeev–Popov ghost in the Landau gauge.

$$\begin{aligned} M^{(\mathrm{Landau})}_{\alpha x,\beta y}\left(A_{\gamma\mu}\right) &= \frac{\delta F_\alpha\left(A_{\gamma\mu}(x)\right)}{\delta A_{\gamma\mu}(x)}\left(-D^{\mathrm{adj}}_\mu\right)_{\gamma\beta}\delta^4(x-y) \\ &= -\partial^\mu\left(\partial_\mu\delta_{\alpha\beta} + C_{\alpha\beta\gamma}A_{\gamma\mu}(x)\right)\delta^4(x-y) \\ &\equiv M^{\mathrm{L}}_{\alpha,\beta}\left(A_{\gamma\mu}(x)\right)\delta^4(x-y). \end{aligned} \tag{10.10.62}$$

The Faddeev–Popov ghost part of $I_{\mathrm{eff}}[A_{\gamma\mu}, \overline{c}_\alpha, c_\beta]$ is given by

$$\begin{aligned} I^{\mathrm{ghost}}_{\mathrm{eff}}\left[A_{\gamma\mu}, \overline{c}_\alpha, c_\beta\right] &= \int d^4x\overline{c}_\alpha(x)M^{\mathrm{L}}_{\alpha,\beta}\left(A_{\gamma\mu}(x)\right)c_\beta(x) \\ &= \int d^4x\partial^\mu\overline{c}_\alpha(x)\left(D^{\mathrm{adj}}_\mu\right)_{\alpha\beta}c_\beta(x). \end{aligned} \tag{10.10.63}$$

From this, we find that the Faddeev–Popov ghost in the Landau gauge is the massless scalar Fermion and that Green's function $D^{(C)}_{\mathrm{L}\,\alpha,\beta}(x-y)$ is given by

$$D^{(C)}_{\mathrm{L}\,\alpha,\beta}(x-y) = \delta_{\alpha\beta}\int\frac{d^4k}{(2\pi)^4}\exp\left[-ik(x-y)\right]\frac{1}{k^2}. \tag{10.10.64}$$

The effective interaction Lagrangian density $\mathcal{L}^{\mathrm{int}}_{\mathrm{eff}}$ is given by the following expression and there emerges the oriented ghost–ghost–gauge coupling in the internal loop,

$$\begin{aligned}
\mathcal{L}_{\text{eff}}^{\text{int}} &= \mathcal{L}_{\text{eff}} - \mathcal{L}_{\text{eff}}^{\text{quad}} \\
&= \mathcal{L}_{\text{gauge}}\left(F_{\gamma\mu\nu}(x)\right) + \overline{c}_{\alpha}(x) M_{\alpha,\beta}^{\text{L}}\left(A_{\gamma\mu}(x)\right) c_{\beta}(x) \\
&\quad - \frac{1}{2} A_{\alpha}^{\mu}(x)\delta_{\alpha\beta}\left(\eta_{\mu\nu}\partial^2 - \partial_{\mu}\partial_{\nu}\right) A_{\beta}^{\nu}(x) - \overline{c} \\
&\quad {}_{\alpha}(x)\delta_{\alpha\beta}\left(-\partial^2\right) c_{\beta}(x) \\
&= \frac{1}{2} C_{\alpha\beta\gamma}\left(\partial_{\mu} A_{\alpha\nu}(x) - \partial_{\nu} A_{\alpha\mu}(x)\right) A_{\beta}^{\mu}(x) A_{\gamma}^{\nu}(x) \\
&\quad - \frac{1}{4} C_{\alpha\beta\gamma} C_{\alpha\delta\varepsilon} A_{\beta\mu}(x) A_{\gamma\nu}(x) A_{\delta}^{\mu}(x) A_{\varepsilon}^{\nu}(x) \\
&\quad + C_{\alpha\beta\gamma}\partial^{\mu}\overline{c}_{\alpha}(x) A_{\gamma\mu}(x) c_{\beta}(x).
\end{aligned} \tag{10.10.65}$$

(3) *Covariant gauge*: Kernel $K_{\alpha\mu,\beta\nu}(x-y;\xi)$ of the quadratic part of the effective action functional $I_{\text{eff}}[A_{\gamma\mu}, \overline{c}_{\alpha}, c_{\beta}]$ with respect to the gauge field $A_{\gamma\mu}(x)$ in the covariant gauge, (10.10.38),

$$\begin{aligned}
I_{\text{eff}}^{\text{quad. in } A_{\mu}^{\gamma}}\left[A_{\gamma\mu}, \overline{c}_{\alpha}, c_{\beta}\right] &= \int d^4x \left\{\mathcal{L}_{\text{gauge}}\left(F_{\gamma\mu\nu}(x)\right) - \frac{1}{2} F_{\alpha}^2\left(A_{\gamma\mu}(x)\right)\right\}^{\text{quad. in } A_{\mu}^{\gamma}} \\
&\equiv -\frac{1}{2}\int d^4x d^4y A_{\alpha}^{\mu}(x) K_{\alpha\mu,\beta\nu}(x-y;\xi) A_{\beta}^{\nu}(y),
\end{aligned} \tag{10.10.66}$$

is given by

$$K_{\alpha\mu,\beta\nu}(x-y;\xi) = \delta_{\alpha\beta}\left\{-\eta_{\mu\nu}\partial^2 + (1-\xi)\partial_{\mu}\partial_{\nu}\right\}\delta^4(x-y), \tag{10.10.67}$$

and is invertible for $\xi > 0$. The "free" Green's function $D_{\alpha\mu,\beta\nu}^{(\text{cov})}(x-y;\xi)$ of the gauge field $A_{\gamma\mu}(x)$ in the covariant gauge satisfies the following equation:

$$\delta_{\alpha\beta}\left\{\eta^{\mu\nu}\partial^2 - (1-\xi)\partial^{\mu}\partial^{\nu}\right\} D_{\beta\nu,\gamma\lambda}^{(\text{cov})}(x-y;\xi) = \delta_{\alpha\gamma}\eta_{\lambda}^{\mu}\delta^4(x-y). \tag{10.10.68}$$

We Fourier-transform the "free" Green's function $D_{\alpha\mu,\beta\nu}^{(\text{cov})}(x-y;\xi)$ as

$$D_{\alpha\mu,\beta\nu}^{(\text{cov})}(x-y;\xi) = \delta_{\alpha\beta}\int \frac{d^4k}{(2\pi)^4}\exp\left[-ik(x-y)\right] D_{\mu,\nu}^{(\text{cov})}(k;\xi). \tag{10.10.69}$$

We have the momentum space Green's function $D_{\mu,\nu}^{(\text{cov})}(k;\xi)$ satisfying

$$\left\{-\eta^{\mu\nu}k^2 + (1-\xi)k^{\mu}k^{\nu}\right\} D_{\nu,\lambda}^{(\text{cov})}(k;\xi) = \eta_{\lambda}^{\mu}. \tag{10.10.70a}$$

We define the four-dimensional projection operators, $\Lambda_{\text{T}}(k)$ and $\Lambda_{\text{L}}(k)$, by

$$\Lambda_{\text{T}\,\mu,\nu}(k) = \eta_{\mu,\nu} - \frac{k_{\mu}k_{\nu}}{k^2}, \quad \Lambda_{\text{L}\,\mu,\nu}(k) = \frac{k_{\mu}k_{\nu}}{k^2}, \tag{10.10.71}$$

$$\Lambda_{\mathrm{T}}^2(k) = \Lambda_{\mathrm{T}}(k), \quad \Lambda_{\mathrm{L}}^2(k) = \Lambda_{\mathrm{L}}(k), \quad \Lambda_{\mathrm{T}}(k)\Lambda_{\mathrm{L}}(k) = \Lambda_{\mathrm{L}}(k)\Lambda_{\mathrm{T}}(k) = 0. \quad (10.10.72)$$

Equation (10.10.70a) is written in terms of $\Lambda_{\mathrm{T}}(k)$ and $\Lambda_{\mathrm{L}}(k)$ as

$$\left(-k^2\right)\left(\Lambda_{\mathrm{T}}^{\mu,\nu}(k) + \xi\,\Lambda_{\mathrm{L}}^{\mu,\nu}(k)\right) D_{\nu,\lambda}^{(\mathrm{cov})}(k;\xi) = \eta_{\lambda}^{\mu}. \quad (10.10.70b)$$

Expressing $D_{\nu,\lambda}^{(\mathrm{cov})}(k;\xi)$ as

$$D_{\nu,\lambda}^{(\mathrm{cov})}(k;\xi) = \Lambda_{\mathrm{T}\,\nu,\lambda}(k)A(k^2) + \Lambda_{\mathrm{L}\,\nu,\lambda}(k)B(k^2;\xi), \quad (10.10.73)$$

and making use of (10.10.72), we obtain

$$\left(-k^2 A(k^2)\right)\Lambda_{\mathrm{T}\,\lambda}^{\mu}(k) + \left(-\xi k^2 B(k^2;\xi)\right)\Lambda_{\mathrm{L}\,\lambda}^{\mu}(k) = \eta_{\lambda}^{\mu}. \quad (10.10.74)$$

Multiplying by $\Lambda_{\mathrm{T}\,\tau,\mu}(k)$ and $\Lambda_{\mathrm{L}\,\tau,\mu}(k)$ on the left-hand side of (10.10.74), respectively, we obtain

$$\left(-k^2 A(k^2)\right)\Lambda_{\mathrm{T}\,\tau,\lambda}(k) = \Lambda_{\mathrm{T}\,\tau,\lambda}(k), \quad \left(-\xi k^2 B(k^2;\xi)\right)\Lambda_{\mathrm{L}\,\tau,\lambda}(k) = \Lambda_{\mathrm{L}\,\tau,\lambda}(k), \quad (10.10.75)$$

from which, we obtain

$$A\left(k^2\right) = -\frac{1}{k^2}, \quad B\left(k^2;\xi\right) = -\frac{1}{\xi}\frac{1}{k^2}, \quad (10.10.76)$$

$$\begin{aligned} D_{\nu,\lambda}^{(\mathrm{cov})}(k;\xi) &= -\frac{1}{k^2}\left\{\left(\eta_{\nu,\lambda} - \frac{k_\nu k_\lambda}{k^2}\right) + \frac{1}{\xi}\frac{k_\nu k_\lambda}{k^2}\right\} \\ &= -\frac{1}{k^2}\left\{\eta_{\nu,\lambda} - \left(1 - \frac{1}{\xi}\right)\frac{k_\nu k_\lambda}{k^2}\right\}. \end{aligned} \quad (10.10.77)$$

The parameter ξ shifts the longitudinal component of the covariant gauge Green's function. In the limit $\xi \to \infty$, the covariant gauge Green's function $D_{\nu,\lambda}^{(\mathrm{cov})}(k;\xi)$ coincides with the Landau gauge Green's function $D_{\nu,\lambda}^{(\mathrm{Landau})}(k)$, and, at $\xi = 1$, the covariant gauge Green's function $D_{\nu,\lambda}^{(\mathrm{cov})}(k;\xi)$ coincides with the 't Hooft–Feynman gauge Green's function $D_{\nu,\lambda}^{(\mathrm{Feynman})}(k)$,

$$D_{\nu,\lambda}^{(\mathrm{cov})}(k;\infty) = -\frac{1}{k^2}\left(\eta_{\nu,\lambda} - \frac{k_\nu k_\lambda}{k^2}\right) = D_{\nu,\lambda}^{(\mathrm{Landau})}(k), \quad (10.10.78a)$$

$$D_{\nu,\lambda}^{(\mathrm{cov})}(k;1) = -\frac{1}{k^2}\eta_{\nu,\lambda} = D_{\nu,\lambda}^{(\mathrm{Feynman})}(k). \quad (10.10.78b)$$

We calculate the kernel $M_{\alpha x,\beta y}^{(\mathrm{cov})}(A_{\gamma\mu})$ of the Faddeev–Popov ghost in the covariant gauge.

$$\begin{aligned} M_{\alpha x,\beta y}^{(\mathrm{cov})}\left(A_{\gamma\mu}\right) &= \frac{\delta F_\alpha\left(A_{\gamma\mu}(x)\right)}{\delta A_{\gamma\mu}(x)}\left(-D_\mu^{\mathrm{adj}}\right)_{\gamma\beta}\delta^4(x-y) = \sqrt{\xi}\,M_{\alpha x,\beta y}^{(\mathrm{Landau})}\left(A_{\gamma\mu}\right) \\ &= \sqrt{\xi}\,M_{\alpha,\beta}^{\mathrm{L}}\left(A_{\gamma\mu}(x)\right)\delta^4(x-y). \end{aligned} \quad (10.10.79)$$

We absorb $\sqrt{\xi}$ factor in the normalization of the Faddeev–Popov ghost field, $\overline{c}_\alpha(x)$ and $c_\beta(x)$. The ghost part of the effective action functional is then given by

$$I_{\text{eff}}^{\text{ghost}}\left[A_{\gamma\mu},\overline{c}_\alpha,c_\beta\right]=\int d^4x\overline{c}_\alpha(x)M^{\text{L}}_{\alpha,\beta}(A_{\gamma\mu}(x))c_\beta(x)$$

$$=\int d^4x\partial^\mu\overline{c}_\alpha(x)\left(D^{\text{adj}}_\mu\right)_{\alpha\beta}c_\beta(x), \tag{10.10.80}$$

just like in the Landau gauge, (10.10.63). From this, we find that the Faddeev–Popov ghost in the covariant gauge is the massless scalar Fermion whose Green's function $D^{(C)}_{\text{L}\,\alpha,\beta}(x-y)$ is given by

$$D^{(C)}_{\text{L}\,\alpha,\beta}(x-y)=\delta_{\alpha\beta}\int\frac{d^4k}{(2\pi)^4}\exp\left[-ik(x-y)\right]\frac{1}{k^2}. \tag{10.10.81}$$

We calculate the interaction Lagrangian density $\mathcal{L}^{\text{int}}_{\text{eff}}$ in the covariant gauge. We have

$$\mathcal{L}_{\text{eff}}=\mathcal{L}_{\text{gauge}}\left(F_{\gamma\mu\nu}(x)\right)-\frac{1}{2}F^2_\alpha\left(A_{\gamma\mu}(x)\right)+\overline{c}_\alpha(x)M^{\text{L}}_{\alpha,\beta}\left(A_{\gamma\mu}(x)\right)c_\beta(x), \tag{10.10.82}$$

$$\mathcal{L}^{\text{quad}}_{\text{eff}}=-\frac{1}{2}A^\mu_\alpha(x)\delta_{\alpha\beta}\left\{-\eta_{\mu\nu}\partial^2+(1-\xi)\partial_\mu\partial_\nu\right\}A^\nu_\beta(x)+\overline{c}_\alpha(x)\delta_{\alpha\beta}\left(-\partial^2\right)c_\beta(x). \tag{10.10.83}$$

From (10.10.82) and (10.10.83), we obtain $\mathcal{L}^{\text{int}}_{\text{eff}}$ as

$$\begin{aligned}\mathcal{L}^{\text{int}}_{\text{eff}}&=\mathcal{L}_{\text{eff}}-\mathcal{L}^{\text{quad}}_{\text{eff}}\\&=\frac{1}{2}C_{\alpha\beta\gamma}\left(\partial_\mu A_{\alpha\nu}(x)-\partial_\nu A_{\alpha\mu}(x)\right)A^\mu_\beta(x)A^\nu_\gamma(x)\\&\quad-\frac{1}{4}C_{\alpha\beta\gamma}C_{\alpha\delta\varepsilon}A_{\beta\mu}(x)A_{\gamma\nu}(x)A^\mu_\delta(x)A^\nu_\varepsilon(x)+C_{\alpha\beta\gamma}\partial^\mu\overline{c}_\alpha A_{\gamma\mu}(x)c_\beta(x).\end{aligned} \tag{10.10.84a}$$

The generating functional $Z_F[J_{\gamma\mu},\zeta_\alpha,\overline{\zeta}_\beta]$ ($W_F[J_{\gamma\mu},\zeta_\alpha,\overline{\zeta}_\beta]$) of (the connected parts of) the "full" Green's functions in the covariant gauge is given by

$$\begin{aligned}Z_F\left[J_{\gamma\mu},\zeta_\alpha,\overline{\zeta}_\beta\right]&\equiv\exp\left[iW_F\left[J_{\gamma\mu},\zeta_\alpha,\overline{\zeta}_\beta\right]\right]\\&\equiv\left\langle 0,\text{out}\middle|\text{T}\left(\exp\left[i\int d^4z\left\{J_{\gamma\mu}(z)\hat{A}^\mu_\gamma(z)\right.\right.\right.\right.\\&\quad\left.\left.\left.\left.+\ \overline{c}_\alpha(z)\zeta_\alpha(z)+\overline{\zeta}_\beta(z)c_\beta(z)\right\}\right]\right)\middle|0,\text{in}\right\rangle\end{aligned}$$

$$= \int \mathcal{D}[A_{\gamma\mu}]\mathcal{D}[\overline{c}_\alpha]\mathcal{D}[c_\beta]\exp\left[i\int d^4z\left\{\mathcal{L}_{\text{eff}} + J_{\gamma\mu}(z)A^\mu_\gamma(z) + \overline{c}_\alpha(z)\zeta_\alpha(z) + \overline{\zeta}_\beta(z)c_\beta(z)\right\}\right]$$

$$= \exp\left[iI^{\text{int}}_{\text{eff}}\left[\frac{1}{i}\frac{\delta}{\delta J^\mu_\gamma}, i\frac{\delta}{\delta\zeta_\alpha}, \frac{1}{i}\frac{\delta}{\delta\overline{\zeta}_\beta}\right]\right] Z_{F,0}\left[J_{\gamma\mu}, \zeta_\alpha, \overline{\zeta}_\beta\right]. \tag{10.10.85}$$

We have $I^{\text{int}}_{\text{eff}}[A_{\gamma\mu}, \overline{c}_\alpha, c_\beta]$ as

$$I^{\text{int}}_{\text{eff}}\left[A_{\gamma\mu}, \overline{c}_\alpha, c_\beta\right] = \int d^4x\mathcal{L}^{\text{int}}_{\text{eff}}\left((10.10.84\text{a})\right). \tag{10.10.84b}$$

The generating functional $Z_{F,0}[J_{\gamma\mu}, \zeta_\alpha, \overline{\zeta}_\beta]$ is given by

$$Z_{F,0}\left[J_{\gamma\mu}, \zeta_\alpha, \overline{\zeta}_\beta\right] = \int \mathcal{D}[A_{\gamma\mu}]\mathcal{D}[\overline{c}_\alpha]\mathcal{D}[c_\beta]\exp\left[i\int d^4z\left\{\mathcal{L}^{\text{quad}}_{\text{eff}}((83)) + J_{\gamma\mu}(z)A^\mu_\gamma(z) + \overline{c}_\alpha(z)\zeta_\alpha(z) + \overline{\zeta}_\beta(z)c_\beta(z)\right\}\right]$$

$$= \exp\left[i\int d^4x d^4y\left\{-\frac{1}{2}J^\mu_\alpha(x)D^{(\text{cov})}_{\alpha\mu,\beta\nu}(x-y;\xi)J^\nu_\beta(y) - \overline{\zeta}_\beta(x)D^{(C)}_{\alpha\beta}(x-y)\zeta_\alpha(y)\right\}\right]. \tag{10.10.86}$$

Equation (10.10.85) is the starting point of Feynman–Dyson expansion of the "full" Green's function with the "free" Green's functions and the effective interaction vertices, (10.10.84a). Noting the fact that the Faddeev–Popov ghost appears only in the internal loops, we might as well set $\zeta_\alpha(z) = \overline{\zeta}_\beta(z) = 0$ in $Z_F[J_{\gamma\mu}, \zeta_\alpha, \overline{\zeta}_\beta]$ and define $Z_F[J_{\gamma\mu}]$ by

$$Z_F[J_{\gamma\mu}] \equiv \exp\left[iW_F\left[J_{\gamma\mu}\right]\right] \equiv \left\langle 0, \text{out}\left|\mathrm{T}\left(\exp\left[i\int d^4z J_{\gamma\mu}(z)\hat{A}^\mu_\gamma(z)\right]\right)\right|0, \text{in}\right\rangle$$

$$= \int \mathcal{D}[A_{\gamma\mu}]\mathcal{D}[\overline{c}_\alpha]\mathcal{D}[c_\beta]\exp\left[i\int d^4z\left\{\mathcal{L}_{\text{eff}} + J_{\gamma\mu}(z)A^\mu_\gamma(z)\right\}\right]$$

$$= \int \mathcal{D}[A_{\gamma\mu}]\Delta_F[A_{\gamma\mu}]\exp\left[i\int d^4z\left\{\mathcal{L}_{\text{gauge}}\left(F_{\gamma\mu\nu}(z)\right) - \frac{1}{2}F^2_\alpha\left(A_{\gamma\mu}(z)\right) + J_{\gamma\mu}(z)A^\mu_\gamma(z)\right\}\right]. \tag{10.10.87}$$

This generating functional of the (connected part of) Green's functions can be used for the proof of the gauge independence of the physical S-matrix.

10.11
Path Integral Quantization of Gauge Field II

We discuss the *method with which we give a mass term to the gauge field without violating gauge invariance*. We write the n-component real scalar field $\phi_i(x)$, $i = 1, \ldots, n$, as

$$\tilde{\phi}(x) = \begin{pmatrix} \phi_1(x) \\ \cdot \\ \cdot \\ \phi_n(x) \end{pmatrix}, \quad \tilde{\phi}^{\mathrm{T}}(x) = \begin{pmatrix} \phi_1(x), & \ldots & \phi_n(x) \end{pmatrix}, \tag{10.11.1}$$

and express the globally G invariant matter field Lagrangian density as

$$\mathcal{L}_{\text{matter}}(\tilde{\phi}(x), \partial_\mu \tilde{\phi}(x)) = \frac{1}{2} \partial_\mu \tilde{\phi}^{\mathrm{T}}(x) \partial^\mu \tilde{\phi}(x) - V(\tilde{\phi}(x)). \tag{10.11.2}$$

Here we assume the following three conditions:

(1) G is a semisimple N-parameter Lie group whose N generators we write as T_α, $\alpha = 1, \ldots, N$.

(2) Under the global G transformation, $\tilde{\phi}(x)$ transform with the n-dimensional reducible representation, θ_α, $\alpha = 1, \ldots, N$,

$$\delta \phi_i(x) = i \varepsilon_\alpha (\theta_\alpha)_{ij} \phi_j(x), \quad \alpha = 1, \ldots, N, \quad i, j = 1, \ldots, n. \tag{10.11.3}$$

(3) $V(\tilde{\phi}(x))$ is the globally G invariant quartic polynomial in $\tilde{\phi}(x)$ whose the global G invariance condition is given by

$$\frac{\partial V\left(\tilde{\phi}(x)\right)}{\partial \phi_i(x)} (\theta_\alpha)_{ij} \phi_j(x) = 0, \quad \alpha = 1, \ldots, N. \tag{10.11.4}$$

We assume that the minimizing $\tilde{\phi}(x)$ of $V(\tilde{\phi}(x))$ exists and is given by the constant vector $\tilde{v}$,

$$\left. \frac{\partial V\left(\tilde{\phi}(x)\right)}{\partial \phi_i(x)} \right|_{\tilde{\phi}(x) = \tilde{v}} = 0, \quad i = 1, \ldots, n. \tag{10.11.5}$$

We differentiate the global G invariance condition, (10.11.4), with respect to $\phi_k(x)$ and set $\tilde{\phi}(x) = \tilde{v}$, obtaining the broken symmetry condition,

$$\left. \frac{\partial^2 V\left(\tilde{\phi}(x)\right)}{\partial \phi_k(x) \partial \phi_i(x)} \right|_{\tilde{\phi}(x) = \tilde{v}} (\theta_\alpha)_{ij} v_j = 0, \quad \alpha = 1, \ldots, N, \quad i, j, k = 1, \ldots, n. \tag{10.11.6}$$

We expand $V(\tilde{\phi}(x))$ around $\tilde{\phi}(x) = \tilde{v}$ and choose $V(\tilde{v}) = 0$ as the origin of energy.

$$V\left(\tilde{\phi}(x)\right) = \frac{1}{2}\left(\tilde{\phi}(x) - \tilde{v}\right)^{\mathrm{T}} \left(M^2\right) \left(\tilde{\phi}(x) - v\right) + \text{higher order term.} \quad (10.11.7a)$$

We set

$$\left(M^2\right)_{i,j} \equiv \left.\frac{\partial^2 V\left(\tilde{\phi}(x)\right)}{\partial\phi_i(x)\partial\phi_j(x)}\right|_{\tilde{\phi}(x)=\tilde{v}}, \quad i,j = 1,\ldots,n, \quad (10.11.7b)$$

where M^2 is the mass matrix of the n-component real scalar field. We express the broken symmetry condition, (10.11.6), in terms of the mass matrix M^2 as

$$\left(M^2\right)_{i,j} (\theta_\alpha \tilde{v})_j = 0, \quad \alpha = 1,\ldots,N, \quad i,j = 1,\ldots,n. \quad (10.11.8)$$

We let the stability group of the vacuum, i.e., the symmetry group of the ground state $\tilde{\phi}(x) = \tilde{v}$, be the M-dimensional subgroup $S \subset G$. When θ_α is the realization on the scalar field $\tilde{\phi}(x)$ of the generator T_α belonging to the stability group S, this θ_α annihilates the "vacuum" $\tilde{v}$

$$(\theta_\alpha \tilde{v}) = 0 \quad \text{for} \quad \theta_\alpha \in S, \quad (10.11.9)$$

and we have the invariance of $\tilde{v}$ expressed as

$$\exp\left[i \sum_{\theta_\alpha \in S} \varepsilon_\alpha \theta_\alpha\right] \tilde{v} = \tilde{v}, \quad \text{stability group of the vacuum.} \quad (10.11.10)$$

As for the M realizations $\theta_\alpha \in S$ on the scalar field $\tilde{\phi}(x)$ of the M generators $T_\alpha \in S$, the broken symmetry condition, (10.11.8), is satisfied automatically due to the stability condition, (10.11.9), and we do not get new information from (10.11.8). As for the remaining $(N - M)$ realizations $\theta_\alpha \notin S$ on the scalar field $\tilde{\phi}(x)$ of the $(N - M)$ broken generators $T_\alpha \notin S$ which break the stability condition, (10.11.9), we get the following information from (10.11.8), namely, the mass matrix $\left(M^2\right)_{i,j}$ has the $(N - M)$ nontrivial eigenvectors belonging to the eigenvalue 0

$$\theta_\alpha \tilde{v} \neq 0, \quad \theta_\alpha \notin S. \quad (10.11.11)$$

Here we show that $\{\theta_\alpha \tilde{v}, \theta_\alpha \notin S\}$ span the $(N - M)$-dimensional vector space. We define the $N \times N$ matrix $\mu^2_{\alpha,\beta}$ by

$$\mu^2_{\alpha,\beta} \equiv (\theta_\alpha \tilde{v}, \theta_\beta \tilde{v}) \equiv \sum_{i=1}^{n} (\theta_\alpha \tilde{v})_i^\dagger (\theta_\beta \tilde{v})_i, \quad \alpha, \beta = 1,\ldots,N. \quad (10.11.12a)$$

From the Hermiticity of θ_α, we have

$$\mu^2_{\alpha,\beta} = \left(\tilde{v}, \theta_\alpha \theta_\beta \tilde{v}\right) = \sum_{i=1}^{n} v_i^* \left(\theta_\alpha \theta_\beta \tilde{v}\right)_i, \tag{10.11.12b}$$

$$\mu^2_{\alpha,\beta} - \mu^2_{\beta,\alpha} = \left(\tilde{v}, \left[\theta_\alpha, \theta_\beta\right] \tilde{v}\right) = iC_{\alpha\beta\gamma} \left(\tilde{v}, \theta_\gamma \tilde{v}\right) = 0, \tag{10.11.13}$$

i.e., we know that $\mu^2_{\alpha,\beta}$ is a real symmetric matrix. The last equality of (10.11.13) follows from the antisymmetry of θ_α. Next we let $\tilde{\mu}^2_{\alpha,\beta}$ be the restriction of $\mu^2_{\alpha,\beta}$ to the subspace $\{\theta_\alpha \tilde{v}, \theta_\alpha \notin S\}$. The $\tilde{\mu}^2_{\alpha,\beta}$ is the real symmetric $(N-M) \times (N-M)$ matrix and diagonalizable. We let the $(N-M) \times (N-M)$ orthogonal matrix that diagonalizes $\tilde{\mu}^2_{\alpha,\beta}$ to be O, and let $\tilde{\mu}'^2_{\alpha,\beta}$ to be the diagonalized $\tilde{\mu}^2_{\alpha,\beta}$.

$$\tilde{\mu}'^2_{\alpha,\beta} = \left(O\tilde{\mu}^2 O\right)_{\alpha,\beta} = \left((O\theta)_\alpha \tilde{v}, (O\theta)_\beta \tilde{v}\right) = \delta_{\alpha\beta} \cdot \tilde{\mu}'^2_{(\alpha)}. \tag{10.11.14}$$

The diagonal element $\tilde{\mu}'^2_{(\alpha)}$ is given by

$$\tilde{\mu}'^2_{(\alpha)} = \left((O\theta)_\alpha \tilde{v}, (O\theta)_\alpha \tilde{v}\right) = \sum_{i=1}^{n} \left((O\theta)_\alpha \tilde{v}\right)_i^\dagger \left((O\theta)_\alpha \tilde{v}\right)_i. \tag{10.11.15}$$

From the definition of the stability group of the vacuum and the definition of the linear independence in the N-dimensional vector space spanned by the realization θ_α, we obtain the following three statements:

(1)

$$\forall \theta_\alpha \notin S : (O\theta)_\alpha \tilde{v} \neq 0.$$

(2) $(N-M)$ diagonal elements, $\tilde{\mu}'^2_{(\alpha)}$, of $\tilde{\mu}'^2_{\alpha,\beta}$ are all positive.

(3) $(N-M)$ θ_α's, $\theta_\alpha \notin S$, are linearly independent.

Hence we understand that $\{\theta_\alpha \tilde{v} : \forall \theta_\alpha \notin S\}$ span the $(N-M)$-dimensional vector space. In this way, we obtain the following theorem.

Goldstone's theorem: When the global symmetry induced by $(N-M)$ generators T_α corresponding to $\{\theta_\alpha : \theta_\alpha \notin S\}$ is spontaneously broken $(\theta_\alpha \tilde{v} \neq 0)$ by the vacuum $\tilde{v}$, the mass matrix $\left(M^2\right)_{i,j}$ has $(N-M)$ eigenvectors $\{\theta_\alpha \tilde{v} : \theta_\alpha \notin S\}$ belonging to the eigenvalue 0, and these vectors $\{\theta_\alpha \tilde{v} : \theta_\alpha \notin S\}$ span the $(N-M)$-dimensional vector space of Nambu–Goldstone boson (massless excitation). The remaining $n-(N-M)$ scalar fields are massive.

From the preceding argument, we find that $N \times N$ matrix $\mu^2_{\alpha,\beta}$ is the rank $(N-M)$ and has the $(N-M)$ positive eigenvalues $\tilde{\mu}'^2_{(\alpha)}$ and the M zero eigenvalues. Now, we rearrange the generators T_α in the order of the M unbroken generators

$T_\alpha \in S$ and the $(N-M)$ broken generators $T_\alpha \notin S$. We get $\mu^2_{\alpha,\beta}$ in the block diagonal form

$$\mu^2_{\alpha,\beta} = \begin{pmatrix} 0, & 0, \\ 0, & \tilde{\mu}^2_{\alpha,\beta} \end{pmatrix}, \tag{10.11.16}$$

where the upper diagonal block corresponds to the M-dimensional vector space of the stability group of the vacuum, S, and the lower diagonal block corresponds to the $(N-M)$-dimensional vector space of Nambu–Goldstone boson.

The purpose of the introduction of the $N \times N$ matrix $\mu^2_{\alpha,\beta}$ is twofold.

(1) $\{\theta_\alpha \tilde{v} : \theta_\alpha \notin S\}$ span the $(N-M)$-dimensional vector space.
(2) The mass matrix of the gauge fields to be introduced by the application of Weyl's gauge principle in the presence of spontaneous symmetry breaking is given by $\mu^2_{\alpha,\beta}$.

Having disposed of the first point, we move on to the discussion of the second point.

Higgs–Kibble mechanism: We now extend the global G invariance which is spontaneously broken by the "vacuum" $\tilde{v}$ of the matter field Lagrangian density, $\mathcal{L}_{\text{matter}}\left(\tilde{\phi}(x), \partial_\mu \tilde{\phi}(x)\right)$, to the local G invariance with Weyl's gauge principle. The total Lagrangian density $\mathcal{L}_{\text{tot}}$ of the matter-gauge system after the gauge extension is given by

$$\begin{aligned}
\mathcal{L}_{\text{tot}} &= \mathcal{L}_{\text{gauge}}\left(F_{\gamma\mu\nu}(x)\right) + \mathcal{L}_{\text{scalar}}\left(\tilde{\phi}(x), D_\mu \tilde{\phi}(x)\right) \\
&= -\frac{1}{4} F_{\gamma\mu\nu}(x) F_\gamma^{\mu\nu}(x) + \frac{1}{2}\left\{\left(\partial_\mu + i\theta_\alpha A_{\alpha\mu}(x)\right)\tilde{\phi}(x)\right\}_i^{\text{T}} \\
&\quad \times \left\{\left(\partial^\mu + i\theta_\beta A_\beta^\mu(x)\right)\tilde{\phi}(x)\right\}_i - V\left(\tilde{\phi}(x)\right) \\
&= -\frac{1}{4} F_{\gamma\mu\nu}(x) F_\gamma^{\mu\nu}(x) + \frac{1}{2}\left\{\partial_\mu \tilde{\phi}^{\text{T}}(x) - i\tilde{\phi}^{\text{T}}(x)\theta_\alpha A_{\alpha\mu}(x)\right\}_i \\
&\quad \times \left\{\partial^\mu \tilde{\phi}(x) + i\theta_\beta A_\beta^\mu(x)\tilde{\phi}(x)\right\}_i - V\left(\tilde{\phi}(x)\right).
\end{aligned} \tag{10.11.17}$$

We parameterize the n-component scalar field $\tilde{\phi}(x)$ as

$$\tilde{\phi}(x) = \exp\left[i \sum_{\theta_\alpha \notin S} \frac{\xi_\alpha(x)\theta_\alpha}{v_\alpha}\right]\left(\tilde{v} + \tilde{\eta}(x)\right). \tag{10.11.18}$$

We remark that the $(N-M)$ $\xi_\alpha(x)$'s correspond to the $(N-M)$ broken generators $T_\alpha \notin S$, and the $\tilde{\eta}(x)$ has the $(n-(N-M))$ nonvanishing components and is orthogonal to Nambu–Goldstone direction, $\{\theta_\alpha \tilde{v} : \theta_\alpha \notin S\}$. We have the vacuum expectation values of $\{\xi_\alpha(x)\}_{\alpha=1}^{N-M}$ and $\{\eta_i(x)\}_{i=1}^{n-(N-M)}$ equal to 0. From the global G invariance of $V\left(\tilde{\phi}(x)\right)$, we have

$$V\left(\tilde{\phi}(x)\right) = V\left(\tilde{v} + \tilde{\eta}(x)\right), \tag{10.11.19}$$

and find that $V(\tilde{\phi}(x))$ is independent of $\{\xi_\alpha(x)\}_{\alpha=1}^{N-M}$. Without the gauge extension, we had the $\xi_\alpha(x)$-dependence from the derivative term in $\mathcal{L}_{\text{scalar}}$

$$\frac{1}{2}\partial_\mu\tilde{\phi}^{\mathrm{T}}(x)\partial^\mu\tilde{\phi}(x) = \frac{1}{2}\partial_\mu\xi_\alpha(x)\partial^\mu\xi_\alpha(x) + \cdots . \tag{10.11.20}$$

Thus $\{\xi_\alpha(x)\}_{\alpha=1}^{N-M}$ would be the massless Nambu–Goldstone boson with the gradient coupling with the other fields. As in (10.11.17), after the gauge extension, we have the freedom of the gauge transformation and are able to eliminate Nambu–Goldstone boson fields $\{\xi_\alpha(x)\}_{\alpha=1}^{N-M}$ completely. We employ the local phase transformation and the nonlinear gauge transformation of the following form:

$$\tilde{\phi}'(x) = \exp\left[-i\sum_{\theta_\alpha\notin S}\frac{\xi_\alpha(x)\theta_\alpha}{v_\alpha}\right]\tilde{\phi}(x) = \tilde{v} + \tilde{\eta}(x), \tag{10.11.21a}$$

$$\begin{aligned}\theta_\gamma A'_{\gamma\mu}(x) = {}& \exp\left[-i\sum_{\theta_\alpha\notin S}\frac{\xi_\alpha(x)\theta_\alpha}{v_\alpha}\right]\left\{\theta_\gamma A_{\gamma\mu}(x) + \exp\left[i\sum_{\theta_\beta\notin S}\frac{\xi_\beta(x)\theta_\beta}{v_\beta}\right]\right.\\ &\left.\times\left(i\partial_\mu\exp\left[-i\sum_{\theta_\beta\notin S}\frac{\xi_\beta(x)\theta_\beta}{v_\beta}\right]\right)\right\}\exp\left[i\sum_{\theta_\alpha\notin S}\frac{\xi_\alpha(x)\theta_\alpha}{v_\alpha}\right].\end{aligned} \tag{10.11.21b}$$

As a result of these local G transformations, we managed to eliminate $\{\xi_\alpha(x)\}_{\alpha=1}^{N-M}$ completely and have the total Lagrangian density $\mathcal{L}_{\text{tot}}$ as

$$\begin{aligned}\mathcal{L}_{\text{tot}} = {}& -\frac{1}{4}F'_{\gamma\mu\nu}(x)F_\gamma^{\prime\mu\nu}(x) + \frac{1}{2}\left\{\left(\partial_\mu + i\theta_\alpha A'_{\alpha\mu}(x)\right)\left(\tilde{v} + \tilde{\eta}(x)\right)\right\}^{\mathrm{T}}\\ &\times\left\{\left(\partial^\mu + i\theta_\alpha A_\alpha^{\prime\mu}(x)\right)\left(\tilde{v} + \tilde{\eta}(x)\right)\right\} - V\left(\tilde{v} + \tilde{\eta}(x)\right).\end{aligned} \tag{10.11.22}$$

After the gauge transformations, (10.11.21a) and (10.11.21b), we have the covariant derivative as

$$\left(\partial_\mu + i\theta_\alpha A'_{\alpha\mu}(x)\right)\left(\tilde{v} + \tilde{\eta}(x)\right) = \partial_\mu\tilde{\eta}(x) + i\left(\theta_\alpha\tilde{v}\right)A'_{\alpha\mu}(x) + i\left(\theta_\alpha\tilde{\eta}(x)\right)A'_{\alpha\mu}(x). \tag{10.11.23}$$

From the definitions of $\mu^2_{\alpha,\beta}$, (10.11.12a) and (10.11.12b), we have

$$\frac{1}{2}\left\{\left(\partial_\mu + i\theta_\alpha A'_{\alpha\mu}(x)\right)\left(\tilde{v}+\tilde{\eta}(x)\right)\right\}^{\mathrm{T}}\left\{\left(\partial^\mu + i\theta_\beta A'^{\mu}_{\beta}(x)\right)\left(\tilde{v}+\tilde{\eta}(x)\right)\right\}$$
$$= \frac{1}{2}\left(\partial_\mu\tilde{\eta}^{\mathrm{T}}(x)\partial^\mu\tilde{\eta}(x) + \mu^2_{\alpha,\beta}A'_{\alpha\mu}(x)A'^{\mu}_{\beta}(x)\right) + \partial_\mu\tilde{\eta}^{\mathrm{T}}(x)i\left(\theta_\alpha\tilde{v}\right)A'^{\mu}_{\alpha}(x)$$
$$+ \partial_\mu\tilde{\eta}^{\mathrm{T}}(x)i\left(\theta_\alpha\tilde{\eta}(x)\right)A'^{\mu}_{\alpha}(x) + \left(\tilde{\eta}(x),\theta_\alpha\theta_\beta\tilde{v}\right)A'_{\alpha\mu}(x)A'^{\mu}_{\beta}(x)$$
$$+ \frac{1}{2}\left(\tilde{\eta}(x),\theta_\alpha\theta_\beta\tilde{\eta}(x)\right)A'_{\alpha\mu}(x)A'^{\mu}_{\beta}(x). \qquad (10.11.24)$$

We expand $V(\tilde{\phi}'(x))$ around $\tilde{\phi}'(x) = \tilde{v}$ and obtain

$$V\left(\tilde{v}+\tilde{\eta}(x)\right) = \frac{1}{2}\left(M^2\right)_{ij}\tilde{\eta}_i(x)\tilde{\eta}_j(x) + \frac{1}{3!}f_{ijk}\tilde{\eta}_i(x)\tilde{\eta}_j(x)\tilde{\eta}_k(x)$$
$$+ \frac{1}{4!}f_{ijkl}\tilde{\eta}_i(x)\tilde{\eta}_j(x)\tilde{\eta}_k(x)\tilde{\eta}_l(x). \qquad (10.11.25a)$$

We have

$$\left(M^2\right)_{ij} = \left.\frac{\partial^2 V\left(\tilde{\phi}(x)\right)}{\partial\phi_i(x)\partial\phi_j(x)}\right|_{\tilde{\phi}(x)=\tilde{v}}, \qquad (10.11.25b)$$

$$f_{ijk} = \left.\frac{\partial^3 V\left(\tilde{\phi}(x)\right)}{\partial\phi_i(x)\partial\phi_j(x)\partial\phi_k(x)}\right|_{\tilde{\phi}(x)=\tilde{v}}, \qquad (10.11.25c)$$

$$f_{ijkl} = \left.\frac{\partial^4 V\left(\tilde{\phi}(x)\right)}{\partial\phi_i(x)\partial\phi_j(x)\partial\phi_k(x)\partial\phi_l(x)}\right|_{\tilde{\phi}(x)=\tilde{v}}. \qquad (10.11.25d)$$

From (10.11.24) and (10.11.25a), the total Lagrangian density $\mathcal{L}_{\text{tot}}$ after the local G transformation is

$$\mathcal{L}_{\text{tot}} = -\frac{1}{4}F'_{\gamma\mu\nu}(x)F'^{\mu\nu}_{\gamma}(x) + \frac{1}{2}\mu^2_{\alpha,\beta}A'_{\alpha\mu}(x)A'^{\mu}_{\beta}(x) + \partial_\mu\tilde{\eta}(x)i\left(\theta_\alpha\tilde{v}\right)A'^{\mu}_{\alpha}(x)$$
$$+ \frac{1}{2}\partial_\mu\tilde{\eta}(x)\partial^\mu\tilde{\eta}(x) - \frac{1}{2}\left(M^2\right)_{i,j}\tilde{\eta}_i(x)\tilde{\eta}_j(x) + \partial_\mu\tilde{\eta}(x)i\left(\theta_\alpha\tilde{\eta}(x)\right)A'^{\mu}_{\alpha}(x)$$
$$+ \tilde{\eta}(x)(\theta_\alpha\theta_\beta\tilde{v})A'_{\alpha\mu}(x)A'^{\mu}_{\beta}(x) - \frac{1}{3!}f_{ijk}\tilde{\eta}_i(x)\tilde{\eta}_j(x)\tilde{\eta}_k(x)$$
$$+ \frac{1}{2}\tilde{\eta}(x)\left(\theta_\alpha\theta_\beta\tilde{\eta}(x)\right)A'_{\alpha\mu}(x)A'^{\mu}_{\beta}(x) - \frac{1}{4!}f_{ijkl}\tilde{\eta}_i(x)\tilde{\eta}_j(x)\tilde{\eta}_k(x)\tilde{\eta}_l(x). \quad (10.11.26)$$

From (10.11.26), we obtain as the quadratic part of $\mathcal{L}_{\text{tot}}$,

$$\mathcal{L}_{\text{tot}}^{\text{quad}} = -\frac{1}{4}\left(\partial_\mu A'_{\alpha\nu}(x) - \partial_\nu A'_{\alpha\mu}(x)\right)\left(\partial^\mu A_\alpha^{'\nu}(x) - \partial^\nu A_\alpha^{'\mu}(x)\right)$$
$$+\frac{1}{2}\mu_{\alpha,\beta}^2 A'_{\alpha\mu}(x)A_\beta^{'\mu}(x) + \frac{1}{2}\partial_\mu\tilde{\eta}(x)\partial^\mu\tilde{\eta}(x) - \frac{1}{2}\left(M^2\right)_{i,j}\tilde{\eta}_i(x)\tilde{\eta}_j(x)$$
$$+\partial_\mu\tilde{\eta}(x)i(\theta_\alpha\tilde{v})A_\alpha^{'\mu}(x). \qquad (10.11.27)$$

Were it not for the last term in (10.11.27) which represents the mixing of the gauge field and the scalar field, we can regard (10.11.27) as the ''free'' Lagrangian density of the following fields:

(1) M massless gauge fields corresponding to the M unbroken generators, $T_\alpha \in S$, belonging to the stability group of the vacuum,
(2) $(N - M)$ massive vector fields corresponding to the $(N - M)$ broken generators, $T_\alpha \notin S$, with the mass eigenvalues, $\tilde{\mu}_{(\alpha)}^{'2}$, $\alpha = M + 1, \ldots, N$,
(3) $(n - (N - M))$ massive scalar fields with the mass matrix, $\left(M^2\right)_{i,j}$.

The $(N - M)$ Nambu–Goldstone boson fields $\{\xi_\alpha(x)\}_{\alpha=1}^{N-M}$ get eliminated from the particle spectrum by the gauge transformations, (10.11.21a) and (10.11.21b), and absorbed as the longitudinal mode of the gauge fields which corresponds to the $(N - M)$ broken generators $T_\alpha \notin S$. The said $(N - M)$ gauge fields become the $(N - M)$ massive vector fields with two transverse modes and one longitudinal mode. We call this mass-generating mechanism for the gauge fields as Higgs–Kibble mechanism. We make the lists of the degrees of freedom of the matter-gauge system before and after the gauge transformations, (10.11.21a) and (10.11.21b):

Before the gauge transformation	Degrees of freedom
N massless gauge fields	$2N$
$(N - M)$ Goldstone boson fields	$N - M$
$(n - (N - M))$ massive scalar fields	$n - (N - M)$
Total degrees of freedom	$n + 2N$

and

After the gauge transformation	Degrees of freedom
M massless gauge fields	$2M$
$(N - M)$ massive vector fields	$3(N - M)$
$(n - (N - M))$ massive scalar fields	$n - (N - M)$
Total degrees of freedom	$n + 2N$

There are no changes in the total degrees of freedom, $n + 2N$. Before the gauge transformation, the local G invariance of $\mathcal{L}_{\text{tot}}$, (10.11.17), is manifest, whereas after the gauge transformation, the particle spectrum content of $\mathcal{L}_{\text{tot}}$, (10.11.26), is manifest and the local G invariance of $\mathcal{L}_{\text{tot}}$ is hidden. In this way, we can give the mass term to the gauge field without violating the local G invariance and verify that the mass matrix of the gauge field is indeed given by $\mu^2_{\alpha,\beta}$.

Path Integral Quantization of Gauge Field in R_ξ-Gauge: We shall employ the R_ξ-gauge as the gauge-fixing condition, and carry out the path integral quantization of the matter-gauge system described by the Lagrangian density, (10.11.26). We can also eliminate the mixed term $\partial_\mu \tilde{\eta}(x) i(\theta_\alpha \tilde{v}) A_\alpha^{'\mu}(x)$ of the scalar fields $\tilde{\eta}(x)$ and the gauge fields $A'_{\alpha\mu}(x)$ in $\mathcal{L}_{\text{tot}}^{\text{quad}}$. R_ξ-gauge is the gauge-fixing condition involving Higgs scalar field and the gauge field , $\{\tilde{\eta}(x), A'_{\gamma\mu}(x)\}$, linearly. We will use the general notation

$$\{\phi_a\} = \left\{\eta_i(x),\ A'_{\gamma\mu}(x)\right\}, \quad a = (i, x),\ (\gamma\mu, x). \tag{10.11.28}$$

As the general linear gauge-fixing condition, we employ

$$F_\alpha(\{\phi_a\}) = a_\alpha(x), \quad \alpha = 1, \ldots, N. \tag{10.11.29}$$

We assume that

$$F_\alpha\left(\{\phi_a^g\}\right) = a_\alpha(x), \quad \alpha = 1, \ldots, N, \quad \{\phi_a\} \text{ fixed}, \tag{10.11.30}$$

has the unique solution $g(x) \in G$. We parameterize the element $g(x)$ in the neighborhood of the identity element of G by

$$g(x) = 1 + i\varepsilon_\alpha(x)\theta_\alpha + O\left(\varepsilon^2\right), \tag{10.11.31}$$

with $\varepsilon_\alpha(x)$ arbitrary infinitesimal function independent of $\{\phi_a\}$. The Faddeev–Popov determinant $\Delta_F[\{\phi_a\}]$ of the gauge-fixing condition, (10.11.29), is defined by

$$\Delta_F[\{\phi_a\}] \int \prod_x dg(x) \prod_{\alpha,x} \delta\left(F_\alpha\left(\{\phi_a^g\}\right) - a_\alpha(x)\right) = 1, \tag{10.11.32}$$

and is invariant under the linear gauge transformation, (10.11.31),

$$\Delta_F\left[\{\phi_a^g\}\right] = \Delta_F\left[\{\phi_a\}\right]. \tag{10.11.33}$$

According to the first formula of Faddeev–Popov, the vacuum-to-vacuum transition amplitude of the matter-gauge system governed by the Lagrangian density, $\mathcal{L}_{\text{tot}}$((10.11.26)), is given by

$$\begin{aligned} \mathcal{Z}_F(a_\alpha(x)) \equiv \left\langle 0, \text{out} \middle| 0, \text{in} \right\rangle_{F,a} = \int \mathcal{D}[\{\phi_a\}] \Delta_F[\{\phi_a\}] \exp\left[iI_{\text{tot}}[\{\phi_a\}]\right] \\ \times \prod_{\alpha,x} \delta\left(F_\alpha(\{\phi_a\}) - a_\alpha(x)\right), \end{aligned} \tag{10.11.34}$$

$$\mathcal{D}[\{\phi_a\}] \equiv \mathcal{D}\left[\eta_i(x)\right]\mathcal{D}\left[A'_{\gamma\mu}(x)\right], \tag{10.11.35}$$

$$I_{\text{tot}}[\{\phi_a\}] = \int d^4x \mathcal{L}_{\text{tot}}((10.11.26)). \tag{10.11.36}$$

Since $\Delta_F[\{\phi_a\}]$ in $\left\langle 0, \text{out} \middle| 0, \text{in} \right\rangle_{F,a}$ gets multiplied by $\delta(F_\alpha(\{\phi_a\}) - a_\alpha(x))$, it is sufficient to calculate $\Delta_F[\{\phi_a\}]$ for $\{\phi_a\}$ which satisfies (10.11.29). We will parameterize $g(x)$ as in (10.11.31) so that

$$\Delta_F[\{\phi_a\}] = \text{Det} M_F(\{\phi_a\}), \tag{10.11.37}$$

$$\{M_F(\{\phi_a\})\}_{\alpha x,\beta y} = \left.\frac{\delta F_\alpha\left(\{\phi_a^g(x)\}\right)}{\delta\varepsilon_\beta(y)}\right|_{g=1}, \quad \alpha,\beta = 1,\ldots,N, \tag{10.11.38}$$

$$F_\alpha(\{\phi_a^g\}) = F_\alpha(\{\phi_a\}) + M_F(\{\phi_a\})_{\alpha x,\beta y}\varepsilon_\beta(y) + O(\varepsilon^2). \tag{10.11.39}$$

We now consider the nonlinear gauge transformation g_0 parameterized by

$$\varepsilon_\alpha(x;\{\phi_a\}) = \left\{M_F^{-1}(\{\phi_a\})\right\}_{\alpha x,\beta y}\lambda_\beta(y) \tag{10.11.40}$$

with $\lambda_\beta(y)$ arbitrary infinitesimal function independent of $\{\phi_a\}$. Under this nonlinear gauge transformation, we have

(1) $I_{\text{tot}}[\{\phi_a\}]$ is gauge invariant,

(2)

$$\mathcal{D}[\{\phi_a\}]\Delta_F[\{\phi_a\}] = \text{gauge invariant measure}, \tag{10.11.41}$$

(3) The gauge-fixing condition $F_\alpha(\{\phi_a\})$ gets transformed into

$$F_\alpha\left(\{\phi_a^{g_0}\}\right) = F_\alpha(\{\phi_a\}) + \lambda_\alpha(x) + O\left(\lambda^2\right). \tag{10.11.42}$$

Since the value of the functional integral remains unchanged under the change of the function variables, the value of $\mathcal{Z}_F(a_\alpha(x))$ remains unchanged. With the choice as

$$\lambda_\alpha(x) = \delta a_\alpha(x), \quad \alpha = 1,\ldots,N, \tag{10.11.43}$$

we have

$$\mathcal{Z}_F(a_\alpha(x)) = \mathcal{Z}_F(a_\alpha(x) + \delta a_\alpha(x)) \quad \text{or} \quad \frac{d}{da_\alpha(x)}\mathcal{Z}_F(a_\alpha(x)) = 0. \tag{10.11.44}$$

Since we find that $\mathcal{Z}_F(a_\alpha(x))$ is independent of $a_\alpha(x)$, we can introduce an arbitrary weighting functional $H[a_\alpha(x)]$ for $\mathcal{Z}_F(a_\alpha(x))$ and path-integrate with respect to $a_\alpha(x)$, obtaining as the weighted $\mathcal{Z}_F(a_\alpha(x))$

$$\mathcal{Z}_F \equiv \int \prod_{\alpha,x} da_\alpha(x) H[a_\alpha(x)]\mathcal{Z}_F(a_\alpha(x))$$

$$= \int \mathcal{D}[\{\phi_a\}]\,\Delta_F[\{\phi_a\}]\,H\left[F_\alpha(\{\phi_a\})\right]\exp\left[iI_{\text{tot}}[\{\phi_a\}]\right]. \tag{10.11.45}$$

As the weighting functional $H[a_\alpha(x)]$, we use the quasi-Gaussian functional,

$$H[a_\alpha(x)] = \exp\left[-\frac{i}{2}\int d^4x a_\alpha^2(x)\right] \tag{10.11.46}$$

and obtain $\mathcal{Z}_F$ as

$$\begin{aligned}\mathcal{Z}_F &= \int \mathcal{D}[\{\phi_a\}]\Delta_F[\{\phi_a\}]\exp\left[i\int d^4x\{\mathcal{L}_{\text{tot}}((10.11.26)) - \frac{1}{2}F_\alpha^2(\{\phi_a\})\}\right]\\ &= \int \mathcal{D}[\{\phi_a\}]\mathcal{D}[\overline{c}_\alpha]\mathcal{D}[c_\beta]\exp\left[i\int d^4x\left\{\mathcal{L}_{\text{tot}}((10.11.26)) - \frac{1}{2}F_\alpha^2(\{\phi_a\})\right.\right.\\ &\quad \left.\left.+\ \overline{c}_\alpha(x)M_F(\{\phi_a(x)\})c_\beta(x)\right\}\right].\end{aligned} \tag{10.11.47}$$

Since the gauge-fixing condition, (10.11.29), is linear in $\{\phi_a(x)\}$, we have

$$\{M_F(\{\phi_a\})\}_{\alpha x,\beta y} = \delta^4(x-y)M_F(\{\phi_a(x)\})_{\alpha,\beta}. \tag{10.11.48}$$

Summarizing the results, we have

$$\left\langle 0,\text{out}\middle|\, 0,\text{in}\right\rangle_F = \int \mathcal{D}[\{\phi_a\}]\mathcal{D}[\overline{c}_\alpha]\mathcal{D}[c_\beta]\exp[iI_{\text{eff}}[\{\phi_a\},\overline{c}_\alpha,c_\beta]] \tag{10.11.49a}$$

with the effective action functional $I_{\text{eff}}[\{\phi_a\},\overline{c}_\alpha,c_\beta]$ given by

$$I_{\text{eff}}[\{\phi_a\},\overline{c}_\alpha,c_\beta] = \int d^4x\mathcal{L}_{\text{eff}}(\{\phi_a(x)\},\overline{c}_\alpha(x),c_\beta(x)). \tag{10.11.49b}$$

The effective Lagrangian density $\mathcal{L}_{\text{eff}}(\{\phi_a(x)\},\overline{c}_\alpha(x),c_\beta(x))$ is given by

$$\begin{aligned}\mathcal{L}_{\text{eff}}(\{\phi_a(x)\},\overline{c}_\alpha(x),c_\beta(x)) &= \mathcal{L}_{\text{tot}}(\{\phi_a(x)\};(10.11.26)) - \frac{1}{2}F_\alpha^2(\{\phi_a(x)\})\\ &\quad + \overline{c}_\alpha(x)M_F(\{\phi_a(x)\})_{\alpha,\beta}c_\beta(x).\end{aligned} \tag{10.11.49c}$$

R_ξ**-gauge**: In order to eliminate the mixed term $\partial_\mu\tilde{\eta}(x)i(\theta_\alpha\tilde{v})A_\alpha^{'\mu}(x)$ in the quadratic part of the Lagrangian density $\mathcal{L}_{\text{tot}}^{\text{quad}}$ in (10.11.27), we choose

$$F_\alpha(\{\phi_a(x)\}) = \sqrt{\xi}\left(\partial_\mu A_\alpha^{'\mu}(x) - \frac{1}{\xi}\tilde{\eta}(x)i(\theta_\alpha\tilde{v})\right),\quad \xi > 0. \tag{10.11.50}$$

We have the exponent of the quasi-Gaussian functional as

$$\frac{1}{2}F_\alpha^2\left(\{\phi_a(x)\}\right) = \frac{\xi}{2}\left(\partial_\mu A_\alpha^{'\mu}(x)\right)^2 - \tilde{\eta}(x)i(\theta_\alpha\tilde{v})\,\partial_\mu A_\alpha^{'\mu}(x) - \frac{1}{2\xi}\left\{(\theta_\alpha\tilde{v})\,\tilde{\eta}(x)\right\}^2 \tag{10.11.51}$$

so that the mixed terms add up to the four divergence,

$$\left\{\mathcal{L}_{\text{tot}}^{\text{quad}} - \frac{1}{2}F_{\alpha}^{2}(\{\phi_a(x)\})\right\}_{\text{mixed}} = \partial_\mu \tilde{\eta}(x) i(\theta_\alpha \tilde{v}) A_{\alpha}^{'\mu}(x) + \tilde{\eta}(x) i(\theta_\alpha \tilde{v}) \partial_\mu A_{\alpha}^{'\mu}(x)$$

$$= \partial_\mu \left\{\tilde{\eta}(x) i(\theta_\alpha \tilde{v}) A_{\alpha}^{'\mu}(x)\right\}. \tag{10.11.52}$$

Namely, in the R_ξ-gauge, the mixed term does not contribute to the effective action functional $I_{\text{eff}}[\{\phi_a\}, \overline{c}_\alpha, c_\beta]$. We obtain the Faddeev–Popov determinant in the R_ξ-gauge from the transformation laws of $A'_{\alpha\mu}(x)$ and $\tilde{\eta}(x)$.

$$\delta A'_{\alpha\mu}(x) = -\left(D_{\mu}^{\text{adj}}\varepsilon(x)\right)_{\alpha}, \quad \alpha = 1, \ldots, N; \quad \delta\tilde{\eta}(x) = i\varepsilon_\alpha(x)\left\{\theta_\alpha(\tilde{v} + \tilde{\eta}(x))\right\}. \tag{10.11.53}$$

$$\{M_F(\{\phi_a\})\}_{\alpha x, \beta y} \equiv \left.\frac{\delta F_\alpha\left(\{\phi_a^g(x)\}\right)}{\delta\varepsilon_\beta(y)}\right|_{g=1} \equiv \delta^4(x-y) M_F(\{\phi_a(x)\})_{\alpha,\beta}$$

$$= \delta^4(x-y)\sqrt{\xi}\left\{-\partial^\mu\left(D_{\mu}^{\text{adj}}\right)_{\alpha,\beta} - \frac{1}{\xi}\mu_{\alpha,\beta}^2 - \frac{1}{\xi}\left(\tilde{v}, \theta_\alpha\theta_\beta\tilde{\eta}(x)\right)\right\}. \tag{10.11.54}$$

We absorb $\sqrt{\xi}$ in the normalization of the Faddeev–Popov ghost fields

$$\{\overline{c}_\alpha(x), c_\beta(x)\}.$$

We then have the Faddeev–Popov ghost Lagrangian density as

$$\mathcal{L}_{\text{ghost}}(\overline{c}_\alpha(x), c_\beta(x)) \equiv \overline{c}_\alpha(x) M_F(\{\phi_a(x)\})_{\alpha,\beta} c_\beta(x)$$

$$= \partial_\mu \overline{c}_\alpha(x) \partial^\mu c_\alpha(x) - \frac{1}{\xi}\mu_{\alpha,\beta}^2 \overline{c}_\alpha(x) c_\beta(x)$$

$$+ C_{\gamma\alpha\beta}\partial_\mu \overline{c}_\alpha(x) A_{\gamma}^{'\mu}(x) c_\beta(x) - \frac{1}{\xi}\overline{c}_\alpha(x)(\tilde{v}, \theta_\alpha\theta_\beta\tilde{\eta}(x)) c_\beta(x). \tag{10.11.55}$$

From $\mathcal{L}_{\text{tot}}((10.11.26))$, the gauge-fixing term, $-\frac{1}{2}F_\alpha^2(\{\phi_a(x)\})$, (10.11.51), and $\mathcal{L}_{\text{ghost}}((10.11.55))$, we have the effective Lagrangian density $\mathcal{L}_{\text{eff}}((10.11.49\text{c}))$ as

$$\mathcal{L}_{\text{eff}}((10.11.49\text{c})) = \mathcal{L}_{\text{tot}}((10.11.26)) - \frac{1}{2}F_\alpha^2(\{\phi_a(x)\}) + \mathcal{L}_{\text{ghost}}((10.11.55))$$

$$= \mathcal{L}_{\text{eff}}^{\text{quad}} + \mathcal{L}_{\text{eff}}^{\text{int}}. \tag{10.11.56}$$

We have $\mathcal{L}_{\text{eff}}^{\text{quad}}$ as

$$\begin{aligned}\mathcal{L}_{\text{eff}}^{\text{quad}} &= -\frac{1}{2}A'_{\alpha\mu}(x)\left[\delta_{\alpha\beta}\left\{-\eta^{\mu\nu}\partial^2+(1-\xi)\partial^\mu\partial^\nu\right\}-\mu_{\alpha,\beta}^2\eta^{\mu\nu}\right]A'_{\beta\nu}(x)\\ &+\frac{1}{2}\eta_i(x)\left\{-\delta_{ij}\partial^2-\left(M^2\right)_{ij}+\frac{1}{\xi}\left(\theta_\alpha\tilde{v}\right)_i\left(\theta_\alpha\tilde{v}\right)_j\right\}\eta_j(x)\\ &+\bar{c}_\alpha(x)\left\{-\delta_{\alpha\beta}\partial^2-\frac{1}{\xi}\mu_{\alpha,\beta}^2\right\}c_\beta(x).\end{aligned} \tag{10.11.57}$$

We have $\mathcal{L}_{\text{eff}}^{\text{int}}$ as

$$\begin{aligned}\mathcal{L}_{\text{eff}}^{\text{int}} = \mathcal{L}_{\text{eff}}-\mathcal{L}_{\text{eff}}^{\text{quad}} &= \frac{1}{2}C_{\alpha\beta\gamma}A'_{\beta\mu}(x)A'_{\gamma\nu}(x)\left(\partial^\mu A'^{\nu}_{\alpha}(x)-\partial^\nu A'^{\mu}_{\alpha}(x)\right)\\ &-\frac{1}{4}C_{\alpha\beta\gamma}C_{\alpha\delta\varepsilon}A'_{\beta\mu}(x)A'_{\gamma\nu}(x)A'^{\mu}_{\delta}(x)A'^{\nu}_{\varepsilon}(x)+\partial_\mu\tilde{\eta}(x)i\left(\theta_\alpha\tilde{\eta}(x)\right)A'^{\mu}_{\alpha}(x)\\ &+\tilde{\eta}(x)\left(\theta_\alpha\theta_\beta\tilde{v}\right)A'_{\alpha\mu}(x)A'^{\mu}_{\alpha}(x)-\frac{1}{3!}f_{ijk}\eta_i(x)\eta_j(x)\eta_k(x)\\ &+\frac{1}{2}\tilde{\eta}(x)\left(\theta_\alpha\theta_\beta\tilde{\eta}(x)\right)A'_{\alpha\mu}(x)A'^{\mu}_{\beta}(x)-\frac{1}{4!}f_{ijkl}\eta_i(x)\eta_j(x)\eta_k(x)\eta_l(x)\\ &+C_{\alpha\beta\gamma}\partial_\mu\bar{c}_\alpha(x)c_\beta(x)A'^{\mu}_{\gamma}(x)-\frac{1}{\xi}\bar{c}_\alpha(x)\left(\tilde{v},\theta_\alpha\theta_\beta\tilde{\eta}(x)\right)c_\beta(x).\end{aligned} \tag{10.11.58}$$

From $\mathcal{L}_{\text{eff}}^{\text{quad}}$((10.11.57)), we have the equations satisfied by the "free" Green's functions

$$\left\{D^{(A')}_{\alpha\mu,\beta\nu}(x-y),\, D^{(\eta)}_{i,j}(x-y),\, D^{(C)}_{\alpha,\beta}(x-y)\right\},$$

of the gauge fields, Higgs scalar fields, and the Faddeev–Popov ghost fields, $\{A'_{\alpha\mu}(x);\, \eta_i(x);\, \bar{c}_\alpha(x);\, c_\beta(x)\}$, as

$$\begin{aligned}&-\left[\delta_{\alpha\beta}\left\{\left(-\eta^{\mu\nu}\partial^2+\partial^\mu\partial^\nu\right)-\xi\partial^\mu\partial^\nu\right\}-\mu_{\alpha,\beta}^2\eta^{\mu\nu}\right]D^{(A')}_{\beta\nu,\gamma\lambda}(x-y)\\ &\quad=\delta_{\alpha\gamma}\eta^{\mu}_{\lambda}\delta^4(x-y),\end{aligned} \tag{10.11.59a}$$

$$\left\{-\delta_{ij}\partial^2-\left(M^2\right)_{ij}+\frac{1}{\xi}\left(\theta_\alpha\tilde{v}\right)_i\left(\theta_\alpha\tilde{v}\right)_j\right\}D^{(\eta)}_{j,k}(x-y)=\delta_{ik}\delta^4(x-y), \tag{10.11.59b}$$

$$\left\{-\delta_{\alpha\beta}\partial^2-\frac{1}{\xi}\mu_{\alpha,\beta}^2\right\}D^{(C)}_{\beta,\gamma}(x-y)=\delta_{\alpha\gamma}\delta^4(x-y). \tag{10.11.59c}$$

We Fourier-transform the "free" Green's functions as

$$\begin{pmatrix} D^{(A')}_{\alpha\mu,\beta\nu}(x-y)\\ D^{(\eta)}_{i,j}(x-y)\\ D^{(C)}_{\alpha,\beta}(x-y)\end{pmatrix}=\int\frac{d^4k}{(2\pi)^4}\exp\left[-ik(x-y)\right]\begin{pmatrix} D^{(A')}_{\alpha\mu,\beta\nu}(k)\\ D^{(\eta)}_{i,j}(k)\\ D^{(C)}_{\alpha,\beta}(k)\end{pmatrix}. \tag{10.11.60}$$

We have the momentum space "free" Green's functions satisfying

$$-\left[\delta_{\alpha\beta}k^2\left\{\left(\eta^{\mu\nu}-\frac{k^\mu k^\nu}{k^2}\right)+\xi\frac{k^\mu k^\nu}{k^2}\right\}-\mu^2_{\alpha,\beta}\eta^{\mu\nu}\right]D^{(A')}_{\beta\nu,\gamma\lambda}(k)=\delta_{\alpha\gamma}\eta^{\mu}_{\lambda}, \tag{10.11.61a}$$

$$\left\{\delta_{ij}k^2-\left(M^2\right)_{ij}+\frac{1}{\xi}\left(\theta_\alpha\tilde{v}\right)_i\left(\theta_\alpha\tilde{v}\right)_j\right\}D^{(\eta)}_{j,k}(k)=\delta_{ik}, \tag{10.11.61b}$$

$$\left\{\delta_{\alpha\beta}k^2-\frac{1}{\xi}\mu^2_{\alpha,\beta}\right\}D^{(C)}_{\beta,\gamma}(k)=\delta_{\alpha\gamma}. \tag{10.11.61c}$$

We obtain the momentum space "free" Green's functions as

$$\begin{aligned}D^{(A')}_{\alpha\mu,\beta\nu}(k)&=-\left(\eta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)\left(\frac{1}{k^2-\mu^2}\right)_{\alpha,\beta}-\frac{1}{\xi}\frac{k_\mu k_\nu}{k^2}\left(\frac{1}{k^2-\mu^2/\xi}\right)_{\alpha,\beta}\\&=-\eta_{\mu\nu}\left(\frac{1}{k^2-\mu^2}\right)_{\alpha,\beta}-\left(\frac{1}{\xi}-1\right)k_\mu k_\nu\left(\frac{1}{k^2-\mu^2}\cdot\frac{1}{k^2-\mu^2/\xi}\right)_{\alpha,\beta},\end{aligned} \tag{10.11.62a}$$

$$\begin{aligned}D^{(\eta)}_{i,j}(k)&=(1-P)_{i,k}\left(\frac{1}{k^2-M^2}\right)_{k,j}-\left(\theta_\alpha\tilde{v}\right)_i\left(\frac{1}{\mu^2}\cdot\frac{1}{k^2-\mu^2/\xi}\right)_{\alpha,\beta}\left(\theta_\beta\tilde{v}\right)_j\\&=\left(\frac{1}{k^2-M^2}\right)_{i,j}-\left(\theta_\alpha\tilde{v}\right)_i\frac{1}{\xi}\cdot\frac{1}{k^2}\cdot\left(\frac{1}{k^2-\mu^2/\xi}\right)_{\alpha,\beta}\left(\theta_\beta\tilde{v}\right)_j,\end{aligned} \tag{10.11.62b}$$

$$D^{(C)}_{\alpha,\beta}(k)=\left(\frac{1}{k^2-\mu^2/\xi}\right)_{\alpha,\beta}. \tag{10.11.62c}$$

Here $P_{i,j}$ is the projection operator onto the $(N-M)$-dimensional subspace spanned by Nambu–Goldstone boson.

$$P_{i,j}=\sum_{\theta_\alpha,\theta_\beta\notin S}\left(\theta_\alpha\tilde{v}\right)_i\left(\frac{1}{\mu^2}\right)_{\alpha,\beta}\left(\theta_\beta\tilde{v}\right)^\dagger_j,\quad i,j=1,\ldots,n, \tag{10.11.63a}$$

$$P_{i,j}\left(\theta_\gamma\tilde{v}\right)_j=\left(\theta_\gamma\tilde{v}\right)_i,\quad\left(\theta_\gamma\tilde{v}\right)^\dagger_i P_{i,j}=\left(\theta_\gamma\tilde{v}\right)^\dagger_j,\quad\theta_\gamma\notin S. \tag{10.11.63b}$$

The R_ξ-gauge is not only the gauge that eliminates the mixed term of $A'_{\alpha\mu}(x)$ and $\tilde{\eta}(x)$ in the quadratic part of the effective Lagrangian density, but also the gauge that connects the Unitarity gauge ($\xi=0$), the 't Hooft–Feynman gauge ($\xi=1$) and the Landau gauge ($\xi\to\infty$) continuously in ξ.

(1) *Unitarity gauge* ($\xi=0$):

$$D^{(A')}_{\alpha\mu,\beta\nu}(k)=-\left\{\left(\eta_{\mu\nu}-\frac{k_\mu k_\nu}{\mu^2}\right)\cdot\left(\frac{1}{k^2-\mu^2}\right)\right\}_{\alpha,\beta}, \tag{10.11.64a}$$

$$D^{(\eta)}_{i,j}(k)=\left(\frac{1}{k^2-M^2}\right)_{i,j}+\frac{1}{k^2}\left(\theta_\alpha\tilde{v}\right)_i\cdot\left(\frac{1}{\mu^2}\right)_{\alpha\beta}\cdot\left(\theta_\beta\tilde{v}\right)_j, \tag{10.11.64b}$$

$$D^{(C)}_{\alpha,\beta}(k)\propto\xi\longrightarrow 0,\quad\text{infinitely massive ghost.} \tag{10.11.64c}$$

(2) *'t Hooft–Feynman gauge* ($\xi = 1$):

$$D^{(A')}_{\alpha\mu,\beta\nu}(k) = -\eta_{\mu\nu}\left(\frac{1}{k^2-\mu^2}\right)_{\alpha,\beta}, \tag{10.11.65a}$$

$$D^{(\eta)}_{i,j}(k) = \left(\frac{1}{k^2-M^2}\right)_{i,j} - (\theta_\alpha\tilde{v})_i\frac{1}{k^2}\left(\frac{1}{k^2-\mu^2}\right)_{\alpha\beta}\left(\theta_\beta\tilde{v}\right)_j, \tag{10.11.65b}$$

$$D^{(C)}_{\alpha,\beta}(k) = \left(\frac{1}{k^2-\mu^2}\right)_{\alpha,\beta}, \quad \text{massive ghost at } k^2=\mu^2. \tag{10.11.65c}$$

(3) *Landau gauge* ($\xi \to \infty$):

$$D^{(A')}_{\alpha\mu,\beta\nu}(k) = -\left(\eta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)\left(\frac{1}{k^2-\mu^2}\right)_{\alpha,\beta}, \tag{10.11.66a}$$

$$D^{(\eta)}_{i,j}(k) = \left(\frac{1}{k^2-M^2}\right)_{i,j}, \tag{10.11.66b}$$

$$D^{(C)}_{\alpha,\beta}(k) = \frac{\delta_{\alpha\beta}}{k^2}, \quad \text{massless ghost at } k^2=0. \tag{10.11.66c}$$

We have the generating functional of (the connected part of) the "full" Green's functions as

$$\begin{aligned}
Z_F\left[\{J_a\},\zeta_\alpha,\overline{\zeta}_\beta\right] &\equiv \exp\left[iW_F\left[\{J_a\},\zeta_\alpha,\overline{\zeta}_\beta\right]\right] \equiv \Big\langle 0,\text{out}\Big|\mathrm{T}\Big(\exp\Big[i\int d^4z\{J_a(z)\hat{\phi}_a(z) \\
&\quad + \overline{c}_\alpha(z)\zeta_\alpha(z)+\overline{\zeta}_\beta(z)c_\beta(z)\}\Big]\Big)\Big|0,\text{in}\Big\rangle \\
&= \int\mathcal{D}[\phi_a]\mathcal{D}[\overline{c}_\alpha]\mathcal{D}[c_\beta]\exp\Big[i\int d^4z\,\{\mathcal{L}_{\text{eff}}\left(\{\phi_a(z)\},\overline{c}_\alpha(z),c_\beta(z)\right) \\
&\quad + J_a(z)\phi_a(z)+\overline{c}_\alpha(z)\zeta_\alpha(z)+\overline{\zeta}_\beta c_\beta(z)\}\Big] \\
&= \exp\left[iI^{\text{int}}_{\text{eff}}\left[\left\{\frac{1}{i}\frac{\delta}{\delta J_a}\right\},i\frac{\delta}{\delta\zeta_\alpha},\frac{1}{i}\frac{\delta}{\delta\overline{\zeta}_\beta}\right]\right]Z_{F,0}\left[\{J_a\},\zeta_\alpha,\overline{\zeta}_\beta\right].
\end{aligned} \tag{10.11.67}$$

$Z_{F,0}[\{J_a\},\zeta_\alpha,\overline{\zeta}_\beta]$ and $I^{\text{int}}_{\text{eff}}[\{\phi_a\},\overline{c}_\alpha,c_\beta]$ are, respectively, given by

$$\begin{aligned}
Z_{F,0}\left[\{J_a\},\zeta_\alpha,\overline{\zeta}_\beta\right] &= \int\mathcal{D}[\phi_a]\mathcal{D}[\overline{c}_\alpha]\mathcal{D}[c_\beta]\exp\Big[i\int d^4z\,\Big\{\mathcal{L}^{\text{quad}}_{\text{eff}}((10.11.57)) \\
&\quad + J_a(z)\phi_a(z)+\overline{c}_\alpha(z)\zeta_\alpha(z)+\overline{\zeta}_\beta(z)c_\beta(z)\Big\}\Big]
\end{aligned}$$

$$= \exp\left[i \int d^4x d^4y \left\{ -\frac{1}{2} J_\alpha^\mu(x) D_{\alpha\mu,\beta\nu}^{(A')}(x-y) J_\beta^\nu(y) \right.\right.$$
$$\left.\left. -\frac{1}{2} J_i(x) D_{i,j}^{(\eta)}(x-y) J_j(y) - \overline{\zeta}_\beta(x) D_{\beta,\alpha}^{(C)}(x-y)\zeta_\alpha(y) \right\}\right], \tag{10.11.68}$$

and

$$I_{\text{eff}}^{\text{int}}[\{\phi_a\}, \overline{c}_\alpha, c_\beta] = \int d^4x \mathcal{L}_{\text{eff}}^{\text{int}}((10.11.58)). \tag{10.11.69}$$

Since the Faddeev–Popov ghost, $\{\overline{c}_\alpha(x), c_\beta(x)\}$, appears only in the internal loop, we might as well set $\zeta_\alpha(z) = \overline{\zeta}_\beta(z) = 0$ in $Z_F[\{J_a\}, \zeta_\alpha, \overline{\zeta}_\beta]$, (10.11.67), and define $Z_F[\{J_a\}]$ by

$$\begin{aligned}
Z_F[\{J_a\}] &\equiv \exp[iW_F[\{J_a\}]] \equiv \left\langle 0, \text{out} \middle| \mathrm{T}\left(\exp\left[i \int d^4z J_a(z)\hat{\phi}_a(z)\right]\right) \middle| 0, \text{in} \right\rangle \\
&= \int \mathcal{D}[\phi_a]\mathcal{D}[\overline{c}_\alpha]\mathcal{D}[c_\beta] \exp\left[i \int d^4z \left\{ \mathcal{L}_{\text{eff}}(\{\phi_a(z)\}, \overline{c}_\alpha(z), c_\beta(z)) + J_a(z)\phi_a(z) \right\} \right] \\
&= \int \mathcal{D}[\phi_a]\Delta_F[\{\phi_a\}] \exp\left[i \int d^4z \left\{ \mathcal{L}_{\text{tot}}(\{\phi_a(z)\}) - \frac{1}{2}F_\alpha^2(\{\phi_a\}) + J_a(z)\phi_a(z) \right\} \right] \\
&= Z_F[\{J_a\}, \zeta_\alpha = 0, \overline{\zeta}_\beta = 0].
\end{aligned} \tag{10.11.70}$$

This generating functional of the (connected part of) Green's functions can be used for the proof of the ξ-independence of the physical S-matrix.

10.12
BRST Invariance and Renormalization

In this section, we discuss BRST invariance and renormalization of non-Abelian gauge field in the covariant gauge in interaction with the fermion field. We refer the reader for the necessity to introduce various gauge-fixing conditions and the requisite introduction of Faddeev–Popov ghost fields, $c(x)$ and $\overline{c}(x)$, to the monograph by *M. Masujima* cited in Bibliography.

In order to establish the notation for this section, we state the various formulas:

$$F_{\alpha,x} = F_{\alpha,x}(\phi),$$

$$\left\langle 0, \text{out} \middle| 0, \text{in} \right\rangle_F = \int \mathcal{D}[A_{\gamma\mu}]\mathcal{D}[\overline{c}_\alpha]\mathcal{D}[c_\beta] \exp[iI_{\text{eff}}],$$

$$I_{\text{eff}} = \int d^4x \mathcal{L}_{\text{eff}},$$

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4}F_{\alpha\mu\nu}F_\alpha^{\mu\nu} - \overline{\psi}_n(x)\left\{ i\gamma^\mu\left(\partial_\mu\delta_{n,m} + i(t_\alpha)_{n,m}A_{\alpha\mu}(x)\right) - m\delta_{n,m}\right\}\psi_m(x)$$

$$-\frac{1}{2}F_\alpha^2\left(A_{\gamma\mu}(x)\right)+\overline{c}_\alpha(x)M_{\alpha,\beta}^F\left(A_{\gamma\mu}(x)\right)c_\beta(x),$$

$$\delta F_{\alpha,x}=\int d^4y\varepsilon_\beta(y)M_{\alpha,\beta}^F,$$

$$M_{\alpha,\beta}^F=\left.\frac{\delta F_\alpha\left(A_{\gamma\mu}^g(x)\right)}{\delta\varepsilon_\beta(y)}\right|_{g=1}.$$

Non-Abelian gauge field theory is renormalizable by power counting. The infinities that arise in the theory of the most general Lagrangian density of the renormalizable form with the operators of the dimension ≤ 4 with usual linear symmetry can be eliminated by renormalization, or cancelled by the introduction of the counter terms of the most general form which respects the same symmetry as the Lagrangian density does.

What kind of symmetry do we have for I_{eff} or $\mathcal{L}_{\text{eff}}$ after the introduction of the gauge-fixing term by $F_{\alpha,x}$ and the Faddeev–Popov ghost Lagrangian density by $M_{\alpha,\beta}^F$? Global symmetry of I_{eff} of supersymmetry type with the anticommuting infinitesimal c-number θ, with

$$\{\theta,\overline{c}_\alpha\}=\{\theta,c_\beta\}=\{\theta,\psi\}=[\theta,A_{\gamma\mu}]=0,$$

is said to be the BRST transformation when

$$\begin{aligned}
\delta A_{\alpha\mu} &= -\theta D_\mu c_\alpha=-\theta(\partial_\mu c_\alpha-C_{\alpha\beta\gamma}c_\beta A_{\gamma\mu}),\\
\delta\psi &= it_\alpha(-\theta c_\alpha)\psi,\\
\delta c_\alpha &= -(\theta/2)C_{\alpha\beta\gamma}c_\beta c_\gamma,\\
\delta\overline{c}_\alpha &= -\theta F_{\alpha,x}.
\end{aligned}\tag{10.12.1}$$

We have two theorems which we state with the proof.

Theorem 1.
The effective action functional I_{eff} is BRST-invariant.

Proof of Theorem 1: We observe that, under BRST transformation, (10.12.1), we have

$$-\frac{1}{2}F_\alpha^2\left(A_{\gamma\mu}(x)\right)\quad\Longrightarrow\quad -F_\alpha\left(A_{\gamma\mu}(x)\right)M_{\alpha,\beta}^F\left(-\theta c_\beta(x)\right),$$

while

$$\overline{c}_\alpha(x)\underbrace{\left[M_{\alpha,\beta}^F\left(A_{\gamma\mu}(x)\right)c_\beta(x)\right]}_{\delta F_\alpha\left(A_{\gamma\mu}(x)\right)}\Longrightarrow -\theta F_\alpha\left(A_{\gamma\mu}(x)\right)M_{\alpha,\beta}^F c_\beta(x)+0.$$

These two terms cancel each other.

Also we observe that

$$\delta\left[-\frac{1}{4}F_{\alpha\mu\nu}F_{\alpha}^{\mu\nu}-\overline{\psi}_n(x)\left\{i\gamma^{\mu}\left(\partial_{\mu}\delta_{n,m}+i(t_{\alpha})_{n,m}A_{\alpha\mu}(x)\right)-m\delta_{n,m}\right\}\psi_m(x)\right]=0,$$

since the expression inside $[\cdots]$ above is independent of $\overline{c}_{\alpha}(x)$ and $c_{\beta}(x)$, and is gauge invariant with the choice

$$\varepsilon_{\alpha}(x)=\theta c_{\alpha}(x),$$

which is the bosonic c-number. Thus the effective action functional I_{eff} is BRST invariant. □

Theorem 2.
For the covariant gauge,

$$F_{\alpha}\left(A_{\gamma\mu}(x)\right)=\sqrt{\xi}\,\partial^{\mu}A_{\alpha\mu}(x),\quad \alpha=1,\ldots,N,\quad 0<\xi<\infty, \tag{10.12.2}$$

I_{eff} is the most general renormalizable action functional with the operators of the dimension ≤ 4, which is Lorentz invariant, global gauge invariant, and consistent with the ghost number conservation and the invariance under the ghost translation,

$$\overline{c}_{\alpha}\longrightarrow\overline{c}_{\alpha}+\text{constant}. \tag{10.12.3}$$

Proof of Theorem 2: (1) The terms with the four ghost fields, $\overline{c}(x)\overline{c}(x)c(x)c(x)$, are ruled out by power counting since $\overline{c}$ is always accompanied by ∂_{μ}. (2) The terms with two ghost fields, $\overline{c}(x)c(x)$, are of the form, $\overline{c}cG(A,\psi)$. We note

$$\delta A\Longrightarrow c,\quad \delta\psi\Longrightarrow c,\quad \delta c\Longrightarrow cc.$$

Thus, we have

$$\delta[cG(A,\psi)]\propto cc\Longrightarrow\delta[cG(A,\psi)]=0.$$

Essentially we have

$$\overline{c}cG(A,\psi)\sim\overline{c}_{\alpha}(x)\left[M_{\alpha,\beta}^{F}\left(A_{\gamma\mu}(x)\right)c_{\beta}(x)\right].$$

We note that the BRST invariance under $\delta\overline{c}$ requires the term of the form, $F_{\alpha}^{2}\left(A_{\gamma\mu}(x)\right)$. We have the BRST invariance of the effective action functional for the ghost field part

$$I_{\text{eff}}^{\text{ghost}}=\int d^4x\left[-\frac{1}{2}F_{\alpha}^{2}\left(A_{\gamma\mu}(x)\right)+\overline{c}_{\alpha}(x)M_{\alpha,\beta}^{F}\left(A_{\gamma\mu}(x)\right)c_{\beta}(x)\right].$$

(3) The term involving A's, ψ's,..., only is BRST invariant (gauge invariant) with the choice

$$\varepsilon_{\alpha}(x)=\theta c_{\alpha}(x).$$

Thus we have the effective action functional I_{eff} as stated at the beginning of this section. □

Theorem 1 was proven by the gauge invariance of I_{eff}. Theorem 2 was proven by considering the terms with four ghosts, two ghosts and no ghosts with the use of the gauge invariance of I_{eff}.

We next consider the nonlinear realization of symmetry. The Green's functions may not have the same symmetry as the Lagrangian density does when the symmetry is realized nonlinearly. We let

$$\begin{aligned} \phi_n(x) \quad &: \quad \text{the generic fields for } A_{\alpha\mu}(x),\ \psi_n(x), \text{ etc.,} \\ W[J] \quad &: \quad \text{generating functional of the connected part of Green's function.} \end{aligned} \tag{10.12.4}$$

We write

$$\exp[iW[J]] = \int [\mathcal{D}\phi] \exp\left[iI[\phi] + i\int d^4x J_n(x)\phi_n(x) \right]. \tag{10.12.5}$$

Then we have

$$\left\langle \phi_n(x) \right\rangle_J = \left(-i\frac{\delta}{\delta J_n(x)} \exp[iW[J]] \right) \Big/ \exp[iW[J]] = \frac{\delta W[J]}{\delta J_n(x)} = \overline{\phi}_n(x), \tag{10.12.6a}$$

which is a functional of $J_n(x)$ and hence we write

$$J_n(x) = J_{n,\overline{\phi}}(x). \tag{10.12.6b}$$

We consider the Legendre transform of $W[J]$ as

$$\Gamma[\overline{\phi}] = W[J_{\overline{\phi}}] - \int d^4x J_{n,\overline{\phi}}(x)\overline{\phi}_n(x), \tag{10.12.7}$$

which is the sum of all one-particle-irreducible diagrams. The relationship between $\Gamma[\overline{\phi}]$ and $W[J_{\overline{\phi}}]$ is given by

$$\frac{\delta\Gamma[\overline{\phi}]}{\delta\overline{\phi}_n(x)} = -J_{n,\overline{\phi}}(x). \tag{10.12.8}$$

We suppose that the following is the symmetry of the action functional,

$$\phi_n(x) \longrightarrow \phi_n(x) + \varepsilon\Phi_n(\phi; x). \tag{10.12.9}$$

We have

$$I\left[\phi + \varepsilon\Phi_n(\phi; x)\right] = I[\phi], \tag{10.12.10a}$$

$$[\mathcal{D}\phi] = \prod_{n,x} d\phi_n(x) = \mathcal{D}\left[\phi + \varepsilon\Phi(\phi; x)\right], \tag{10.12.10b}$$

$$\exp[iW[J]] = \int [\mathcal{D}\phi] \exp\left[iI[\phi] + i\int d^4x J_n(x)\phi_n(x)\right] \tag{10.12.10c}$$

$$= \int [\mathcal{D}\phi] \exp\left[iI[\phi] + i\int d^4x J_n(x)\left(\phi_n(x) + \varepsilon\Phi_n(\phi; x)\right)\right]. \tag{10.12.10d}$$

Expanding Eq. (10.12.10d) to the lowest order in ε, we have the identity

$$\int [\mathcal{D}\phi] \int d^4x J_n(x)\Phi_n(\phi; x) \exp\left[iI[\phi] + i\int d^4x J_n(x)\phi_n(x)\right]$$

$$= \int d^4x J_n(x) \left\langle \Phi_n(\phi; x) \right\rangle_{J_{\overline{\phi}}} = 0. \tag{10.12.11}$$

Since we know

$$\frac{\delta\Gamma[\overline{\phi}]}{\delta\overline{\phi}_n(x)} = -J_{n,\overline{\phi}}(x), \tag{10.12.8}$$

we have

$$\int d^4x \frac{\delta\Gamma[\overline{\phi}]}{\delta\overline{\phi}_n(x)} \left\langle \Phi_n(\phi; x) \right\rangle_{J_{\overline{\phi}}} = 0. \tag{10.12.12}$$

Namely, $\Gamma[\overline{\phi}]$ is invariant under

$$\delta\overline{\phi}_n(x) = \left\langle \Phi_n(\phi; x) \right\rangle_{J_{\overline{\phi}}}. \tag{10.12.13}$$

In general, we have

$$\left\langle \Phi_n(\phi; x) \right\rangle_{J_{\overline{\phi}}} = \frac{\int [\mathcal{D}\phi]\Phi_n(\phi; x) \exp\left[iI[\phi] + i\int d^4x J_n(x)\phi_n(x)\right]}{\int [\mathcal{D}\phi] \exp\left[iI[\phi] + i\int d^4x J_n(x)\phi_n(x)\right]} \neq \Phi_n\left(\overline{\phi}(x); x\right). \tag{10.12.14a}$$

Only for the linear realization of symmetry, we have

$$\left\langle \Phi_n(\phi; x) \right\rangle_{J_{\overline{\phi}}} = \Phi_n\left(\overline{\phi}(x); x\right). \tag{10.12.14b}$$

The infinite part of $\Gamma[\overline{\phi}] = \Gamma_\infty$ is the term of the minimum dimensionality and hence, by the loop expansion, we have

$$\Gamma_\infty = \Gamma_{\text{min}}. \tag{10.12.15a}$$

Above identity (10.12.12) should hold for each dimensionality and hence, for the term for the minimum dimension, we have

$$\int d^4x \frac{\delta\Gamma_\infty[\overline{\phi}]}{\delta\overline{\phi}_n(x)} \left\langle \Phi_n(\phi;x) \right\rangle_{J_{\overline{\phi}},\,\min} = 0. \tag{10.12.15b}$$

Namely, $\Gamma_\infty[\overline{\phi}]$ is invariant under

$$\delta\overline{\phi}_n(x) = \left\langle \Phi_n(\phi;x) \right\rangle_{J_{\overline{\phi}},\,\min}. \tag{10.12.16}$$

The BRST invariance is the nonlinear realization of the gauge invariance . We have

$$\delta\phi_n(x) = \theta\Phi_n(\phi;x), \tag{10.12.17}$$

where θ is of the dimension 1 and $\Phi_n(\phi;x)$ is dimension ≤ 3. As for the minimum dimension, we write

$$\begin{aligned} \left\langle D_\mu c_\alpha \right\rangle_{\min} &= Z\left(\partial_\mu \tilde{c}_\alpha - C'_{\alpha\beta\gamma}\tilde{c}_\beta \tilde{A}_{\gamma\mu}\right), \\ \left\langle \psi c_\alpha \right\rangle_{\min} &= Z'\tilde{\psi}\tilde{c}_\alpha, \\ \left\langle C_{\alpha\beta\gamma} c_\beta c_\gamma \right\rangle_{\min} &= C''_{\alpha\beta\gamma}\tilde{c}_\beta\tilde{c}_\gamma, \end{aligned} \tag{10.12.18}$$

while

ghost number conservation $\sim$ linear realization,
quark number conservation $\sim$ linear realization.

Theorem 3.
Let $I_\infty[\overline{\phi}]$ be the most general action functional with dimensionality ≤ 4 of $\tilde{A}_{\alpha\mu}(x)$, $\tilde{c}_\alpha(x)$, $\tilde{\overline{c}}_\alpha(x)$, and $\tilde{\psi}_n(x)$, which is invariant under usual linear symmetries (the Lorentz invariance, the global gauge invariance, the ghost number conservation, and the ghost translation $\overline{c}_\alpha \to \overline{c}_\alpha$+constant). Let $\delta\tilde{A}_{\gamma\mu}$, $\delta\tilde{\psi}$, $\delta\tilde{c}_\alpha$, and $\delta\tilde{\overline{c}}_\alpha$ be the most general infinitesimal transformations with minimum dimensionality, and be the symmetry transformation of $I_\infty[\overline{\phi}]$. Then

(1) *$(\delta\tilde{A}_{\gamma\mu})_{\min}$, $(\delta\tilde{\psi})_{\min}$, $(\delta\tilde{c}_\alpha)_{\min}$, and $(\delta\tilde{\overline{c}}_\alpha)_{\min}$ are given in terms of $\tilde{A}_{\alpha\mu}(x)$, $\tilde{c}_\alpha(x)$, $\tilde{\overline{c}}_\alpha(x)$ and $\tilde{\psi}_n(x)$ by usual equations for BRST transformation except that $C_{\alpha\beta\gamma}$ and t_α get multiplied with Z_c,*

$$C_{\alpha\beta\gamma} \longrightarrow Z_c C_{\alpha\beta\gamma} \quad t_\alpha \longrightarrow t_\alpha Z_c. \tag{10.12.19}$$

The BRST transformation is given by

$$\begin{aligned} \left(\delta\tilde{A}_{\gamma\mu}\right)_{\min} &= -\theta\left(\partial_\mu\tilde{c}_\alpha - Z_c C_{\alpha\beta\gamma}\tilde{c}_\beta\tilde{A}_{\gamma\mu}\right), \\ \left(\delta\tilde{\psi}\right)_{\min} &= it_\alpha Z_c\left(-\theta\tilde{c}_\alpha\right)\tilde{\psi}, \\ \left(\delta\tilde{c}_\alpha\right)_{\min} &= -(\theta/2)\, Z_c C_{\alpha\beta\gamma}\tilde{c}_\beta\tilde{c}_\gamma, \\ \left(\delta\tilde{\overline{c}}_\alpha\right)_{\min} &= -\theta\left(\tilde{F}_{\alpha,x}\right)_{\min}. \end{aligned} \tag{10.12.20}$$

(2) $I_\infty[\tilde{A}, \tilde{\psi}, \tilde{c}, \tilde{\bar{c}}]$ *is given by the renormalized quantities as*

$$I_\infty\left[\tilde{A}, \tilde{\psi}, \tilde{c}, \tilde{\bar{c}}\right] = \int d^4x \left[-\frac{Z_A}{4}\left(\partial_\mu A^{\mathrm{R}}_{\alpha\nu}(x) - \partial_\nu A^{\mathrm{R}}_{\alpha\mu} - Z_{\mathrm{c}} C^{\mathrm{R}}_{\alpha\beta\gamma} A^{\mathrm{R}}_{\beta\mu}(x) A^{\mathrm{R}}_{\gamma\nu}(x)\right)^2 \right.$$
$$- \frac{Z_\xi}{2\xi^{\mathrm{R}}}\left(\partial_\mu A^{\mu\mathrm{R}}_\alpha\right)^2 + Z_\psi \overline{\psi}^{\mathrm{R}}_n(x)\left\{i\gamma^\mu\left(\partial_\mu + it^{\mathrm{R}}_\alpha Z_{\mathrm{c}} A^{\mathrm{R}}_{\alpha\mu}(x)\right) - m\delta_{n,m}\right\}$$
$$\left.\times \psi^{\mathrm{R}}_m(x) + \int d^4x \left[Z_{\mathrm{c}} \partial_\mu \bar{c}^{\mathrm{R}}_\beta \left(\partial^\mu c^{\mathrm{R}}_\beta - Z_{\mathrm{c}} C^{\mathrm{R}}_{\alpha\beta\gamma} A^{\mu\mathrm{R}}_\gamma c^{\mathrm{R}}_\alpha\right)\right]\right]. \tag{10.12.21}$$

Proof of Theorem 3: We define the unrenormalized quantities as

$$\begin{aligned}
A^{\mathrm{R}}_{\alpha\mu}(x) &= Z_A^{-1/2} A_{\alpha\mu}(x), \\
\psi^{\mathrm{R}}(x) &= Z_\psi^{-1/2} \psi(x), \\
c^{\mathrm{R}}_\alpha &= Z_{\mathrm{c}}^{-1/2} c_\alpha(x), \\
C^{\mathrm{R}}_{\alpha\beta\gamma} &= Z_{\mathrm{c}}^{-1} Z_A^{+1/2} \tilde{C}_{\alpha\beta\gamma}, \\
t^{\mathrm{R}}_\alpha &= Z_{\mathrm{c}}^{-1} Z_A^{+1/2} \tilde{t}_\alpha, \\
g^{\mathrm{R}} &= Z_{\mathrm{c}}^{-1} Z_A^{+1/2} \tilde{g}, \\
\xi^{\mathrm{R}} &= Z_\xi Z_A^{-1} \xi.
\end{aligned} \tag{10.12.22}$$

We express $I_\infty[\tilde{A}, \tilde{\psi}, \tilde{c}, \tilde{\bar{c}}]$ in terms of the unrenormalized quantities

$$I_\infty\left[\tilde{A}, \tilde{\psi}, \tilde{c}, \tilde{\bar{c}}\right] = \int d^4x \left[-\frac{1}{4}\left(\partial_\mu A_{\alpha\nu}(x) - \partial_\nu A_{\alpha\mu} - \tilde{C}_{\alpha\beta\gamma} A_{\beta\mu}(x) A_{\gamma\nu}(x)\right)^2 \right.$$
$$- \frac{1}{2\xi}\left(\partial_\mu A^\mu_\alpha\right)^2 + \overline{\psi}_n(x)\left\{i\gamma^\mu\left(\partial_\mu + i\tilde{t}_\alpha A_{\alpha\mu}(x)\right) - m\delta_{n,m}\right\}$$
$$\left.\times \psi_m(x) + \int d^4x \left[\partial_\mu \bar{c}_\beta \left(\partial^\mu c_\beta - \tilde{C}_{\alpha\beta\gamma} A^\mu_\gamma c_\alpha\right)\right]\right], \tag{10.12.23}$$

which is the form of the action functional we started out with. The effective Lagrangian density has the sufficient symmetry structure to absorb all the infinities. The most general form of the renormalization counter terms is given by replacing Z's with δZ's where we had the renormalization constants

$$Z = 1 + \delta Z. \tag{10.12.24}$$

Thus the system is renormalizable. □

What is essential to this proof are the loop expansion of $\Gamma[\overline{\phi}]$ and the nonlinear realization of the gauge invariance. The BRST invariance is the last remnant of the gauge invariance after the introduction of the gauge-fixing term and the Faddeev–Popov ghost term.

10.13 Asymptotic Disaster in *QED*

In this section, we derive the set of completely renormalized Schwinger–Dyson equations, which is free from overlapping divergence for Abelian gauge field in interaction with the fermion field. With tri-Γ approximation, we demonstrate asymptotic disaster of Abelian gauge field in interaction with the fermion field. Asymptotic disaster of Abelian gauge field was discovered in the mid-1950s by Gell–Mann and Low and independently by Landau, Abrikosov, Galanin, and Khalatnikov. Soon after this discovery was made, quantum field theory was once abandoned for a decade, and dispersion theory became fashionable.

We shall begin with the definition of generating functional of Green's functions

$$Z_F[J, \eta, \overline{\eta}] = \int \mathcal{D}[\phi_a] \exp\left[i \int d^4x \mathcal{L}_{\text{eff}}(\phi_a) \right], \tag{10.13.1}$$

where the effective Lagrangian density is given by

$$\begin{aligned}\mathcal{L}_{\text{eff}}(\phi_a) = & -(1/2)A(1)[D_{A^2}]_0^{-1}(1,2)A(2) + \overline{\psi}(1)[D_{\psi\overline{\psi}}]_0^{-1}(1,2)\psi(2) \\ & + ig_0\Gamma^{(0)}[1,2,3]\overline{\psi}(1)\psi(2)A(3) \\ & + \overline{\psi}(1)\eta_{\overline{\psi}}(1) + \overline{\eta}_{\psi}(1)\psi(1) + A(1)J_A(1),\end{aligned} \tag{10.13.2}$$

and the differential operators and the vertex operator are given by

$$\begin{aligned}[D_{A^2}]_{0\,\mu\nu}^{-1} &= \delta^4(x-y)\left[-\eta_{\mu\nu}\partial^2 + (1-\xi)\partial_\mu\partial_\nu\right], \\ [D_{\psi\overline{\psi}}]_0^{-1} &= \delta^4(x-y)(i\gamma^\mu\partial_\mu - m_0), \\ (\Gamma^{(0)})_{\alpha\beta\mu}(x,y,z) &= -\delta^4(x-z)\delta^4(z-y)(\gamma_\mu)_{\alpha\beta}.\end{aligned} \tag{10.13.3}$$

The "full" Green's functions and the "full" vertex operator are defined by

$$D_{A^2}(1,2) = \frac{1}{i}\frac{\delta^2}{\delta J_A(1)\delta J_A(2)} \ln Z_F[J, \eta, \overline{\eta}], \tag{10.13.4a}$$

$$D_{\psi\overline{\psi}}(1,2) = \frac{1}{i}\frac{\delta^2}{\delta\overline{\eta}_\psi(1)\delta\eta_{\overline{\psi}}(2)} \ln Z_F[J, \eta, \overline{\eta}], \tag{10.13.4b}$$

$$\Gamma[1,2,3] = -\frac{\delta}{(ig_0)\delta\left\langle A(3)\right\rangle} D_{\psi\overline{\psi}}^{-1}(1,2). \tag{10.13.4c}$$

The set of functional equations for $\langle A(2)\rangle$, and $\langle\psi(2)\rangle$ are derived by the standard method as

$$[D_{A^2}]_0^{-1}(1,2)\left\langle A(2)\right\rangle + g_0\Gamma^{(0)}(1,2,3)\left[D_{\psi\overline{\psi}}(3,2) - \frac{1}{i}\left\langle\psi(3)\right\rangle\left\langle\overline{\psi}(2)\right\rangle\right] = J_A(1), \tag{10.13.5a}$$

$$[D_{\psi\overline{\psi}}]_0^{-1}(1,2)\left\langle\psi(2)\right\rangle - g_0\Gamma^{(0)}(1,2,3)\left[\frac{\delta\left\langle\psi(2)\right\rangle}{\delta J_A(3)} - \frac{1}{i}\left\langle\psi(2)\right\rangle\left\langle A(3)\right\rangle\right] = \eta_{\overline{\psi}}(1). \tag{10.13.5b}$$

Schwinger–Dyson equations are obtained as

$$\begin{aligned} [D_{A^2}]^{-1}(1,2) &= [D_{A^2}]_0^{-1}(1,2) - \Pi_{A^2}(1,2), \\ [D_{\psi\overline{\psi}}]^{-1}(1,2) &= [D_{\psi\overline{\psi}}]_0^{-1}(1,2) - \Sigma_{\psi\overline{\psi}}(1,2). \end{aligned} \tag{10.13.6}$$

The proper self-energy parts for the gauge field and the fermion field are defined by

$$\begin{aligned} \Pi_{A^2}(1,\overline{1}) &= -ig_0^2\Gamma^{(0)}(1,2,3)D_{\psi\overline{\psi}}(3,\overline{3})\Gamma(\overline{3},\overline{2},\overline{1})D_{\psi\overline{\psi}}(\overline{2},2), \\ \Sigma_{\psi\overline{\psi}}(1,\overline{1}) &= ig_0^2\Gamma^{(0)}(1,2,3)D_{\psi\overline{\psi}}(2,\overline{2})\Gamma(\overline{3},\overline{2},\overline{1})D_{A^2}(\overline{3},3). \end{aligned} \tag{10.13.7}$$

We renormalize the physical quantities by

$$\Gamma^{\mathrm{R}} = Z_1\Gamma, \quad D^{\mathrm{R}}_{\psi\overline{\psi}} = Z_2{}^{-1}D_{\psi\overline{\psi}}, \quad D^{\mathrm{R}}_{A^2} = Z_3^{-1}D_{A^2}. \tag{10.13.8}$$

In terms of the renormalized quantities,

$$\begin{aligned} [D^{\mathrm{R}}_{A^2}]^{-1}(1,2) &= Z_3[D_{A^2}]_0^{-1}(1,2) - \Pi'_{A^2}(1,2), \\ [D^{\mathrm{R}}_{\psi\overline{\psi}}]^{-1}(1,2) &= Z_2[D_{\psi\overline{\psi}}]_0^{-1}(1,2) - \Sigma'_{\psi\overline{\psi}}(1,2). \end{aligned} \tag{10.13.9}$$

The primes indicate the proper self-energy parts after the renormalization and we made use of Ward–Takahashi identities

$$Z_1 = Z_2, \quad g = Z_1^{-1}Z_2Z_3^{1/2}g_0. \tag{10.13.10}$$

From the elimination of the ultraviolet divergences in Green's functions , we have

$$\begin{aligned} Z_3 &= 1 + \partial\Pi'_{A^2}(k_0^2)\diagup\partial k_0^2, \\ Z_2 &= 1 + \partial\Sigma'_{\psi\overline{\psi}}(k_0^2)\diagup\partial k_0^2. \end{aligned} \tag{10.13.11}$$

The renormalized Schwinger–Dyson equations are given by

$$\begin{aligned} [D^{\mathrm{R}}_{A^2}]^{-1}(1,2) &= [D_{A^2}]_0^{-1}(1,2) - \tilde{\Pi}^{\mathrm{R}}_{A^2}(1,2), \\ [D^{\mathrm{R}}_{\psi\overline{\psi}}]^{-1}(1,2) &= [D_{\psi\overline{\psi}}]_0^{-1}(1,2) - \tilde{\Sigma}^{\mathrm{R}}_{\psi\overline{\psi}}(1,2), \end{aligned} \tag{10.13.12}$$

where the proper self-energy parts in the above equations are given by

$$\begin{aligned} \tilde{\Pi}^{\mathrm{R}}_{A^2}(k^2) &= \tilde{\Pi}'_{A^2}(k^2) - k^2\partial\tilde{\Pi}'_{A^2}(k_0^2)\diagup\partial k_0^2, \\ \tilde{\Sigma}^{\mathrm{R}}_{\psi\overline{\psi}}(k^2) &= \tilde{\Sigma}'_{\psi\overline{\psi}}(k^2) - k^2\partial\tilde{\Sigma}'_{\psi\overline{\psi}}(k_0^2)\diagup\partial k_0^2. \end{aligned} \tag{10.13.13}$$

These proper self-energy parts do not contain the ultraviolet divergences. We note that k_0^2 is the normalization point and that $\tilde{\Pi}'_{A^2}$ and $\tilde{\Sigma}'_{\psi\overline{\psi}}$ are those parts of the proper self-energy parts which are subject to the renormalization.

From the consideration of the overlapping divergences, we obtain

$$\Gamma^{\mathrm{R}}(1,2,3) = Z_1\Gamma^{(0)}(1,2,3) + ig^2\Gamma^{\mathrm{R}}(1,\overline{2},\overline{3})D^{\mathrm{R}}_{\psi\overline{\psi}}(\overline{2},\overline{1})P(\overline{1},2,3,\overline{4})D^{\mathrm{R}}_{A^2}(\overline{4},\overline{3}), \tag{10.13.14a}$$

$$P(1,2,3,4) = W(1,2,3,4) - ig^2 W(1,\overline{2},\overline{3},4)D^{\mathrm{R}}_{A^2}(\overline{4},\overline{3})D^{\mathrm{R}}_{\psi\overline{\psi}}(\overline{2},\overline{1})P(\overline{1},2,3,\overline{4}), \tag{10.13.14b}$$

$$W(1,2,3,4) = \frac{\delta\Gamma^{\mathrm{R}}(1,2,4)}{(ig)\delta\left\langle A^{\mathrm{R}}(3)\right\rangle} + \Gamma^{\mathrm{R}}(1,\overline{2},3)D^{\mathrm{R}}_{\psi\overline{\psi}}(\overline{2},\overline{1})\Gamma^{\mathrm{R}}(\overline{1},2,4). \tag{10.13.14c}$$

We finally consider the asymptotic behavior of Green's functions and the vertex operator. We shall consider the vertex operator

$$\Gamma^{\mathrm{R}}(1,2,3) = Z_1\Gamma^{(0)}(1,2,3) + ig^2\Gamma^{\mathrm{R}}(1,\overline{2},\overline{3})D^{\mathrm{R}}_{\psi\overline{\psi}}(\overline{2},\overline{1})P(\overline{1},2,3,\overline{4})D^{\mathrm{R}}_{A^2}(\overline{4},\overline{3}), \tag{10.13.15}$$

and the set of equations for the derivatives of Green's functions

$$\begin{aligned} d[D^{\mathrm{R}}_{A^2}]^{-1}(p^2)/dp^2 &= 1 - d\Pi^{\mathrm{R}}_{A^2}(p^2)/dp^2, \\ d[D^{\mathrm{R}}_{\psi\overline{\psi}}]^{-1}(p^2)/dp^2 &= 1 - d\Sigma^{\mathrm{R}}_{\psi\overline{\psi}}(p^2)/dp^2. \end{aligned} \tag{10.13.16}$$

By tri-Γ approximation, we mean to replace $Z_1\Gamma^{(0)}$ with Γ^{R} to determine the vertex operator of the fermion field by solving simpler equation given by

$$\begin{aligned} \Gamma^{\mathrm{R}}(1,2,3) = Z_1\Gamma^{(0)}(1,2,3) &+ ig^2\Gamma^{\mathrm{R}}(1,\overline{2},\overline{3})D^{\mathrm{R}}_{\psi\overline{\psi}}(\overline{2},\overline{1})\Gamma^{\mathrm{R}}(\overline{1},\bar{\bar{\bar{2}}},4) \\ &\times D^{\mathrm{R}}_{\psi\psi}(\bar{\bar{\bar{2}}},\bar{\bar{\bar{1}}})\Gamma^{\mathrm{R}}(\bar{\bar{\bar{1}}},2,3)D^{\mathrm{R}}_{A^2}(4,\overline{3}). \end{aligned} \tag{10.13.17}$$

In order to separate the tensor structure of various quantities, we choose the Feynman gauge which leads to the following equations:

$$\begin{aligned} D^{\mathrm{R}}_{A^2\ \mu\nu}(p^2) &= \eta_{\mu\nu}D^{\mathrm{R}}_{A^2}(p^2), \\ D^{\mathrm{R}}_{\psi\overline{\psi}\ \alpha\beta}(p^2) &\approx (\gamma_\mu)_{\alpha\beta}p^\mu D^{\mathrm{R}}_{\psi\overline{\psi}}(p^2), \\ (\Gamma^{\mathrm{R}})_{\alpha\beta\mu}(p+k,p,k) &= (\gamma_\mu)_{\alpha\beta}\Gamma^{\mathrm{R}}(p+k,p,k). \end{aligned} \tag{10.13.18}$$

We seek the solutions of the following forms:

$$\begin{aligned} D^{\mathrm{R}}_{A^2}(p^2) &= d_{A^2}(p^2)/p^2, \\ D^{\mathrm{R}}_{\psi\overline{\psi}}(p^2) &= h_{\psi\overline{\psi}}(p^2)/p^2, \\ \Gamma^{\mathrm{R}}(p+k,p,k) &\longrightarrow \Gamma(q^2). \end{aligned} \tag{10.13.19}$$

We regard $d_{A^2}(p^2)$, $h_{\psi\overline{\psi}}(p^2)$, and $\Gamma(q^2)$ as slowly varying functions of p^2 and q^2, namely, the derivative of these functions is close to zero.

Equations (10.13.16) and (10.13.17) take the following form of a set of integral equations:

$$\frac{1}{d_{A^2}(\varsigma)} = 1 + \frac{4}{3}\frac{g^2}{16\pi^2}\int_{\varsigma}^{0} dz\Gamma^2(z)h_{\psi\overline{\psi}}^2(z), \tag{10.13.20}$$

$$\frac{1}{h_{\psi\overline{\psi}}(\varsigma)} = 1 - \frac{g^2}{16\pi^2}\int_{\varsigma}^{0} dz\Gamma^2(z)h_{\psi\overline{\psi}}^2(z)d_{A^2}(z), \tag{10.13.21}$$

$$\Gamma(\varsigma) = 1 + \frac{g^2}{16\pi^2}\int_{\varsigma}^{0} dz\Gamma^3(z)h_{\psi\overline{\psi}}^2(z)d_{A^2}(z), \quad \varsigma = \ln(p^2/m^2). \tag{10.13.22}$$

These equations are completely equivalent to the following set of differential equations:

$$\frac{1}{d_{A^2}}\frac{d}{d\varsigma}(d_{A^2}) = \frac{4}{3}\frac{g^2}{16\pi^2}\Gamma^2 h_{\psi\overline{\psi}}^2 d_{A^2}, \tag{10.13.23}$$

$$\frac{1}{h_{\psi\overline{\psi}}}\frac{d}{d\varsigma}(h_{\psi\overline{\psi}}) = -\frac{g^2}{16\pi^2}\Gamma^2 h_{\psi\overline{\psi}}^2 d_{A^2}, \tag{10.13.24}$$

$$\frac{1}{\Gamma}\frac{d}{d\varsigma}(\Gamma) = -\frac{g^2}{16\pi^2}\Gamma^2 h_{\psi\overline{\psi}}^2 d_{A^2}, \tag{10.13.25}$$

with the boundary conditions

$$d_{A^2}(0) = h_{\psi\overline{\psi}}(0) = 1. \tag{10.13.26}$$

The set of preceding nonlinear differential equations, (10.13.23)–(10.13.25), with the boundary conditions, (10.13.26), can be solved explicitly as

$$h_{\psi\overline{\psi}} = \Gamma, \quad d_{A^2} = \Gamma^{-4/3}, \quad \Gamma = \left[1 - (4/3)\frac{g^2}{16\pi^2}\varsigma\right]^{3/4}. \tag{10.13.27}$$

The connection between the bare or running coupling constant (or the bare charge) and the observed coupling constant (or the observed charge) in this theory takes the following form:

$$g_0^2(\Lambda^2) = \frac{g^2}{1 - (4/3)\left(g^2/16\pi^2\right)\ln\left(\Lambda^2/m^2\right)}. \tag{10.13.28}$$

The bare or running coupling constant gets large as the cut-off parameter Λ^2 gets large. Thus, at high energies or at short distances, the theory becomes a strong coupling theory. Equation (10.13.28) can be solved for the observed coupling constant (or the observed charge) as

$$g^2 = \frac{g_0^2(\Lambda^2)}{1 + (4/3)\left(g_0^2(\Lambda^2)/16\pi^2\right)\ln\left(\Lambda^2/m^2\right)}. \tag{10.13.29}$$

The observed coupling constant (or the observed charge) goes to zero if the local limit $p^2 = \Lambda^2 \to \infty$ is taken.

10.14 Asymptotic Freedom in *QCD*

In this section, we derive the set of completely renormalized Schwinger–Dyson equations which is free from overlapping divergence for non-Abelian gauge field in interaction with the fermion field. With tri-Γ approximation, we demonstrate asymptotic freedom of non-Abelian gauge field in interaction with the fermion field. This property arises from the non-Abelian nature of the gauge group and such property is not present for Abelian gauge field like *QED*. Actually, no quantum field theory is asymptotically free without non-Abelian gauge field .

We shall begin with the definition of generating functional of Green's functions,

$$Z_F[J,\eta,\overline{\eta}] = \int \mathcal{D}[\phi_a]\exp\left[i\int d^4x\mathcal{L}_{\text{eff}}(\phi_a)\right], \tag{10.14.1}$$

where the effective Lagrangian density is given by

$$\begin{aligned}
\mathcal{L}_{\text{eff}}(\phi_a) = &-(1/2)A(1)[D_{A^2}]_0^{-1}(1,2)A(2) \\
&+(ig_0/3!)\Gamma_{A^3}^{(0)}[1,2,3]A(1)A(2)A(3) \\
&+\left((ig_0)^2/4!\right)\Gamma_{A^4}^{(0)}[1,2,3,4]A(1)A(2)A(3)A(4) \\
&+\overline{c}(1)\,[D_{c\overline{c}}]_0^{-1}\,(1,2)c(2) + ig_0\Gamma_{c\overline{c}A}^{(0)}[1,2,3]\overline{c}(1)c(2)A(3) \\
&+\overline{\psi}(1)[D_{\psi\overline{\psi}}]_0^{-1}(1,2)\psi(2) + ig_0\Gamma_{\psi\overline{\psi}A}^{(0)}[1,2,3]\overline{\psi}(1)\psi(2)A(3) \\
&+\overline{c}(1)\eta_{\overline{c}}(1) + \overline{\eta}_c(1)c(1) + \overline{\psi}(1)\eta_{\overline{\psi}}(1) + \overline{\eta}_{\psi}(1)\psi(1) + A(1)J_A(1),
\end{aligned} \tag{10.14.2}$$

and the differential operators are given by

$$\begin{aligned}
\left([D_{A^2}]_0^{-1}\right)_{\mu\nu}^{ab} &= \delta^4(x-y)\delta^{ab}\left[-\eta_{\mu\nu}\partial^2 + (1-\xi)\partial_\mu\partial_\nu\right], \\
\left([D_{c\overline{c}}]_0^{-1}\right)^{ab} &= \delta^4(x-y)\delta^{ab}\left(-\partial^2\right), \\
\left([D_{\psi\overline{\psi}}]_0^{-1}\right)^{ab} &= \delta^4(x-y)\delta^{ab}(i\gamma^\mu\partial_\mu - m_0).
\end{aligned} \tag{10.14.3}$$

In the above, $c(1)$ and $\overline{c}(1)$ are the Faddeev–Popov ghost fields.

The vertex operators are given by

$$\begin{aligned}
\left(\Gamma^{(0)}_{A^3}\right)^{abc}_{\mu\nu\lambda}(x,y,z) &= if^{abc}\left\{\eta_{\mu\nu}\left[-2\delta^4(x-z)\partial^y_\lambda\delta^4(y-x)\right.\right.\\
&\quad\left.+\delta^4(x-y)\partial^x_\lambda\delta^4(x-z)\right]\\
&\quad+\eta_{\mu\lambda}\left[-2\delta^4(x-y)\partial^x_\nu\delta^4(x-z)\right.\\
&\quad\left.+\delta^4(x-z)\partial^y_\nu\delta^4(x-y)\right]\\
&\quad+\eta_{\nu\lambda}\left[\delta^4(x-z)\partial^y_\mu\delta^4(y-x)\right.\\
&\quad\left.\left.+\delta^4(x-y)\partial^x_\mu\delta^4(x-z)\right]\right\},\\
\left(\Gamma^{(0)}_{c\bar{c}A}\right)^{abc}_{\mu}(x,y,z) &= -if^{abc}\left(\partial^z_\mu\delta^4(z-y)\right)\delta^4(x-z),\\
\left(\Gamma^{(0)}_{\psi\bar{\psi}A}\right)^{abc}_{\alpha\beta\mu}(x,y,z) &= -\delta^4(x-z)\delta^4(z-y)(\gamma_\mu)_{\alpha\beta}(t^c)^{ab},\\
\left(\Gamma^{(0)}_{A^4}\right)^{abcd}_{\mu\nu\delta\sigma}(z_1,z_2,z_3,z_4) &= \delta^4(z_1-z_2)\delta^4(z_1-z_3)\delta^4(z_1-z_4)\\
&\quad\times\left[f^{pab}f^{pcd}\left(\eta_{\delta\mu}\eta_{\nu\sigma}-\eta_{\mu\sigma}\eta_{\nu\delta}\right)\right.\\
&\qquad\left.+f^{pbc}f^{pad}\left(\eta_{\delta\sigma}\eta_{\mu\nu}-\eta_{\nu\delta}\eta_{\mu\sigma}\right)\right].
\end{aligned}\tag{10.14.4}$$

The "full" Green's functions and the "full" vertex operators are defined by

$$D_{A^2}(1,2)=\frac{1}{i}\frac{\delta^2}{\delta J_A(1)\delta J_A(2)}\ln Z_F[J,\eta,\overline{\eta}],\tag{10.14.5a}$$

$$D_{c\bar{c}}(1,2)=\frac{1}{i}\frac{\delta^2}{\delta\overline{\eta}_c(1)\delta\eta_{\bar{c}}(2)}\ln Z_F[J,\eta,\overline{\eta}],\tag{10.14.5b}$$

$$D_{\psi\overline{\psi}}(1,2)=\frac{1}{i}\frac{\delta^2}{\delta\overline{\eta}_\psi(1)\delta\eta_{\overline{\psi}}(2)}\ln Z_F[J,\eta,\overline{\eta}],\tag{10.14.5c}$$

$$\Gamma_{A^3}[1,2,3]=-\frac{\delta}{(ig_0)\delta\left\langle A(3)\right\rangle}D^{-1}_{A^2}(1,2),\tag{10.14.5d}$$

$$\Gamma_{A^4}[1,2,3,4]=-\frac{\delta^2}{(ig_0)^2\delta\left\langle A(3)\right\rangle\delta\left\langle A(4)\right\rangle}D^{-1}_{A^2}(1,2),\tag{10.14.5e}$$

$$\Gamma_{c\bar{c}A}[1,2,3]=-\frac{\delta}{(ig_0)\delta\left\langle A(3)\right\rangle}D^{-1}_{c\bar{c}}(1,2),\tag{10.14.5f}$$

$$\Gamma_{\psi\overline{\psi}A}[1,2,3]=-\frac{\delta}{(ig_0)\delta\left\langle A(3)\right\rangle}D^{-1}_{\psi\overline{\psi}}(1,2).\tag{10.14.5g}$$

The set of functional equations for $\langle A(2)\rangle$, $\langle c(2)\rangle$, and $\langle\psi(2)\rangle$ are derived by the standard method as

$$[D_{A^2}]_0^{-1}(1,2)\left\langle A(2)\right\rangle - \frac{g_0}{2!}\Gamma^{(0)}_{A^3}(1,2,3)\left[D_{A^2}(2,3) - \frac{1}{i}\left\langle A(2)\right\rangle\left\langle A(3)\right\rangle\right]$$
$$+\frac{(ig_0)^2}{3!}\Gamma^{(0)}_{A^4}(1,2,3,4)\left[\frac{\delta D_{A^2}(2,3)}{\delta J_A(4)} - \frac{1}{i}\left\langle A(2)\right\rangle D_{A^2}(3,4) - \frac{1}{i}\left\langle A(3)\right\rangle D_{A^2}(2,4)\right.$$
$$\left. - \frac{1}{i}\left\langle A(4)\right\rangle D_{A^2}(2,3) - \left\langle A(2)\right\rangle\left\langle A(3)\right\rangle\left\langle A(4)\right\rangle\right]$$
$$+ g_0\Gamma^{(0)}_{Ac\bar{c}}(1,2,3)\left[D_{c\bar{c}}(3,2) - \frac{1}{i}\left\langle c(3)\right\rangle\left\langle \bar{c}(2)\right\rangle\right]$$
$$+ g_0\Gamma^{(0)}_{A\psi\bar{\psi}}(1,2,3)\left[D_{\psi\bar{\psi}}(3,2) - \frac{1}{i}\left\langle \psi(3)\right\rangle\left\langle \bar{\psi}(2)\right\rangle\right] = J_A(1), \tag{10.14.6a}$$

$$[D_{c\bar{c}}]_0^{-1}(1,2)\left\langle c(2)\right\rangle - g_0\Gamma^{(0)}_{c\bar{c}A}(1,2,3)\left[\frac{\delta\left\langle c(2)\right\rangle}{\delta J_A(3)} - \frac{1}{i}\left\langle c(2)\right\rangle\left\langle A(3)\right\rangle\right] = \eta_{\bar{c}}(1), \tag{10.14.6b}$$

$$[D_{\psi\bar{\psi}}]_0^{-1}(1,2)\left\langle \psi(2)\right\rangle - g_0\Gamma^{(0)}_{\psi\bar{\psi}A}(1,2,3)\left[\frac{\delta\left\langle \psi(2)\right\rangle}{\delta J_A(3)} - \frac{1}{i}\left\langle \psi(2)\right\rangle\left\langle A(3)\right\rangle\right] = \eta_{\bar{\psi}}(1). \tag{10.14.6c}$$

The set of equations for the unrenormalized Green's functions is obtained from Eqs. (10.14.6a)–(10.14.6c) by taking the functional derivatives of the latter with respect to the external hooks. Schwinger–Dyson equations are obtained as

$$[D_{A^2}]^{-1}(1,2) = [D_{A^2}]_0^{-1}(1,2) - (ig_0)\Gamma^{(0)}_{A^3}(1,2,3)\left\langle A(3)\right\rangle$$
$$-\frac{(ig_0)}{2}\Gamma^{(0)}_{A^4}(1,2,3,4)\left\langle A(3)\right\rangle\left\langle A(4)\right\rangle - \Pi_{A^2}(1,2), \tag{10.14.7a}$$

$$[D_{c\bar{c}}]^{-1}(1,2) = [D_{c\bar{c}}]_0^{-1}(1,2) - \Sigma_{c\bar{c}}(1,2), \tag{10.14.7b}$$

$$[D_{\psi\bar{\psi}}]^{-1}(1,2) = [D_{\psi\bar{\psi}}]_0^{-1}(1,2) - \Sigma_{\psi\bar{\psi}}(1,2). \tag{10.14.7c}$$

The proper self-energy part for the gauge field is defined by

$$\Pi_{A^2}(1,\bar{1}) = i\frac{g_0^2}{2}\Gamma^{(0)}_{A^4}(1,2,3,\bar{1})D_{A^2}(2,3)$$
$$+i\frac{g_0^2}{2}\Gamma^{(0)}_{A^3}(1,2,3)D_{A^2}(2,\bar{3})\Gamma_{A^3}(\bar{3},\bar{2},\bar{1})D_{A^2}(\bar{2},3)$$
$$+i\frac{g_0^3}{2}\Gamma^{(0)}_{A^4}(1,2,3,4)\left\langle A(2)\right\rangle D_{A^2}(4,\bar{3})\Gamma_{A^3}(\bar{3},\bar{2},\bar{1})D_{A^2}(\bar{2},3)$$
$$-\frac{g_0^4}{2}\Gamma^{(0)}_{A^4}(1,2,3,4)D_{A^2}(2,\bar{4})\Gamma_{A^3}(\bar{4},5,6)D_{A^2}(5,\bar{2})$$

$$\times\Gamma_{A^3}(\overline{3},\overline{2},\overline{1})D_{A^2}(\overline{3},4)D_{A^2}(6,3)$$

$$-\frac{g_0^4}{3!}\Gamma^{(0)}_{A^4}(1,2,3,4)D_{A^2}(2,\overline{2})\Gamma_{A^4}(\overline{4},\overline{3},\overline{2},\overline{1})D_{A^2}(3,\overline{3})D_{A^2}(\overline{4},4)$$

$$-ig_0^2\Gamma^{(0)}_{Ac\overline{c}}(1,2,3)D_{c\overline{c}}(3,\overline{3})\Gamma_{c\overline{c}A}(\overline{3},\overline{2},\overline{1})D_{c\overline{c}}(\overline{2},2)$$

$$-ig_0^2\Gamma^{(0)}_{A\psi\overline{\psi}}(1,2,3)D_{\psi\overline{\psi}}(3,\overline{3})\Gamma_{\psi\overline{\psi}A}(\overline{3},\overline{2},\overline{1})D_{\psi\overline{\psi}}(\overline{2},2).$$

The proper self-energy parts for the Faddeev–Popov ghost field and the fermion field are defined by

$$\Sigma_{c\overline{c}}(1,\overline{1})=ig_0^2\Gamma^{(0)}_{c\overline{c}A}(1,2,3)D_{c\overline{c}}(2,\overline{2})\Gamma_{Ac\overline{c}}(\overline{3},\overline{2},\overline{1})D_{A^2}(\overline{3},3),$$

$$\Sigma_{\psi\overline{\psi}}(1,\overline{1})=ig_0^2\Gamma^{(0)}_{\psi\overline{\psi}A}(1,2,3)D_{\psi\overline{\psi}}(2,\overline{2})\Gamma_{A\psi\overline{\psi}}(\overline{3},\overline{2},\overline{1})D_{A^2}(\overline{3},3).$$

The external hook for the Faddeev–Popov ghost fields is switched off here.

We renormalize the physical quantities by

$$\Gamma^{\mathrm{R}}=Z_1\Gamma,\quad D^{\mathrm{R}}_{c\overline{c}}=(Z_2^{c\overline{c}})^{-1}D_{\psi\overline{\psi}},\quad D^{\mathrm{R}}_{\psi\overline{\psi}}=(Z_2^{\psi\overline{\psi}})^{-1}D_{\psi\overline{\psi}},\quad D^{\mathrm{R}}_{A^2}=Z_3^{-1}D_{A^2}. \tag{10.14.8}$$

We express Eqs. (10.14.7a)–(10.14.7c) in terms of the renormalized quantities as

$$[D^{\mathrm{R}}_{A^2}]^{-1}(1,2)=Z_3^{A^2}[D_{A^2}]_0^{-1}(1,2)-(ig)Z_1^{A^3}\Gamma^{(0)}_{A^3}(1,2,3)\left\langle A^{\mathrm{R}}(3)\right\rangle$$
$$-\frac{(ig)}{2}Z_1^{A^4}\Gamma^{(0)}_{A^4}(1,2,3,4)\left\langle A^{\mathrm{R}}(3)\right\rangle\left\langle A^{\mathrm{R}}(4)\right\rangle-\Pi'_{A^2}(1,2), \tag{10.14.9a}$$

$$[D^{\mathrm{R}}_{c\overline{c}}]^{-1}(1,2)=Z_2^{c\overline{c}}[D_{c\overline{c}}]_0^{-1}(1,2)-\Sigma'_{c\overline{c}}(1,2), \tag{10.14.9b}$$

$$[D^{\mathrm{R}}_{\psi\overline{\psi}}]^{-1}(1,2)=Z_2^{\psi\overline{\psi}}[D_{\psi\overline{\psi}}]_0^{-1}(1,2)-\Sigma'_{\psi\overline{\psi}}(1,2). \tag{10.14.9c}$$

The primes indicate the proper self-energy parts after the renormalization and we made use of Ward–Takahashi–Slavnov–Taylor identities,

$$\begin{aligned} Z_1^{A^3}/Z_3^{A^2} &= Z_1^{Ac\overline{c}}/Z_2^{c\overline{c}},\\ Z_1^{A^4} &= (Z_1^{A^3})^2/Z_3^{A^2},\\ Z_1^{A^3}/Z_3^{A^2} &= Z_1^{A\psi\overline{\psi}}/Z_2^{\psi\overline{\psi}}, \end{aligned} \tag{10.14.10}$$

$$g^2=g_0^2Z_1^{A^3}(Z_3^{A^2})^3. \tag{10.14.11}$$

From the elimination of the ultraviolet divergences in Green's functions, we have

$$\begin{aligned} Z_3^{A^2} &= 1 + \partial \Pi'_{A^2}(k_0^2) / \partial k_0^2, \\ Z_2^{c\bar{c}} &= 1 + \partial \Sigma'_{c\bar{c}}(k_0^2) / \partial k_0^2, \\ Z_2^{\psi\overline{\psi}} &= 1 + \partial \Sigma'_{\psi\overline{\psi}}(k_0^2) / \partial k_0^2. \end{aligned} \tag{10.14.12}$$

From the elimination of the ultraviolet divergences in the vertex operators, we determine Z_1-factors.

The renormalized Schwinger–Dyson equations are given by

$$[D^{\mathrm{R}}_{A^2}]^{-1}(1,2) = [D_{A^2}]_0^{-1}(1,2) - (ig) Z_1^{A^3} \Gamma^{(0)}_{A^3}(1,2,3) \left\langle A^{\mathrm{R}}(3) \right\rangle - \frac{(ig)}{2} Z_1^{A^4} \Gamma^{(0)}_{A^4}(1,2,3,4) \left\langle A^{\mathrm{R}}(3) \right\rangle \left\langle A^{\mathrm{R}}(4) \right\rangle - \tilde{\Pi}^{\mathrm{R}}_{A^2}(1,2), \tag{10.14.13a}$$

$$[D^{\mathrm{R}}_{c\bar{c}}]^{-1}(1,2) = [D_{c\bar{c}}]_0^{-1}(1,2) - \tilde{\Sigma}^{\mathrm{R}}_{c\bar{c}}(1,2), \tag{10.14.13b}$$

$$[D^{\mathrm{R}}_{\psi\overline{\psi}}]^{-1}(1,2) = [D_{\psi\overline{\psi}}]_0^{-1}(1,2) - \tilde{\Sigma}^{\mathrm{R}}_{\psi\overline{\psi}}(1,2), \tag{10.14.13c}$$

where the proper self-energy parts in the above equations

$$\begin{aligned} \tilde{\Pi}^{\mathrm{R}}_{A^2}(k^2) &= \tilde{\Pi}'_{A^2}(k^2) - k^2 \partial \tilde{\Pi}'_{A^2}(k_0^2) / \partial k_0^2, \\ \tilde{\Sigma}^{\mathrm{R}}_{c\bar{c}}(k^2) &= \tilde{\Sigma}'_{c\bar{c}}(k^2) - k^2 \partial \tilde{\Sigma}'_{c\bar{c}}(k_0^2) / \partial k_0^2, \\ \tilde{\Sigma}^{\mathrm{R}}_{\psi\overline{\psi}}(k^2) &= \tilde{\Sigma}'_{\psi\overline{\psi}}(k^2) - k^2 \partial \tilde{\Sigma}'_{\psi\overline{\psi}}(k_0^2) / \partial k_0^2, \end{aligned} \tag{10.14.14}$$

now do not contain the ultraviolet divergences. Here we note that k_0^2 is the normalization point and that $\tilde{\Pi}'_{A^2}$, $\tilde{\Sigma}'_{c\bar{c}}$, and $\tilde{\Sigma}'_{\psi\overline{\psi}}$ are those parts of the proper self-energy parts which are subject to the renormalization.

From the consideration of the overlapping divergences, we obtain

$$\Gamma^{\mathrm{R}}_{c\bar{c}A}(1,2,3) = Z_1^{c\bar{c}A} \Gamma^{(0)}_{c\bar{c}A}(1,2,3) + ig^2 \Gamma^{\mathrm{R}}_{c\bar{c}A}(1,\overline{2},\overline{3}) D^{\mathrm{R}}_{c\bar{c}}(\overline{2},\overline{1}) P_{c\bar{c}A^2}(\overline{1},2,3,\overline{4}) D^{\mathrm{R}}_{A^2}(\overline{4},\overline{3}), \tag{10.14.15}$$

$$P_{c\bar{c}A^2}(1,2,3,4) = W_{c\bar{c}A^2}(1,2,3,4) - ig^2 W_{c\bar{c}A^2}(1,\overline{2},\overline{3},4) D^{\mathrm{R}}_{A^2}(\overline{4},\overline{3}) D^{\mathrm{R}}_{c\bar{c}}(\overline{2},\overline{1}) P_{c\bar{c}A^2}(\overline{1},2,3,\overline{4}), \tag{10.14.16}$$

$$W_{c\bar{c}A^2}(1,2,3,4) = \frac{\delta \Gamma^{\mathrm{R}}_{c\bar{c}A}(1,2,4)}{(ig)\delta \left\langle A^{\mathrm{R}}(3) \right\rangle} + \Gamma^{\mathrm{R}}_{c\bar{c}A}(1,\overline{2},3) D^{\mathrm{R}}_{c\bar{c}}(\overline{2},\overline{1}) \Gamma^{\mathrm{R}}_{c\bar{c}A}(\overline{1},2,4) + \Gamma^{\mathrm{R}}_{c\bar{c}A}(1,2,\overline{3}) D^{\mathrm{R}}_{A^2}(\overline{3},\overline{2}) \Gamma^{\mathrm{R}}_{c\bar{c}A}(\overline{2},3,4), \tag{10.14.17}$$

$$\Gamma^{\mathrm{R}}_{\psi\overline{\psi}A}(1,2,3) = Z_1^{\psi\overline{\psi}A} \Gamma^{(0)}_{\psi\overline{\psi}A}(1,2,3) + ig^2 \Gamma^{\mathrm{R}}_{\psi\overline{\psi}A}(1,\overline{2},\overline{3}) D^{\mathrm{R}}_{\psi\overline{\psi}}(\overline{2},\overline{1}) P_{\psi\overline{\psi}A^2}(\overline{1},2,3,\overline{4}) D^{\mathrm{R}}_{A^2}(\overline{4},\overline{3}), \tag{10.14.18}$$

$$P_{\psi\overline{\psi}A^2}(1,2,3,4) = W_{\psi\overline{\psi}A^2}(1,2,3,4) \\ - ig^2 W_{\psi\overline{\psi}A^2}(1,\overline{2},\overline{3},4) D^{\mathrm{R}}_{A^2}(\overline{4},\overline{3}) D^{\mathrm{R}}_{\psi\overline{\psi}}(\overline{2},\overline{1}) P_{\psi\overline{\psi}A^2}(\overline{1},2,3,\overline{4}), \tag{10.14.19}$$

$$W_{\psi\overline{\psi}A^2}(1,2,3,4) = \frac{\delta\Gamma^{\mathrm{R}}_{\psi\overline{\psi}A}(1,2,4)}{(ig)\delta\left\langle A^{\mathrm{R}}(3)\right\rangle} + \Gamma^{\mathrm{R}}_{\psi\overline{\psi}A}(1,\overline{2},3) D^{\mathrm{R}}_{\psi\overline{\psi}}(\overline{2},\overline{1}) \Gamma^{\mathrm{R}}_{\psi\overline{\psi}A}(\overline{1},2,4) \\ + \Gamma^{\mathrm{R}}_{\psi\overline{\psi}A}(1,2,\overline{3}) D^{\mathrm{R}}_{A^2}(\overline{3},\overline{2}) \Gamma^{\mathrm{R}}_{A^3}(\overline{2},3,4). \tag{10.14.20}$$

We finally consider the asymptotic behavior of Green's functions and the vertex operators. We shall consider only the Faddeev–Popov ghost vertex operator

$$\Gamma^{\mathrm{R}}_{c\overline{c}A}(1,2,3) = Z_1^{c\overline{c}A}\Gamma^{(0)}_{c\overline{c}A}(1,2,3) \\ +ig^2\Gamma^{\mathrm{R}}_{c\overline{c}A}(1,\overline{2},\overline{3}) D^{\mathrm{R}}_{c\overline{c}}(\overline{2},\overline{1}) P_{c\overline{c}A^2}\left(\overline{1},2,3,\overline{4}\right) D^{\mathrm{R}}_{A^2}\left(\overline{4},\overline{3}\right), \tag{10.14.15}$$

and the set of equations for the derivatives of Green's functions,

$$\begin{aligned} d[D^{\mathrm{R}}_{A^2}]^{-1}(p^2)/dp^2 &= 1 - d\Pi^{\mathrm{R}}_{A^2}(p^2)/dp^2, \\ d[D^{\mathrm{R}}_{c\overline{c}}]^{-1}(p^2)/dp^2 &= 1 - d\Sigma^{\mathrm{R}}_{c\overline{c}}(p^2)/dp^2, \\ d[D^{\mathrm{R}}_{\psi\overline{\psi}}]^{-1}(p^2)/dp^2 &= 1 - d\Sigma^{\mathrm{R}}_{\psi\overline{\psi}}(p^2)/dp^2. \end{aligned} \tag{10.14.21}$$

By tri-Γ approximation, we mean to replace

$$\begin{aligned} Z_1^{\psi\overline{\psi}A}\Gamma^{(0)}_{\psi\overline{\psi}A} &\quad \text{with} \quad \Gamma^{\mathrm{R}}_{\psi\overline{\psi}A}, \\ Z_1^{c\overline{c}A}\Gamma^{(0)}_{c\overline{c}A} &\quad \text{with} \quad \Gamma^{\mathrm{R}}_{c\overline{c}A}, \\ Z_1^{A^3}\Gamma^{(0)}_{A^3} &\quad \text{with} \quad \Gamma^{\mathrm{R}}_{A^3}, \end{aligned} \tag{10.14.22}$$

and to determine the vertex operator of the Faddeev–Popov ghost field by solving simpler equation given by

$$\Gamma^{\mathrm{R}}_{c\overline{c}A}(1,2,3) = Z_1^{c\overline{c}A}\Gamma^{(0)}_{c\overline{c}A}(1,2,3) \\ + ig^2\Gamma^{\mathrm{R}}_{c\overline{c}A}(1,\overline{2},\overline{3}) D^{\mathrm{R}}_{c\overline{c}}(\overline{2},\overline{1}) \Gamma^{\mathrm{R}}_{c\overline{c}A}\left(\overline{1},2,\overline{\overline{\overline{3}}}\right) D^{\mathrm{R}}_{A^2}\left(\overline{\overline{\overline{3}}},\overline{\overline{\overline{2}}}\right) \Gamma^{\mathrm{R}}_{A^3}\left(\overline{\overline{\overline{2}}},3,4\right) D^{\mathrm{R}}_{A^2}(4,\overline{3}) \\ + ig^2\Gamma^{\mathrm{R}}_{c\overline{c}A}(1,\overline{2},\overline{3}) D^{\mathrm{R}}_{c\overline{c}}(\overline{2},\overline{1}) \Gamma^{\mathrm{R}}_{c\overline{c}A}\left(\overline{1},\overline{\overline{\overline{2}}},4\right) D^{\mathrm{R}}_{c\overline{c}}\left(\overline{\overline{\overline{2}}},\overline{\overline{\overline{1}}}\right) \Gamma^{\mathrm{R}}_{c\overline{c}A}\left(\overline{\overline{\overline{1}}},2,3\right) D^{\mathrm{R}}_{A^2}\left(4,\overline{3}\right). \tag{10.14.23}$$

In order to separate the tensor structure of various quantities, we choose the Feynman gauge which leads to the following equations:

$$\begin{aligned} [D^{\mathrm{R}}_{A^2}]^{ab}_{\mu\nu}(p^2) &= \delta^{ab}\eta_{\mu\nu} D^{\mathrm{R}}_{A^2}(p^2), \\ [D^{\mathrm{R}}_{c\overline{c}}]^{ab}(p^2) &= \delta^{ab} D^{\mathrm{R}}_{c\overline{c}}(p^2), \\ [D^{\mathrm{R}}_{\psi\overline{\psi}}]^{ab}_{\alpha\beta}(p^2) &\approx \delta^{ab}(\gamma_\mu)_{\alpha\beta}p^\mu D^{\mathrm{R}}_{\psi\overline{\psi}}(p^2), \end{aligned} \tag{10.14.24}$$

$$
\begin{aligned}
(\Gamma^{\mathrm{R}}_{c\bar{c}A})^{abc}_{\mu}(p+k,p,k) &= f^{abc} p_\mu \Gamma^{\mathrm{R}}_{c\bar{c}A}(p+k,p,k),\\
(\Gamma^{\mathrm{R}}_{A^3})^{abc}_{\kappa\sigma\mu}(p+k,p,k) &= f^{abc}[\eta_{\kappa\sigma}(2p+k)_\mu - \eta_{\kappa\mu}(2k+p)_\sigma\\
&\quad -\eta_{\sigma\mu}(p-k)_\kappa]\Gamma^{\mathrm{R}}_{A^3}(p+k,p,k),\\
(\Gamma^{\mathrm{R}}_{\psi\overline{\psi}A})^{abc}_{\alpha\beta\mu}(p+k,p,k) &= -(t^c)^{ab}(\gamma_\mu)_{\alpha\beta}\Gamma^{\mathrm{R}}_{\psi\overline{\psi}A}(p+k,p,k).
\end{aligned}
\tag{10.14.25}
$$

When these equations are substituted into Eqs. (10.14.21) and (10.14.23), the same tensor structure is reproduced.

We seek the solutions of the following forms:

$$
\begin{aligned}
D^{\mathrm{R}}_{A^2}(p^2) &= d_{A^2}(p^2)/p^2,\\
D^{\mathrm{R}}_{c\bar{c}}(p^2) &= h_{c\bar{c}}(p^2)/p^2,\\
D^{\mathrm{R}}_{\psi\overline{\psi}}(p^2) &= h_{\psi\overline{\psi}}(p^2)/p^2,
\end{aligned}
\tag{10.14.26}
$$

$$
\begin{aligned}
\Gamma^{\mathrm{R}}_{A^3}(p+k,p,k) &\longrightarrow \Gamma_{A^3}(q^2),\\
\Gamma^{\mathrm{R}}_{c\bar{c}A}(p+k,p,k) &\longrightarrow \Gamma_{c\bar{c}A}(q^2),\\
\Gamma^{\mathrm{R}}_{\psi\overline{\psi}A}(p+k,p,k) &\longrightarrow \Gamma_{\psi\overline{\psi}A}(q^2),
\end{aligned}
\tag{10.14.27}
$$

where q^2 is the largest four-momentum squared of the arguments of the $\Gamma^{\mathrm{R}}_{A^3}$, $\Gamma^{\mathrm{R}}_{c\bar{c}A}$ and $\Gamma^{\mathrm{R}}_{\psi\overline{\psi}A}$ functions. We regard the functions $d_{A^2}(p^2)$, $h_{c\bar{c}}(p^2)$, $h_{\psi\overline{\psi}}(p^2)$, $\Gamma_{A^3}(q^2)$, $\Gamma_{c\bar{c}A}(q^2)$, and $\Gamma_{\psi\overline{\psi}A}(q^2)$ as slowly varying functions of p^2 and q^2; namely, the derivative of these functions is close to zero.

Equations (10.14.21) and (10.14.25) take the following form of a set of integral equations:

$$
\begin{aligned}
\frac{1}{d_{A^2}(\varsigma)} &= 1 - \frac{19}{4}\frac{C_2(G)}{3}\frac{g^2}{16\pi^2}\int_\varsigma^0 dz\Gamma^2_{A^3}(z)d^2_{A^2}(z) - \frac{C_2(G)}{12}\frac{g^2}{16\pi^2}\\
&\quad \int_\varsigma^0 dz\Gamma^2_{c\bar{c}A}(z)h^2_{c\bar{c}}(z) + \frac{8T(R)}{6}\frac{g^2}{16\pi^2}\int_\varsigma^0 dz\Gamma^2_{\psi\overline{\psi}A}(z)h^2_{\psi\overline{\psi}}(z),
\end{aligned}
\tag{10.14.28}
$$

$$
\frac{1}{h_{c\bar{c}}(\varsigma)} = 1 - \frac{C_2(G)}{2}\frac{g^2}{16\pi^2}\int_\varsigma^0 dz\Gamma^2_{c\bar{c}A}(z)h_{c\bar{c}}(z)d_{A^2}(z),
\tag{10.14.29}
$$

$$
\frac{1}{h_{\psi\overline{\psi}}(\varsigma)} = 1 - \frac{C_2(G)}{2}\frac{g^2}{16\pi^2}\int_\varsigma^0 dz\Gamma^2_{\psi\overline{\psi}A}(z)h_{\psi\overline{\psi}}(z)d_{A^2}(z),
\tag{10.14.30}
$$

$$
\begin{aligned}
\Gamma_{c\bar{c}A}(\varsigma) &= 1 + \frac{C_2(G)}{8}\frac{g^2}{16\pi^2}\int_\varsigma^0 dz\Gamma^3_{c\bar{c}A}(z)h^2_{c\bar{c}}(z)d_{A^2}(z)\\
&\quad + \frac{3C_2(G)}{8}\frac{g^2}{16\pi^2}\int_\varsigma^0 dz\Gamma^2_{c\bar{c}A}(z)\Gamma_{A^3}(z)d^2_{A^2}(z)h_{c\bar{c}}(z),
\end{aligned}
\tag{10.14.31}
$$

$$
\begin{aligned}
\Gamma_{A^3}(\varsigma)d_{A^2}(\varsigma) &= \Gamma_{c\bar{c}A}(\varsigma)h_{c\bar{c}}(\varsigma),\\
\Gamma_{A^3}(\varsigma)d_{A^2}(\varsigma) &= \Gamma_{\psi\overline{\psi}A}(\varsigma)h_{\psi\overline{\psi}}(\varsigma),
\end{aligned}
\tag{10.14.32}
$$

$$
\varsigma = \ln(p^2/m^2).
\tag{10.14.33}
$$

$C_2(G)$ is the quadratic Casimir operator and $T(R)$ is defined by $\mathrm{Tr}(t^a t^b) = T(R)\delta^{ab}$. Equation (10.14.32) is the direct consequence of the Ward–Takahashi–Slavnov–Taylor identity.

Equations (10.14.28)–(10.14.32) are completely equivalent to the set of differential equations

$$\frac{1}{d_{A^2}}\frac{d}{d\varsigma}(d_{A^2}) = -\frac{19}{4}\frac{C_2(G)}{3}\frac{g^2}{16\pi^2}\Gamma_{A^3}^2 d_{A^2}^3 - \frac{C_2(G)}{12}\frac{g^2}{16\pi^2}\Gamma_{\bar{c}cA}^2 h_{\bar{c}c}^2 d_{A^2} + \frac{8T(R)}{6}\frac{g^2}{16\pi^2}\Gamma_{\psi\overline{\psi}A}^2 h_{\psi\overline{\psi}}^2 d_{A^2}, \tag{10.14.34}$$

$$\frac{1}{\Gamma_{\bar{c}cA}}\frac{d}{d\varsigma}(\Gamma_{\bar{c}cA}) = -\frac{C_2(G)}{8}\frac{g^2}{16\pi^2}\Gamma_{\bar{c}cA}^2 h_{\bar{c}c}^2 d_{A^2} - \frac{3C_2(G)}{8}\frac{g^2}{16\pi^2}\Gamma_{\bar{c}cA}\Gamma_{A^3} d_{A^2}^2 h_{\bar{c}c}, \tag{10.14.35}$$

$$\frac{1}{h_{\bar{c}c}}\frac{d}{d\varsigma}(h_{\bar{c}c}) = -\frac{C_2(G)}{2}\frac{g^2}{16\pi^2}\Gamma_{\bar{c}cA}^2 h_{\bar{c}c}^2 d_{A^2}, \tag{10.14.36}$$

$$\frac{1}{h_{\psi\overline{\psi}}}\frac{d}{d\varsigma}(h_{\psi\overline{\psi}}) = -\frac{C_2(G)}{2}\frac{g^2}{16\pi^2}\Gamma_{\psi\overline{\psi}A}^2 h_{\psi\overline{\psi}}^2 d_{A^2}, \tag{10.14.37}$$

with Ward–Takahashi–Slavnov–Taylor identities,

$$\Gamma_{A^3} d_{A^2} = \Gamma_{\bar{c}cA} h_{\bar{c}c}, \quad \Gamma_{A^3} d_{A^2} = \Gamma_{\psi\overline{\psi}A} h_{\psi\overline{\psi}}, \tag{10.14.38}$$

and the boundary conditions

$$d_{A^2}(0) = h_{\bar{c}c}(0) = h_{\psi\overline{\psi}}(0) = \Gamma_{\bar{c}cA}(0) = 1. \tag{10.14.39}$$

The set of nonlinear differential equations, (10.14.34)–(10.14.38), with the boundary conditions, (10.14.39), can be solved explicitly as

$$\Gamma_{\bar{c}cA} = h_{\bar{c}c} = h_{\psi\overline{\psi}} = \Gamma, \quad d_{A^2} = \Gamma^{\kappa-4}, \quad \Gamma_{A^3} = \Gamma^{6-\kappa}, \tag{10.14.40}$$

where Γ and κ are, respectively, given by

$$\Gamma = \left[1 + \kappa\frac{C_2(G)}{2}\frac{g^2}{16\pi^2}\varsigma\right]^{-1/\kappa} \quad \text{and} \quad \kappa = \frac{22}{3}\left(1 - \frac{4}{11}\frac{T(R)}{C_2(G)}\right). \tag{10.14.41}$$

Here κ is positive since

$$T(R) < \frac{11}{4}C_2(G), \tag{10.14.42}$$

which follows from the group identity,

$$rT(R) = d(R)C_2(G), \tag{10.14.43}$$

where $d(R)$ and r are the dimensionalities of the representation R and the group G.

The connection between the bare or running coupling constant (or the bare charge) and the observed coupling constant (or the observed charge) in this theory takes the following form:

$$g_0^2(\Lambda^2) = \frac{g^2}{1 + \kappa(C_2(G)/2)(g^2/16\pi^2)\ln(\Lambda^2/m^2)}, \tag{10.14.44}$$

which differs from the corresponding expression for the Abelian theories like *QED* by the opposite sign in the denominator. The bare or running coupling constant tends to zero as the cut-off parameter Λ^2 goes to infinity. Thus, at high energies or at short distances, the theory becomes asymptotically free. Equation (10.14.44) can be solved for the observed coupling constant (or the observed charge) as

$$g^2 = \frac{g_0^2(\Lambda^2)}{1 - \kappa(C_2(G)/2)(g_0^2(\Lambda^2)/16\pi^2)\ln(\Lambda^2/m^2)}. \tag{10.14.45}$$

The observed coupling constant (or the observed charge) can assume any value if, in the local limit $p^2 = \Lambda^2 \to \infty$, the bare or running coupling constant (or the bare charge) also tends to zero.

10.15 Renormalization Group Equations

As the renormalization group equation, we have Gell–Mann–Low equation, which originates from the perturbative calculation of the massless *QED* with the use of the mathematical theory of the regular variations . We also have Callan–Symanzik equation,which is slightly different from the former. The relationship between the two approaches is established in this section. We remark that the method of the renormalization group essentially consists of separating out the field components into the rapidly varying components ($k^2 > \Lambda^2$) and the slowly varying components ($k^2 < \Lambda^2$), path-integrating out the rapidly varying components ($k^2 > \Lambda^2$) in the generating functional of the (connected parts of) Green's functions, and focusing our attention to the slowly varying components ($k^2 < \Lambda^2$) to analyze the low energy phenomena at $k^2 < \Lambda^2$.

We first consider the renormalization in the *path integral formalism*, and take the classical self-interacting scalar field as the model

$$\mathcal{L}\left(\phi(x), \partial_\mu\phi(x)\right) = \frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - \frac{1}{2}m^2\phi^2(x) - \frac{g}{4!}\phi^4(x).$$

We carry out the Fourier transformation of the self-interacting scalar field $\phi(x)$ as

$$\phi(x) = \int \frac{d^4k}{(2\pi)^4}\exp[ikx]\tilde{\phi}(k), \quad \text{with} \quad \tilde{\phi}^*(k) = \tilde{\phi}(-k).$$

The action functional is written as

$$I\left[\tilde{\phi}\right] = \int_{|k|<\Lambda} \frac{d^4k}{(2\pi)^4} \left\{ \left(\frac{1}{2}k^2 - m^2 \right) \tilde{\phi}(k)\tilde{\phi}(-k) \right\} + I_{\text{int}}\left[\tilde{\phi}\right],$$

where $I_{\text{int}}[\tilde{\phi}]$ is given by

$$I_{\text{int}}\left[\tilde{\phi}\right] = -\frac{g}{4!}\int_{|k|<\Lambda} \frac{d^4k}{(2\pi)^4}\delta^4(k_1+k_2+k_3+k_4)\tilde{\phi}(k_1)\tilde{\phi}(k_2)\tilde{\phi}(k_3)\tilde{\phi}(k_4).$$

The generating functional of Green's function is given by

$$Z[J] = N\int \mathcal{D}\left[\tilde{\phi}\right] \exp\left[i\int d^4x \left\{ \mathcal{L}\left(\tilde{\phi}(k), ik_\mu\tilde{\phi}(k)\right) + \tilde{J}(k)\tilde{\phi}(k) + \text{“}i\varepsilon\text{-piece”} \right\} \right].$$

Here we have

$$\int \mathcal{D}[\tilde{\phi}] = \prod_{|k|<\Lambda} \int \mathcal{D}\tilde{\phi}(k)\mathcal{D}\tilde{\phi}^*(k).$$

With the scale transformations

$$p = \frac{k}{\Lambda}, \quad \varphi(p) = \Lambda\tilde{\phi}(k), \quad \tilde{m} = \Lambda^{-1}m,$$

the generating functional of Green's function is now given by

$$Z[J] = N\int \mathcal{D}[\varphi] \exp\left[i\int \frac{d^4p}{(2\pi)^4} \left\{ \mathcal{L}(\varphi, i\Lambda p_\mu\varphi) + J\varphi +'' i\varepsilon\text{-piece}'' \right\} \right],$$

with

$$\int \mathcal{D}[\varphi] = \prod_{|p|<1} \int \mathcal{D}\varphi(p)\mathcal{D}\varphi^*(p).$$

The action functional is now given by

$$I[\varphi] = \int_{|p|<1} \frac{d^4p}{(2\pi)^4} \left\{ \left(\frac{1}{2}p^2 - \tilde{m}^2 \right) \varphi(p)\varphi(-p) \right\} + I_{\text{int}}[\varphi],$$

with the interaction term given by

$$I_{\text{int}}[\varphi] = -\frac{g}{4!}\int_{|p|<1} \frac{d^4p}{(2\pi)^4}\delta^4(p_1+p_2+p_3+p_4)\varphi(p_1)\varphi(p_2)\varphi(p_3)\varphi(p_4).$$

We now separate the field variable into the slowly varying component and the rapidly varying component as

$$\varphi(p) = \varphi_{\text{slow}}(p) + \varphi_{\text{rapid}}(p)$$

with

$$\begin{aligned} \varphi_{\text{slow}}(p) &= 0 \quad \text{unless} \quad |p| < 1/b, \\ \varphi_{\text{rapid}}(p) &= 0 \quad \text{unless} \quad (1/b) \leq |p| \leq 1, \end{aligned}$$

$$b > 1.$$

The generating functional of Green's function is rewritten as

$$Z[J] = N \int \mathcal{D}[\varphi_{\text{slow}}]\mathcal{D}\left[\varphi_{\text{ rapid}}\right] \exp\left[iI[\varphi_{\text{slow}}(p) + \varphi_{\text{rapid}}(p)] + \text{``}i\varepsilon\text{-piece''}\right],$$

and we perform the path integration with respect to φ_{rapid}. We define the new action functional $\tilde{I}[\varphi_{\text{slow}}]$, which depends only on φ_{slow} by

$$\exp\left[i\tilde{I}[\varphi_{\text{slow}}]\right] = \int \mathcal{D}\left[\varphi_{\text{ rapid}}\right] \exp\left[iI\left[\varphi_{\text{slow}}(p) + \varphi_{\text{rapid}}(p)\right]\right].$$

We expand the new action functional $\tilde{I}[\varphi_{\text{slow}}]$ in the form

$$\tilde{I}[\varphi_{\text{slow}}] = \int_{|p|<1/b} \frac{d^4p}{(2\pi)^4} \left\{\frac{1}{2}zp^2 - r_1\right\} \varphi_{\text{slow}}(p)\varphi_{\text{slow}}(-p) + I_{\text{int}}[\varphi_{\text{slow}}],$$

which defines z, r_1, and the new parameters in $I_{\text{int}}[\varphi_{\text{slow}}]$. We set $z = b^{-\gamma}$, where we call γ the anomalous mass dimension. The generating functional of Green's function is now written as

$$Z[J] = N \int \mathcal{D}[\varphi_{\text{slow}}] \exp\left[i\tilde{I}[\varphi_{\text{slow}}] + \text{``}i\varepsilon\text{-piece''}\}\right].$$

We rescale the cutoff back to 1 by defining

$$p' = bp, \quad \varphi'(p') = b^{-1-\gamma/2}\varphi_{\text{slow}}(p'/b).$$

The action functional can be written as

$$I'[\varphi'] \equiv \tilde{I}[\varphi_{\text{slow}}] = \int_{|p|<1} \frac{d^4p}{(2\pi)^4} \left\{\frac{1}{2}p^2 - r'\right\} \varphi'(p)\varphi'(-p) + I'_{\text{int}}\left[\varphi'\right],$$

where $r' = b^{2+\gamma} r_1$.

We compute the N-point Green's function in the momentum space as

$$G_N(p_1,\dots,p_N;\Lambda,g_0) = \frac{1}{i}\frac{\delta}{\delta J(p_1)}\cdots\frac{1}{i}\frac{\delta}{\delta J(p_N)}Z[J]$$

$$= \frac{\int \mathcal{D}[\tilde{\phi}]\left\{\tilde{\phi}(p_1)\cdots\tilde{\phi}(p_N)\right\}\exp\left[iI[\tilde{\phi}] + \text{“}i\varepsilon\text{-piece”}\right]}{\int \mathcal{D}[\tilde{\phi}]\exp[iI[\tilde{\phi}] + \text{“}i\varepsilon\text{-piece”}]}.$$

Here

$$|p_i| < \Lambda$$

and g_0 represents the bare coupling constant.

In terms of the dimensionless field,

$$\varphi(p/\Lambda) = \Lambda\tilde{\phi}(p)$$

and the dimensionless bare coupling constant g_0, we can write

$$G_N(p_1,\dots,p_N;\Lambda,g_0) = \Lambda^{-N}\mathcal{G}_N\left(\frac{p_1}{\Lambda},\dots,\frac{p_N}{\Lambda};g_0\right),$$

where

$$\mathcal{G}_N(p_1,\dots,p_N;g_0) = \frac{\int \mathcal{D}[\varphi]\left\{\varphi(p_1)\cdots\varphi(p_N)\right\}\exp\left[iI[\varphi] + \text{“}i\varepsilon\text{-piece”}\right]}{\int \mathcal{D}[\varphi]\exp\left[iI[\varphi] + \text{“}i\varepsilon\text{-piece”}\right]}.$$

We are interested only in the N-point Green's function of the slowly varying fields with

$$|p_i| < 1/b.$$

Setting $\varphi(p_i) = \varphi_{\text{slow}}(p_i)$, we have

$$\mathcal{G}_N(p_1,\dots,p_N;g_0)$$

$$= \frac{\int \mathcal{D}[\varphi_{\text{slow}}]\int \mathcal{D}[\varphi_{\text{rapid}}]\left\{\varphi(p_1)\cdots\varphi(p_N)\right\}\exp\left[iI[\varphi_{\text{slow}}+\varphi_{\text{rapid}}] + \text{“}i\varepsilon\text{-piece”}\right]}{\int \mathcal{D}[\varphi_{\text{slow}}]\int \mathcal{D}[\varphi_{\text{rapid}}]\exp\left[iI[\varphi_{\text{slow}}+\varphi_{\text{rapid}}] + \text{“}i\varepsilon\text{-piece”}\right]}$$

$$= \frac{\int \mathcal{D}[\varphi_{\text{slow}}]\left\{\varphi_{\text{slow}}(p_1)\cdots\varphi_{\text{slow}}(p_N)\right\}\exp\left[i\tilde{I}[\varphi_{\text{slow}}] + \text{“}i\varepsilon\text{-piece”}\right]}{\int \mathcal{D}[\varphi_{\text{slow}}]\exp\left[i\tilde{I}[\varphi_{\text{slow}}] + \text{“}i\varepsilon\text{-piece”}\right]}$$

$$= z_0^{-N/2} b^N \frac{\int \mathcal{D}[\varphi'] \left\{\varphi'(bp_1)\cdots\varphi'(bp_N)\right\} \exp\left[iI'[\varphi'] + \text{“}i\varepsilon\text{-piece”}\right]}{\int \mathcal{D}[\varphi'] \exp\left[iI'[\varphi'] + \text{“}i\varepsilon\text{-piece”}\right]}$$

$$= z_0^{-N/2} b^N \mathcal{G}_N(bp_1, \ldots, bp_N; g').$$

To take contact with the perturbative renormalization, we set

$$b = \Lambda/\mu,$$

where μ is the normalization point. Multiplying the above equation by $\Lambda^{-N} = (b\mu)^{-N}$, we obtain

$$G_N(p_1, \ldots, p_N; \Lambda, g_0) = z_0^{-N/2} \mu^{-N} \mathcal{G}_N\left(\frac{p_1}{\mu}, \ldots, \frac{p_N}{\mu}; g'\right).$$

In the perturbative renormalization, the above equation is usually written as

$$G_N(p_1, \ldots, p_N; \Lambda, g_0) = \left[z_0\left(\frac{\Lambda}{\mu}, g_0\right)\right]^{-N/2} G'_N(p_1, \ldots, p_N; \mu, g),$$

where g is the renormalized coupling constant and $G'_N(p_1, \ldots, p_N; \mu, g)$ is the renormalized N-point Green's function. We should observe that the cutoff dependence is isolated in the factor $z_0(\Lambda/\mu, g_0)$.

Renormalization Group of Gell–Mann–Low

We briefly outline the Gell–Mann–Low approach to the renormalization group. We consider the quartic self-interacting neutral scalar field theory as a model,

$$\mathcal{L}\left(\hat{\phi}(x), \partial_\mu \hat{\phi}(x)\right) = -\frac{1}{2}\partial_\mu \hat{\phi}(x)\partial^\mu \hat{\phi}(x) - \frac{1}{2}m^2 \hat{\phi}^2(x) - \frac{g}{4!}\hat{\phi}^4(x).$$

In order to fix the normalization of Green's functions, we give a set of prescriptions. We introduce a parameter μ of the dimension of mass and properly normalize the two-point Green's function at $p^2 = \mu^2$. The coupling constant $g(\mu)$ is introduced as the value of the one-particle-irreducible (1PI) four-point Green's function for $p_i p_j = (\mu^2/3)(4\delta_{ij} - 1)$, $i, j = 1, 2, 3, 4$. The 1PI Green's functions are defined as those Green's functions which cannot be made disjoint by cutting one internal line.

The N-point Green's function is denoted by

$$G^{(N)}\left(p_i; \mu, g(\mu)\right), \quad i = 1, 2, \ldots, N. \tag{10.15.1}$$

When the parameter μ is equal to the physical mass m_{p}, we have the standard mass-shell renormalization and we define

$$g_{\mathrm{p}} = g(m_{\mathrm{p}}). \tag{10.15.2}$$

The scale transformation argument yields the following relationship:

$$G^{(N)}(p_i; m_{\mathrm{p}}, g_{\mathrm{p}}) = Z(\mu)^{-N/2} G^{(N)}\left(p_i; \mu, g(\mu)\right). \tag{10.15.3}$$

The renormalization group is defined as the group of scale transformations of the parameter μ. Equation (10.15.3) shows the characteristic feature of the renormalization group that the change of the scale of μ induces the change of the coupling constant and the change of the overall normalization of Green's functions.

In order to express the scale change of μ, we introduce a new parameter ρ by

$$\rho = \ln\left(\frac{\mu}{m_{\mathrm{p}}}\right), \tag{10.15.4}$$

and rewrite Eq. (10.15.3) as

$$G^{(N)}(p_i; m_{\mathrm{p}}, g_{\mathrm{p}}) = Z(\rho)^{-N/2} G^{(N)}\left(p_i; \mu(\rho), g(\rho)\right). \tag{10.15.5}$$

The left-hand side of Eq. (10.15.5) is independent of ρ so that the derivative of the right-hand side of Eq. (10.15.5) with respect to ρ must vanish. As a function of ρ, we observe that $\mu(\rho)$, $g(\rho)$, and $Z(\rho)$ satisfy the boundary conditions

$$\mu(0) = m_{\mathrm{p}}, \quad g(0) = g_{\mathrm{p}}, \quad Z(0) = 1. \tag{10.15.6}$$

Now $g(\rho)$ is a function of ρ and g_{p}, so that its derivative with respect to ρ is also a function of ρ and g_{p}. Since g_{p} can be expressed in terms of ρ and g, $\partial g/\partial\rho$ can be regarded as a function of $\mu/m_{\mathrm{p}} (= \exp[\rho])$ and g,

$$\frac{\partial}{\partial\rho} g = \beta\left(g, \frac{\mu}{m_{\mathrm{p}}}\right). \tag{10.15.7}$$

Similarly, we have

$$\frac{\partial}{\partial\rho} \ln Z(\rho)^{-1} = \gamma_{\phi}\left(g, \frac{\mu}{m_{\mathrm{p}}}\right). \tag{10.15.8}$$

The vanishing of the derivative of Eq. (10.15.5) with respect to ρ is expressed as

$$\left(\mu\frac{\partial}{\partial\mu} + \beta\left(g, \frac{\mu}{m_{\mathrm{p}}}\right)\frac{\partial}{\partial g} + \frac{1}{2}N\gamma_{\phi}\left(g, \frac{\mu}{m_{\mathrm{p}}}\right)\right) G^{(N)}\left(p_i; \mu, g\right) = 0. \tag{10.15.9}$$

We define a differential operator $\mathcal{D}$ by

$$\mathcal{D} = \mu \frac{\partial}{\partial \mu} + \beta \left(g, \frac{\mu}{m_{\mathrm{p}}} \right) \frac{\partial}{\partial g}. \tag{10.15.10}$$

With $\delta\rho$ as the infinitesimal parameter of the renormalization group, the infinitesimal change of Q, a function of μ and g, under the renormalization group, is given by

$$\delta Q = (\mathcal{D}Q)\delta\rho. \tag{10.15.11}$$

Renormalization Group of Callan–Symanzik

In the Gell–Mann–Low equation briefly discussed above, the physical mass and the physical coupling constant, m_{p} and g_{p}, are held fixed and only the normalization point μ is varied. On the contrary, both the physical mass and the physical coupling constant are varied in the Callan–Symanzik equation. For the reason to be discussed later, we simply denote them by m and g, respectively.

The Callan–Symanzik equationfor the N-point Green's function is given by

$$\left(m \frac{\partial}{\partial m} + \beta(g) \frac{\partial}{\partial g} + \frac{1}{2} N \gamma_{\phi}(g) \right) G^{(N)}(p_i; m, g) = \Delta G^{(N)}(p_i; m, g), \tag{10.15.12}$$

where the right-hand side is the mass-insertion term, and for a model theory under consideration, it is given by

$$\Delta G^{(N)}(p_i; m, g) = \left(-\delta(g) m^2 \frac{\partial}{\partial m^2} \right) G^{(N)}(p_i; m, g),$$

$$\beta(g) = dg(\rho)/d\rho, \quad \gamma_{\phi}(g) = d \ln Z_{\phi}^{-1}(\rho)/d\rho, \quad \delta(g) = dm(\rho)/d\rho.$$

Callan–Symanzik equationsfor the j-fold mass insertion terms in the $\hat{\phi}^4(x)$ theory are given by

$$\left(m \frac{\partial}{\partial m} + \beta(g) \frac{\partial}{\partial g} + \frac{1}{2} N \gamma_{\phi}(g) - j\left(2 - \gamma_{\mathrm{S}}(g)\right) \right) \Delta^j G^{(N)}(p_i; m, g)$$
$$= \Delta^{j+1} G^{(N)}(p_i; m, g). \tag{10.15.13}$$

Here

$$m_0^2 \phi_0^2(x) = Z_{\mathrm{S}} m^2 \phi^2(x), \quad Z m_0 \frac{\partial m}{\partial m_0} = m, \quad \gamma_{\mathrm{S}}(g) = Z m_0 \frac{\partial \ln Z_{\mathrm{S}}}{\partial m_0}.$$

We call Eq. (10.15.13) the generalized Callan–Symanzik equation. In order to cast the generalized Callan–Symanzik equation, (10.15.13), in a homogeneous form, we introduce the generating function of the j-fold mass-insertion terms by

$$G^{(N)}(p_i;\, m, g, K) = \sum_{j=0}^{\infty} \frac{K^j}{j!} \Delta^j G^{(N)}(p_i;\, m, g). \tag{10.15.14}$$

This generating function satisfies a homogeneous equation

$$\left(m\frac{\partial}{\partial m} + \beta(g)\frac{\partial}{\partial g} + \frac{1}{2}N\gamma_{\phi}(g) - \left\{1 + (1-\gamma_{S}(g))K\right\} \frac{\partial}{\partial K} \right) G^{(N)}(p_i;\, m, g, K) = 0. \tag{10.15.15}$$

Although this differential equation is homogeneous, we have introduced three parameters, m, g, and K, to be compared with the two parameters, μ and g, in the Gell–Mann–Low equation, (10.15.9). Next we define the differential operator $\tilde{\mathcal{D}}$ by

$$\tilde{\mathcal{D}} = m\frac{\partial}{\partial m} + \beta(g)\frac{\partial}{\partial g} - \left\{(2+\delta(g))K + 1\right\} \frac{\partial}{\partial K}. \tag{10.15.16}$$

The infinitesimal change of Q, a function of m, g and K, under the Callan–Symanzik type renormalization group is given by

$$\delta Q = (\tilde{\mathcal{D}}Q)\delta\rho. \tag{10.15.17}$$

For example, we have

$$\delta m = m\delta\rho, \quad \delta g = \beta(g)\delta\rho, \quad \delta K = -\left\{\left(2+\delta(g)\right) K + 1\right\} \delta\rho. \tag{10.15.18}$$

The solutions of Eq. (10.15.18) with the initial conditions,

$$\tilde{g}(0) = g, \quad \tilde{K}(0) = K, \tag{10.15.19}$$

are denoted by $m\exp[\rho]$, $\tilde{g}(\rho)$, and $\tilde{K}(\rho)$.

The equation corresponding to Eq. (10.15.3) in the present approach is given by

$$G^{(N)}(p_i;\, m, g, K) = \exp\left[\frac{N}{2}\int_0^{\rho} \gamma_{\phi}\left(\tilde{g}(\rho')\right) d\rho'\right] G^{(N)}\left(p_i;\, m\exp[\rho], \tilde{g}(\rho), \tilde{K}(\rho)\right). \tag{10.15.20}$$

We shall choose a special value of ρ, denoted by $\tilde{\rho}$, such that

$$\tilde{K}(\tilde{\rho}) = 0. \tag{10.15.21}$$

Then this $\tilde{\rho}$ is a function of g and K, and the two-point Green's function for this choice of ρ is given by $G^{(2)}(p_i;\, m\exp[\tilde{\rho}], \tilde{g}(\tilde{\rho}), 0)$, which has a pole at

$$p^2 = m^2 \exp[2\tilde{\rho}]. \tag{10.15.22}$$

The single particle mass, $m\exp[\tilde{\rho}]$, is a function of m, g, and K. From now on, we shall consider that

$$m_{\mathrm{p}} = m\exp[\tilde{\rho}] \tag{10.15.23}$$

denotes the fixed physical mass, and m denotes a movable normalization point. In fact, we have

$$\tilde{\mathcal{D}}m_{\mathrm{p}} = 0, \tag{10.15.24}$$

which is a consequence of the fact that m_{p} is the pole of the two-point Green's function , $G^{(2)}(p_i;\, m\exp[\tilde{\rho}], \tilde{g}(\tilde{\rho}), 0)$. Substituting Eq. (10.15.23) into Eq. (10.15.24), we find

$$\tilde{\mathcal{D}}\tilde{\rho} = -1. \tag{10.15.25}$$

The function $\tilde{\rho}$ is alternatively fixed by Eq. (10.15.25) with the boundary condition, $\tilde{\rho} = 0$ for $K = 0$. Integrating the second equation of Eq. (10.15.18), $\delta g = \beta(g)\delta\rho$, with the initial condition

$$\tilde{g}(0) = g, \quad \tilde{K}(0) = K, \tag{10.15.19}$$

we obtain

$$\tilde{\rho} = \int_{g}^{\tilde{g}(\tilde{\rho})} \frac{dx}{\beta(x)}. \tag{10.15.26}$$

Applying $\tilde{\mathcal{D}}$ on the above, and using Eqs. (10.15.25), (10.15.17), and (10.15.18), we obtain

$$-1 = \tilde{\mathcal{D}}\tilde{\rho} = \frac{1}{\beta(\tilde{g}(\tilde{\rho}))}\tilde{\mathcal{D}}\tilde{g}(\tilde{\rho}) - \frac{1}{\beta(g)}\tilde{\mathcal{D}}g = \frac{1}{\beta(\tilde{g}(\tilde{\rho}))}\tilde{\mathcal{D}}\tilde{g}(\tilde{\rho}) - 1.$$

Hence we have

$$\tilde{\mathcal{D}}\tilde{g}(\tilde{\rho}) = 0. \tag{10.15.27}$$

Namely, the on-shell coupling constant, $g_{\mathrm{p}} = \tilde{g}(\tilde{\rho})$, is also an invariant of the Callan–Symanzik type renormalization group.

Next we shall regard K as a function of g and $\tilde{\rho} = \ln[m_{\mathrm{p}}/m]$, or as a function of g and m/m_{p}. Then Green's functions can be expressed in terms of the parameters, p_i, m, g, and m_{p}, instead of p_i, m, g, and K. We write

$$G^{(N)}\left(p_i;\, m, g, K(g, \frac{m}{m_{\mathrm{p}}})\right) \equiv G^{(N)}(p_i;\, m, g, m_{\mathrm{p}}). \tag{10.15.28}$$

Taking the following equation into consideration:

$$\tilde{\mathcal{D}} m_{\mathrm{p}} = 0, \tag{10.15.24}$$

we have the following equation from Eq. (10.15.15):

$$\begin{aligned}
0 &= \left(\tilde{\mathcal{D}} + \frac{1}{2} N \gamma_\phi(g)\right) G^{(N)}(p_i; m, g, m_{\mathrm{p}}) \\
&= \left(\left(\tilde{\mathcal{D}} m\right)\left(\frac{\partial}{\partial m}\right)_{g,m_{\mathrm{p}}} + \left(\tilde{\mathcal{D}} g\right)\left(\frac{\partial}{\partial g}\right)_{m,m_{\mathrm{p}}} + \left(\tilde{\mathcal{D}} m_{\mathrm{p}}\right)\left(\frac{\partial}{\partial m_{\mathrm{p}}}\right)_{g,m} + \frac{1}{2} N \gamma_\phi(g)\right) \\
&\quad \times G^{(N)}(p_i; m, g, m_{\mathrm{p}}) \\
&= \left(m \frac{\partial}{\partial m} + \beta(g) \frac{\partial}{\partial g} + \frac{1}{2} N \gamma_\phi(g)\right) G^{(N)}(p_i; m, g, m_{\mathrm{p}}).
\end{aligned}$$

We have a homogeneous Callan–Symanzik equation involving m and g alone,

$$\left(m \frac{\partial}{\partial m} + \beta(g) \frac{\partial}{\partial g} + \frac{1}{2} N \gamma_\phi(g)\right) G^{(N)}(p_i; m, g, m_{\mathrm{p}}) = 0. \tag{10.15.29}$$

Lastly we give K as a function of g and $\tilde{\rho}$. From Eqs. (10.15.18), (10.15.19), and (10.15.21), we immediately obtain

$$K = \int_0^{\tilde{\rho}} d\rho \exp\left[\int_0^{\rho} d\rho \left(2 + \delta\left(\tilde{g}(\rho)\right)\right)\right], \tag{10.15.30}$$

where $\tilde{g}(\rho)$ is defined by the following equation:

$$\rho = \int_g^{\tilde{g}(\rho)} \frac{dg'}{\beta(g')}. \tag{10.15.31}$$

Since Green's functions in the two alternate approaches depend on the two parameters differently, we distinguish them by subscripts, "GML" and "CS." The renormalization group equations are given by

$$\left(m \frac{\partial}{\partial m} + \beta\left(g, \frac{m}{m_{\mathrm{p}}}\right) \frac{\partial}{\partial g} + \frac{1}{2} N \gamma_\phi\left(g, \frac{m}{m_{\mathrm{p}}}\right)\right) G_{\mathrm{GML}}^{(N)}(p_i; m, g) = 0, \tag{10.15.9}$$

$$\left(m \frac{\partial}{\partial m} + \beta(g) \frac{\partial}{\partial g} + \frac{1}{2} N \gamma_\phi(g)\right) G_{\mathrm{CS}}^{(N)}(p_i; m, g) = 0. \tag{10.15.29}$$

We assume that these two Green's functions are related to each other as

$$G_{\mathrm{GML}}^{(N)}(p_i; m, g) = Z_2^{N/2} G_{\mathrm{CS}}^{(N)}\left(p_i; m, Z_1^{-1} Z_2^2 g\right). \tag{10.15.32}$$

Here Z_1 and Z_2 are the functions of g and m/m_{p}, and when $m = m_{\mathrm{p}}$, both sides of Eq. (10.15.32) become Green's functions renormalized on the mass shell. Thus,

$$Z_1 = Z_2 = 1 \quad \text{for} \quad m = m_{\mathrm{p}}. \tag{10.15.33}$$

We shall define

$$g' = Z_1^{-1} Z_2^2 g, \tag{10.15.34}$$

$$\mathcal{D} = m\left(\frac{\partial}{\partial m}\right)_g + \beta\left(g, \frac{m}{m_{\mathrm{p}}}\right)\left(\frac{\partial}{\partial g}\right)_m, \tag{10.15.35}$$

$$\tilde{\mathcal{D}}' = m\left(\frac{\partial}{\partial m}\right)_{g'} + \beta(g')\left(\frac{\partial}{\partial g'}\right)_m. \tag{10.15.36}$$

In order to study the transformation (10.15.34), we assume

$$\mathcal{D} = \tilde{\mathcal{D}}'. \tag{10.15.37}$$

We note that

$$\begin{aligned}\frac{\partial}{\partial g'} &= \left(\frac{\partial g'}{\partial g}\right)^{-1}\frac{\partial}{\partial g},\\ m\left(\frac{\partial}{\partial m}\right)_{g'} &= m\left(\frac{\partial}{\partial m}\right)_g - m\left(\frac{\partial g'}{\partial m}\right)_g\left(\frac{\partial g'}{\partial g}\right)^{-1}\frac{\partial}{\partial g}.\end{aligned}$$

Thus we have

$$\tilde{\mathcal{D}}' = m\left(\frac{\partial}{\partial m}\right)_g + \left(\frac{\partial g'}{\partial g}\right)^{-1}\left(-m\frac{\partial g'}{\partial m} + \beta(g')\right)\frac{\partial}{\partial g},$$

and by the assumption (10.15.37), we obtain

$$\left[m\frac{\partial}{\partial m} + \beta\left(g, \frac{m}{m_{\mathrm{p}}}\right)\frac{\partial}{\partial g}\right]g' = \beta(g'), \tag{10.15.38}$$

or

$$\left[m\frac{\partial}{\partial m} + \beta\left(g, \frac{m}{m_{\mathrm{p}}}\right)\frac{\partial}{\partial g}\right]F(g') = 1 \quad \text{with} \quad F(g') = \int^{g'}\frac{dx}{\beta(x)}. \tag{10.15.39}$$

Equation (10.15.39) determines g' as a function of g and m/m_{p} only. The boundary condition is given by

$$g' = g \quad \text{for} \quad m = m_{\mathrm{p}}, \tag{10.15.40}$$

which follows from Eqs. (10.15.33) and (10.15.34).

Lastly, by Eqs. (10.15.9), (10.15.29), (10.15.32), and (10.15.37), we obtain the equation determining Z_2 as

$$\left[m\frac{\partial}{\partial m} + \beta\left(g, \frac{m}{m_{\mathrm{p}}}\right)\frac{\partial}{\partial g} - \gamma_\phi(g') + \gamma_\phi\left(g, \frac{m}{m_{\mathrm{p}}}\right)\right]Z_2 = 0. \tag{10.15.41}$$

Once $g' = Z_1^{-1}Z_2^2 g$ and Z_2 are known, the relationship between the two sets of Green's functions is completely determined by Eq. (10.15.32).

Renormalization Group of Callan–Symanzik for QED

We now briefly mention the renormalization group equation for QED. Lagrangian density for QED is given by

$$\mathcal{L}_{QED} = -\frac{1}{4}F_{0\mu\nu}F_0^{\mu\nu} - \frac{1}{2\alpha_0}(\partial^\mu A_{0\mu})^2 + \overline{\psi}_0(x)(i\gamma^\mu D_\mu - m_0)\psi_0(x), \quad (10.15.42)$$

where the renormalized quantities are defined by

$$\begin{aligned} A_{0\mu}(x) &= Z_3^{1/2}A_\mu(x), \\ \psi_0(x) &= Z_2^{1/2}\psi(x), \\ e_0 &= Z_3^{-1/2}e, \\ \alpha_0 &= Z_3\alpha, \end{aligned} \quad (10.15.43)$$

with the Ward–Takahashi identity,

$$Z_1 = Z_2. \quad (10.15.44)$$

We define the $(n+m)$ point Green's function , $G^{(n,m)}(k_1, \ldots, k_n; p_1, \ldots, p_m; e)$, by

$$G_0^{(n,m)}(k_1, \ldots, k_n; p_1, \ldots, p_m; e)$$

$$= Z_3^{n/2}(\mu)Z_2^{m/2}(\mu)G^{(n,m)}\left(k_1, \ldots, k_n; p_1, \ldots, p_m; e(\mu)\right), \quad (10.15.45)$$

$$e_0 = Z_3^{-1/2}e(\mu), \quad (10.15.46)$$

where n is the number of the photons $A_\mu(x)$ and m is the number of the electrons $\psi(x)$ in the $(n+m)$ point Green's function. We write down the renormalization group equation from the independence of the unrenormalized Green's function upon the normalization point μ,

$$\left(\mu\frac{\partial}{\partial\mu} + \beta\left(e, \frac{\mu}{m_{\text{p}}}\right)\frac{\partial}{\partial e} + \delta\left(e, \frac{\mu}{m_{\text{p}}}, \alpha\right)\frac{\partial}{\partial\alpha} + \gamma^{(n,m)}\right)G^{(n,m)}(k_i, p_j; \mu, e) = 0, \quad (10.15.47)$$

$$\gamma^{(n,m)} = n\gamma_{\text{ph}}\left(e, \frac{\mu}{m_{\text{p}}}\right) + m\gamma_{\text{el}}\left(e, \frac{\mu}{m_{\text{p}}}, \alpha\right). \quad (10.15.48)$$

Callan–Symanzik equationsfor the j-fold mass insertion terms in QED are given by

$$\Delta^{j+1}G^{(n,m)} = \left(m\frac{\partial}{\partial m} + \beta(e)\frac{\partial}{\partial e} + \delta(e,\alpha)\frac{\partial}{\partial\alpha} + \gamma^{(n,m)} - j\left(1 - \gamma_{\text{S}}(e,\alpha)\right)\right)\Delta^j G^{(n,m)}, \quad (10.15.49)$$

with

$$\begin{aligned}
Zm_0(\partial m/m_0) &= m, \\
\beta(e) &= Zm_0(\partial e/\partial m_0), \\
\gamma_{\mathrm{el}}(e,\alpha) &= \tfrac{1}{2}Zm_0(\partial \ln Z_2/\partial m_0), \\
\gamma_{\mathrm{ph}}(e) &= \tfrac{1}{2}Zm_0(\partial \ln Z_3/\partial m_0), \\
\delta(e,\alpha) &= Zm_0(\partial\alpha/\partial m_0), \\
m_0\overline{\psi}_0(x)\psi_0(x) &= Z_{\mathrm{S}} m\overline{\psi}(x)\psi(x), \\
\gamma_{\mathrm{S}}(e,\alpha) &= Zm_0(\partial \ln Z_{\mathrm{S}}/\partial m_0).
\end{aligned} \tag{10.15.50}$$

Combining these equations with the Ward–Takahashi identity,

$$e = Z_3^{1/2} e_0 \quad \text{and} \quad e^2\alpha = e_0^2\alpha_0, \tag{10.15.51}$$

we have

$$\beta(e) = e\gamma_{\mathrm{ph}}(e), \quad \delta(e,\alpha) = -2\alpha\gamma_{\mathrm{ph}}(e). \tag{10.15.52}$$

We define the generating function of the j-fold mass insertion terms by

$$G^{(n,m)}(p_i;\, m, e, K) = \sum_{j=0}^{\infty} \frac{K^j}{j!} \Delta^j G^{(n,m)}(p_i;\, m, e). \tag{10.15.53}$$

We obtain a homogeneous Callan–Symanzik equation for QED,

$$\begin{aligned}
&\left(m\frac{\partial}{\partial m} + \beta(e)\frac{\partial}{\partial e} + \delta(e)\frac{\partial}{\partial \alpha} + \gamma^{(n,m)} - \left\{ 1 + \left(1-\gamma_{\mathrm{S}}(e)\right) K \right\} \frac{\partial}{\partial K} \right) \\
&\quad \times G^{(n,m)}(p_i;\, m, e, K) = 0.
\end{aligned} \tag{10.15.54}$$

10.16 Standard Model

Electro-Weak Unification

With strictly vanishing neutrino masses, the neutrinos always enter in the elementary processes in the left-handed state. So it is natural to consider the left-handed doublets of the leptons and the right-handed singlets of the charged leptons.

$$l_{e\mathrm{L}} = \begin{pmatrix} \nu_{e\mathrm{L}} \\ e_{\mathrm{L}} \end{pmatrix} = \begin{pmatrix} \frac{1-\gamma_5}{2}\nu_e \\ \frac{1-\gamma_5}{2} e \end{pmatrix}, \quad l_{e\mathrm{R}} = e_{\mathrm{R}} = \frac{1+\gamma_5}{2} e. \tag{10.16.1}$$

The most general transformation which commutes with the lepton number, the Lorentz transformations, and the kinematic part of the Lagrangian density is

$$\begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix} \longrightarrow U \cdot \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}, \quad e_R \longrightarrow V \cdot e_R, \tag{10.16.2}$$

where U is the 2×2 unitary matrix and V is the 1×1 unitary matrix. The group G is $U(2) \times U(1)$. But this includes the lepton number phase transformations,

$$\begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix} \longrightarrow \exp[i\alpha] \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}, \quad e_R \longrightarrow \exp[i\alpha] e_R. \tag{10.16.3}$$

Assuming that this is unbroken, we cannot let these transformations be the gauge transformation. We want the group to include the charge phase transformations,

$$\begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & \exp[i\beta] \end{pmatrix} \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}, \quad e_R \longrightarrow \exp[i\beta] e_R. \tag{10.16.4}$$

We take the generators of these transformations as T_1, T_2, T_3, and Q with the realizations given by

$$t_{1L} = \frac{g}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; t_{2L} = \frac{g}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; t_{3L} = \frac{g}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \begin{matrix} t_{jR} = 0, \\ (j = 1, 2, 3), \end{matrix} \tag{10.16.5}$$

$$q_L = e \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad q_R = -e. \tag{10.16.6}$$

If the weak hypercharge is defined by the Gell–Mann–Nishijima relation

$$Y \equiv -2g' \left(\frac{T_3}{g} - \frac{Q}{e} \right), \quad y_L = -g' \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad y_R = -2g', \tag{10.16.7}$$

we have

$$[Y, \vec{T}] = 0. \tag{10.16.8}$$

So the group is $G = SU(2)_{\text{weak isospin}} \times U(1)_{\text{weak hypercharge}}$. We assume that $W_{1\mu}$, $W_{2\mu}$ and $W_{3\mu}$ are the generators of $SU(2)_{\text{weak isospin}}$ and that the neutral gauge field, V_μ, is the generator of $U(1)_{\text{weak hypercharge}}$.

Covariant derivative of the lepton field ψ in terms of $\vec{W}_\mu$ and V_μ is given by

$$D_\mu \psi = \partial_\mu \psi + i\vec{t}\vec{W}_\mu \psi + i\frac{y}{2} V_\mu \psi. \tag{10.16.9}$$

The unbroken linear combination of $\vec{t}$ and y is the electric charge q which is given by

$$q \propto \frac{t_3}{g} + \frac{y}{2g'} \propto g' t_3 + g\frac{y}{2}, \quad \text{or} \quad A_\mu \propto g' W_{3\mu} + gV_\mu. \tag{10.16.10}$$

Upon orthonormalization, we have the two neutral gauge fields,

$$\begin{cases} A_\mu &= \frac{1}{\sqrt{g^2+g'^2}}[g' W_{3\mu} + gV_\mu], \\ Z^0_\mu &= \frac{1}{\sqrt{g^2+g'^2}}[gW_{3\mu} - g' V_\mu], \end{cases} \quad \text{or} \quad \begin{cases} W_{3\mu} &= \frac{1}{\sqrt{g^2+g'^2}}[g' A_\mu + gZ^0_\mu], \\ V_\mu &= \frac{1}{\sqrt{g^2+g'^2}}[gA_\mu - g' Z^0_\mu]. \end{cases} \tag{10.16.11}$$

We remark that these results are independent of spontaneous symmetry breaking, but are only dependent on the charge conservation.

The neutral part of the covariant derivative is given by

$$\begin{aligned} t_3 W_{3\mu} + \frac{y}{2}V_\mu &= t_3 \frac{g' A_\mu + gZ^0_\mu}{\sqrt{g^2+g'^2}} + \frac{y}{2}\frac{gA_\mu - g' Z^0_\mu}{\sqrt{g^2+g'^2}} \\ &= A_\mu \frac{g' t_3 + gy/2}{\sqrt{g^2+g'^2}} + Z^0_\mu \frac{gt_3 - g' y/2}{\sqrt{g^2+g'^2}}. \end{aligned} \tag{10.16.12}$$

Hence the electric charge q is given by

$$q = \frac{g' t_3 + gy/2}{\sqrt{g^2+g'^2}}. \tag{10.16.13}$$

For the electron, we have $t_3 = -g/2, y = -g'$, for the left-handed component, $t_3 = 0, y = -2g'$, for the right-handed component, and $q = -e$, so that the elementary charge is given by

$$e = \frac{gg'}{\sqrt{g^2+g'^2}}. \tag{10.16.14}$$

The charged part of the covariant derivative with $k = 1, 2$, is given by

$$t_k W_{k\mu} = \frac{t_1 - it_2}{\sqrt{2}}\frac{W_{1\mu} + iW_{2\mu}}{\sqrt{2}} + \frac{t_1 + it_2}{\sqrt{2}}\frac{W_{1\mu} - iW_{2\mu}}{\sqrt{2}} = t_- W^+_\mu + t_+ W^-_\mu. \tag{10.16.15}$$

The charged current interaction is given by

$$\frac{g}{\sqrt{2}}\overline{\nu_{eL}}\gamma^\mu e_L W^+_\mu, \quad \frac{g}{\sqrt{2}}\overline{e_L}\gamma^\mu \nu_{eL} W^-_\mu, \quad \frac{g}{\sqrt{2}}\overline{p}\gamma^\mu n W^+_\mu. \tag{10.16.16}$$

The effective interaction is given by

$$\frac{g}{\sqrt{2}}\overline{e_{\mathrm{L}}}\gamma^{\mu}\nu_{e\mathrm{L}}\frac{1}{m_{\mathrm{W}\pm}^{2}}\frac{g}{\sqrt{2}}\overline{p_{\mathrm{L}}}\gamma_{\mu}n_{\mathrm{L}}=\frac{g^{2}}{8m_{\mathrm{W}\pm}^{2}}\bar{e}\gamma^{\mu}(1-\gamma_{5})\nu_{e}\overline{p}\gamma_{\mu}(1-\gamma_{5})n, \quad (10.16.17)$$

to be compared with

$$\frac{G_{\mathrm{F}}}{\sqrt{2}}\bar{e}\gamma^{\mu}(1-\gamma_{5})\nu_{e}\overline{p}\gamma_{\mu}(1-\gamma_{5})n, \quad G_{\mathrm{F}}=1.02\times 10^{-5}m_{\mathrm{p}}^{-2}, \quad (10.16.18)$$

resulting in

$$\frac{g^{2}}{8m_{\mathrm{W}\pm}^{2}}=\frac{G_{\mathrm{F}}}{\sqrt{2}}. \quad (10.16.19)$$

We define the Weinberg angle θ_{W} by

$$g'/g=\tan\theta_{\mathrm{W}}, \quad g=e/\sin\theta_{\mathrm{W}}, \quad g'=e/\cos\theta_{\mathrm{W}}. \quad (10.16.20)$$

Then we have

$$m_{\mathrm{W}\pm}=\sqrt{\frac{g^{2}\sqrt{2}}{8G_{\mathrm{F}}}}=\frac{e}{\sin\theta_{\mathrm{W}}}\sqrt{\frac{\sqrt{2}}{8G_{\mathrm{F}}}}=\frac{37.3\mathrm{Gev}}{\sin\theta_{\mathrm{W}}}. \quad (10.16.21)$$

To give the electron its mass, we introduce the doublet of $SU(2)_{\text{weak isospin}}$ Higgs scalar field,

$$\phi=\begin{pmatrix}\phi^{+}\\ \phi^{0}\end{pmatrix}, \quad \begin{array}{lllllll} t_{3} & = & +g/2, & & y & = & g', \\ t_{3} & = & -g/2, & & y & = & g'. \end{array} \quad (10.16.22)$$

Covariant derivative of Higgs scalar field is given by

$$(D_{\mu}\phi)_{\alpha}=\left(\partial_{\mu}\delta_{\alpha\beta}+i(\vec{t})_{\alpha\beta}\vec{W}_{\mu}+i\frac{y}{2}V_{\mu}\delta_{\alpha\beta}\right)\phi_{\beta}. \quad (10.16.23)$$

As the Higgs scalar Lagrangian density $\mathcal{L}_{\mathrm{Higgs}}$, we choose

$$\mathcal{L}_{\mathrm{Higgs}}=-\left(D_{\mu}\phi(x)\right)^{\dagger}D^{\mu}\phi(x)-V\left(\phi(x)\right), \quad (10.16.24a)$$

$$V(\phi)=-m^{2}\phi^{\dagger}\phi+\frac{\lambda}{2}(\phi^{\dagger}\phi)^{2}, \quad \lambda>0, \quad m>0. \quad (10.16.24b)$$

We generate the masses of the leptons by the Yukawa coupling of the leptons to Higgs scalar field, ϕ,

$$\mathcal{L}_{\mathrm{Yukawa}}^{\mathrm{lepton}}=-\sum_{f=e,\mu,\tau}\left\{G_{f}\bar{l}_{f\mathrm{L}}\phi l_{f\mathrm{R}}+\mathrm{h.c.}\right\}. \quad (10.16.25)$$

Higgs scalar field ϕ develops the vacuum expectation value such that

$$\langle\phi\rangle = \begin{pmatrix} 0 \\ v \end{pmatrix} \quad \text{with} \quad v = m/\sqrt{\lambda}. \tag{10.16.26}$$

Then we have the lepton mass term as

$$\mathcal{L}_{\text{lepton mass}} = -\sum_{l=e,\mu,\tau} \left\{ vG_l \bar{l}_\text{L} l_\text{R} + v^* G_l^* \bar{l}_\text{R} l_\text{L} \right\}. \tag{10.16.27}$$

Thus we obtain

$$m_l = G_l v, \quad m_{\nu_l} = 0, \quad l = e, \mu, \tau. \tag{10.16.28}$$

The covariant derivative term in the Higgs scalar Lagrangian density which depends on the vacuum expectation value $\langle\phi\rangle$ quadratically becomes

$$v^2 \left(\frac{g^2}{4}(W_{1\mu}^2 + W_{2\mu}^2) + \frac{1}{4}(gW_{3\mu} - g'V_\mu)^2 \right). \tag{10.16.29}$$

We have the masses for the intermediate vector bosons, $W_\mu^\pm$ and Z_μ^0, as

$$\begin{aligned} m_{W^\pm} &= \left(1/\sqrt{2}\right) vg, \\ m_{Z^0} &= \left(1/\sqrt{2}\right) v\sqrt{g^2 + g'^2}, \end{aligned} \quad m_\gamma = 0, \quad v \simeq 247 \text{ GeV}. \tag{10.16.30}$$

The lepton–gauge couplings result from the lepton Lagrangian density

$$\mathcal{L}_{\text{lepton}} = \sum_{f=e,\mu,\tau} \left\{ \bar{l}_{f\text{L}} i\gamma^\mu D_\mu l_{f\text{L}} + \bar{l}_{f\text{R}} i\gamma^\mu D_\mu l_{f\text{R}} \right\}. \tag{10.16.31}$$

Writing this Lagrangian density in terms of the physical fields, we obtain

$$\begin{aligned} \mathcal{L}_{\text{lepton-}A_\mu\text{ coupling}} &= \frac{gg'}{\sqrt{g^2 + g'^2}} A_\mu \sum_{l=e,\mu,\tau} \bar{l}\gamma^\mu l, \\ \mathcal{L}_{\text{lepton-}Z_\mu^0\text{ coupling}} &= -\sqrt{g^2 + g'^2} Z_\mu^0 \sum_{f=e,\mu,\tau} \bar{l}_f \gamma^\mu \left(t_3 \frac{1-\gamma_5}{2} - Q_{l_f} \sin^2\theta_\text{W} \right) l_f, \\ \mathcal{L}_{\text{lepton-}W_\mu^\pm\text{ coupling}} &= \frac{g}{\sqrt{2}} \sum_{l=e,\mu,\tau} \left\{ W_\mu^+ \bar{\nu}_l \gamma^\mu \frac{1-\gamma_5}{2} l + \text{h.c.} \right\}. \end{aligned}$$

The total Lagrangian density $\mathcal{L}_{\text{GWS}}$ for the *Glashow–Weinberg–Salam* model without the gauge-fixing term and the Faddeev–Popov ghost term is

$$\mathcal{L}_{\text{GWS}} = -\frac{1}{4}\left(\partial_\mu W_{\alpha\nu} - \partial_\nu W_{\alpha\mu} - g\varepsilon_{\alpha\beta\gamma} W_{\beta\mu} W_{\gamma\nu}\right)^2 - \frac{1}{4}\left(\partial_\mu V_\nu - \partial_\nu V_\mu\right)^2$$

$$+\left(\left(\partial_\mu + i\vec{t}\vec{W}_\mu + i\frac{y}{2}V_\mu\right)\phi\right)^\dagger\left(\partial_\mu + i\vec{t}\vec{W}_\mu + i\frac{y}{2}V_\mu\right)\phi$$

$$+ m^2\phi^\dagger\phi - \frac{\lambda}{2}(\phi^\dagger\phi)^2 + \sum_{f=e,\mu,\tau}\left\{\bar{l}_{f\mathrm{L}} i\gamma^\mu D_\mu l_{f\mathrm{L}} + \bar{l}_{f\mathrm{R}} i\gamma^\mu D_\mu l_{f\mathrm{R}}\right\}$$

$$- \sum_{f=e,\mu,\tau}\left\{G_f \bar{l}_{f\mathrm{L}}\phi l_{f\mathrm{R}} + \text{h.c.}\right\}. \qquad (10.16.32)$$

Standard Model: As for the strong interaction, we choose $SU(3)_{\text{color}}$ as the gauge group. We assume that the quark field $q_n(x)$ transforms under the fundamental representation of $SU(3)_{\text{color}}$,

$$t_\alpha = \frac{g_3}{2}\lambda_\alpha, \quad (t_\alpha^{\text{adj}})_{\beta,\gamma} = -ig_3 f_{\alpha\beta\gamma}, \quad \alpha,\beta,\gamma = 1,\ldots,8. \qquad (10.16.33)$$

We have the quark–gluon Lagrangian density $\mathcal{L}_{\text{quark-gluon}}$ as

$$\mathcal{L}_{\text{quark-gluon}} = -\frac{1}{4}\left(\partial_\mu A_{\alpha\nu}(x) - \partial_\nu A_{\alpha\mu}(x) - g_3 f_{\alpha\beta\gamma}A_{\beta\mu}(x)A_{\gamma\nu}(x)\right)^2$$

$$+ \sum_{q=u,d,c,s,t,b}\bar{q}_n(x)i\gamma^\mu\left(\partial_\mu\delta_{nm} + ig_3\left(\frac{1}{2}\lambda_\alpha\right)_{nm}A_{\alpha\mu}(x)\right)q_m(x), \qquad (10.16.34)$$

where

$$\begin{array}{lll} q_n(x): & \text{Quark (color triplet)}, & n = 1,2,3, \\ A_{\alpha\mu}(x): & \text{Gluon (color octet)}, & \alpha = 1,\ldots,8. \end{array} \qquad (10.16.35)$$

Here the indices n and α refer to the color degrees of freedom of the quark–gluon system.

The interaction Lagrangian density for the quarks and the gluons can be easily read off from Eq. (10.16.34). There also exists the cubic and quartic self-interaction of the gluons. The explicit form of the self-interaction Lagrangian density can be easily read off from Eq. (10.16.34).

The quark degrees of freedom have further gauge couplings upon gauging $SU(2)_{\text{weak isospin}} \times U(1)_{\text{weak hypercharge}}$,

$$\mathcal{L}_{\text{quark-gauge}} = \sum_{j=1,2,3}\overline{(u_j,d_j)_\mathrm{L}}i\gamma^\mu\left(i\vec{t}\vec{W}_\mu + i\frac{y}{2}V_\mu\right)\begin{pmatrix} u_j \\ d_j\end{pmatrix}_\mathrm{L}$$

$$+ \sum_{j=1,2,3}\left\{\overline{u_{j\mathrm{R}}}i\gamma^\mu\left(i\frac{y}{2}V_\mu\right)u_{j\mathrm{R}} + \overline{d_{j\mathrm{R}}}i\gamma^\mu\left(i\frac{y}{2}V_\mu\right)d_{j\mathrm{R}}\right\}. \qquad (10.16.36)$$

Here, we designate

$$\begin{pmatrix} u_j \\ d_j \end{pmatrix}_{j=1,2,3} \sim \begin{array}{lllllllll} j=1, & j=2, & j=3, & & & & & & \\ u_{\rm L}, & c_{\rm L}, & t_{\rm L}, & t_3 & = & +g/2, & y & = & +g'/3, \\ d_{\rm L}, & s_{\rm L}, & b_{\rm L}, & t_3 & = & -g/2, & y & = & +g'/3, \\ u_{\rm R}, & c_{\rm R}, & t_{\rm R}, & t_3 & = & 0, & y & = & +4g'/3, \\ d_{\rm R}, & s_{\rm R}, & b_{\rm R}, & t_3 & = & 0, & y & = & -2g'/3, \end{array} \tag{10.16.37}$$

$$\overline{(u_j, d_j)_{\rm L}} \equiv (\overline{u_{j\rm L}}, \overline{d_{j\rm L}}). \tag{10.16.38}$$

In terms of $W_\mu^\pm$, Z_μ^0, and A_μ, we write the interaction Lagrangian density of the electro-weak coupling of the quarks in the following form:

$$\mathcal{L}_{\text{quark-gauge}} = \mathcal{L}_{\text{quark-}A_\mu\text{ coupling}} + \mathcal{L}_{\text{quark-}Z_\mu^0\text{ coupling}} + \mathcal{L}_{\text{quark-}W_\mu^\pm\text{ coupling}}, \tag{10.16.39}$$

where

$$\mathcal{L}_{\text{quark-}A_\mu\text{ coupling}} = eA_\mu j^\mu_{\rm e.m.}, \tag{10.16.40}$$

$$\mathcal{L}_{\text{quark-}Z_\mu^0\text{ coupling}} = -\frac{g}{\cos\theta_{\rm W}} Z_\mu^0 \left(j^\mu_{\rm L,3} - \sin^2\theta_{\rm W} j^\mu_{\rm e.m.}\right), \tag{10.16.41}$$

$$\mathcal{L}_{\text{quark-}W_\mu^\pm\text{ coupling}} = \frac{g}{\sqrt{2}} \left(W_\mu^+ j^\mu_{\rm L,-} + \text{h.c.}\right), \tag{10.16.42}$$

$$j^\mu_{\rm e.m.} = \sum_{j=1,2,3} \overline{(u_j, d_j)} \gamma^\mu \left(\frac{t_3}{g} + \frac{y}{2g'}\right) \begin{pmatrix} u_j \\ d_j \end{pmatrix}, \quad \overline{(u_j, d_j)} \equiv (\overline{u_j}, \overline{d_j}), \tag{10.16.43}$$

$$j^\mu_{\rm L,3\ or\ \pm} = \sum_{j=1,2,3} \overline{(u_j, d_j)_{\rm L}} \gamma^\mu \frac{t_{3\ \rm or\ \pm}}{g} \begin{pmatrix} u_j \\ d_j \end{pmatrix}_{\rm L}, \quad t_\pm = t_1 \pm i t_2. \tag{10.16.44}$$

We note that the neutral currents, $j^\mu_{\rm e.m.}$ and $j^\mu_{\rm L,3}$, are flavor diagonal, and therefore they do not provide the interactions which bridge over the different generations. The charged current, $j^\mu_{\rm L,\pm}$, is not flavor diagonal, but it still does not provide the interactions which bridge over the different generations.

We observe that the quarks acquire the masses from the Yukawa coupling to Higgs scalar field, ϕ, via

$$\mathcal{L}^{\rm quark}_{\rm Yukawa} = -\sum_{j=1,2,3} \left\{ G_{u_j} \overline{(u_j, d_j)_{\rm L}} \phi^{\rm c} u_{j\rm R} + G_{d_j} \overline{(u_j, d_j)_{\rm L}} \phi d_{j\rm R} + \text{h.c.} \right\}. \tag{10.16.45}$$

As long as the quarks, $q_{\rm L}$, in the left-handed doublets are in the eigenstates of the mass matrix, we cannot provide the interactions which bridge over the different generations, since the Lagrangian density does not contain the term which bridges over the different generations of the quarks. Since the standard model is incapable of providing the interactions which bridge over the different generations of the quarks, we shall mix the bottom components of the doublet of the quarks by hand. We write

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = M \begin{pmatrix} d \\ s \\ b \end{pmatrix}. \tag{10.16.46}$$

The explicit form of the matrix M was given by M. Kobayashi and T. Maskawa as

$$M = \begin{pmatrix} c_1 & -s_1 c_3 & -s_1 s_3 \\ s_1 c_2 & c_1 c_2 c_3 - s_2 s_3 \exp[i\delta] & c_1 c_2 s_3 + s_2 c_3 \exp[i\delta] \\ s_1 s_2 & c_1 s_2 s_3 + c_2 s_3 \exp[i\delta] & c_1 s_2 s_3 - c_2 c_3 \exp[i\delta] \end{pmatrix}, \tag{10.16.47}$$

where

$$c_i = \cos\theta_i, \quad s_i = \sin\theta_i, \quad i = 1, 2, 3. \tag{10.16.48}$$

Since we are mixing the bottom components of the doublet of the quarks by hand, we must determine the mixing angles, θ_i, ($i = 1, 2, 3$), and δ, from the experiment. In this respect, the standard model contains too many adjustable parameters and is far from satisfactory as the ultimate model of the unification of the weak interaction, the electromagnetic interaction and the strong interaction.

We write down the Lagrangian density for the standard model for the sake of completeness, with the gauge-fixing term, without the Faddeev–Popov ghost term, and with the bottom components of the doublet of the quarks Kobayashi–Maskawa rotated:

$$\begin{aligned}
\mathcal{L}_{\text{SM}} = & -\frac{1}{4}\left(\partial_\mu W_{\alpha\nu} - \partial_\nu W_{\alpha\mu} - g\varepsilon_{\alpha\beta\gamma} W_{\beta\mu} W_{\gamma\nu}\right)^2 - \frac{1}{4}\left(\partial_\mu V_\nu - \partial_\nu V_\mu\right)^2 \\
& + \left(\left(\partial_\mu + i\vec{t}\vec{W}_\mu + i\frac{y}{2}V_\mu\right)\phi\right)^\dagger \left(\partial_\mu + i\vec{t}\vec{W}_\mu + i\frac{y}{2}V_\mu\right)\phi + m^2\phi^\dagger\phi \\
& - \frac{\lambda}{2}(\phi^\dagger\phi)^2 \\
& + \sum_{f=e,\mu,\tau}\left\{\bar{l}_{f\text{L}} i\gamma^\mu D_\mu l_{f\text{L}} + \bar{l}_{f\text{R}} i\gamma^\mu D_\mu l_{f\text{R}}\right\} - \sum_{f=e,\mu,\tau}\left\{G_f \bar{l}_{f\text{L}}\phi l_{f\text{R}} + \text{h.c.}\right\} \\
& - \frac{1}{4}\left(\partial_\mu A_{\alpha\nu}(x) - \partial_\nu A_{\alpha\mu}(x) - g_3 f_{\alpha\beta\gamma} A_{\beta\mu}(x) A_{\gamma\nu}(x)\right)^2 \\
& + \sum_{q=u,d',c,s',t,b'} \bar{q}_n(x) i\gamma^\mu \left(\partial_\mu\delta_{nm} + ig_3\left(\frac{1}{2}\lambda_\alpha\right)_{nm} A_{\alpha\mu}(x)\right) q_m(x) \\
& + eA_\mu j^\mu_{\text{e.m.}} - \frac{g}{\cos\theta_{\text{W}}} Z^0_\mu \left(j^\mu_{\text{L},3} - \sin^2\theta_{\text{W}} j^\mu_{\text{e.m.}}\right) + \frac{g}{\sqrt{2}}\left(W^+_\mu j^\mu_{\text{L},-} + \text{h.c.}\right) \\
& - \sum_{j=1,2,3}\left\{G_{u_j}\overline{(u_j, d'_j)_\text{L}}\phi^\text{c} u_{j\text{R}} + G_{d_j}\overline{(u_j, d'_j)_\text{L}}\phi d'_{j\text{R}} + \text{h.c.}\right\} - \frac{\xi}{2}(\partial^\mu A_{\alpha\mu})^2 \\
& - \frac{\xi}{2}\left|\partial^\mu W^+_\mu - \frac{i}{\xi} m_{\text{W}\pm}\phi^+\right|^2 - \frac{\xi}{2}\left|\partial^\mu Z^0_\mu - \frac{i}{\xi}\frac{m_{Z^0}}{\sqrt{2}}\phi^0\right|^2 - \frac{\xi}{2}(\partial^\mu A_\mu)^2,
\end{aligned} \tag{10.16.49}$$

$$\phi = \begin{pmatrix} \phi^{+} \\ \phi^{0} \end{pmatrix}, \quad j^{\mu}_{\text{e.m.}} = \sum_{j=1,2,3} \overline{(u_j, d'_j)}\gamma^{\mu}\left(\frac{t_3}{g} + \frac{y}{2g'}\right)\begin{pmatrix} u_j \\ d'_j \end{pmatrix}, \tag{10.16.50a}$$

$$j^{\mu}_{\text{L},3} = \sum_{j=1,2,3} \overline{(u_j, d'_j)_{\text{L}}}\gamma^{\mu}\frac{t_3}{g}\begin{pmatrix} u_j \\ d'_j \end{pmatrix}_{\text{L}}, \quad j^{\mu}_{\text{L},\pm} = \sum_{j=1,2,3} \overline{(u_j, d'_j)_{\text{L}}}\gamma^{\mu}\frac{t_{\pm}}{g}\begin{pmatrix} u_j \\ d'_j \end{pmatrix}_{\text{L}}. \tag{10.16.50b}$$

Instanton, Strong CP-Violation, and Axion

In the $SU(2)$ gauge field theory, we have Belavin–Polyakov–Schwartz–Tyupkin instanton solution which is a classical solution to field equation in Euclidean space–time. Proper account for the instanton solution in the path integral formalism requires the addition of the strong CP-violating term to the QCD Lagrangian density. Peccei–Quinn axion hypothesis resolves this strong CP-violation problem.

We first discuss the instanton solution. The instanton is the solution to the classical equation, which makes Euclidean action functional stationary and finite. We note that the arbitrary element of the $SU(2)$ gauge group can be expressed as

$$g(x) = \exp[i\omega_a T_a] = a(x) + \vec{b}(x), \quad T_a = \frac{1}{2}\tau_a, \quad a = 1, 2, 3, \tag{10.16.51a}$$

$$g(x)g(x)^{\dagger} = a(x)^2 + \vec{b}(x)^2 = 1. \tag{10.16.51b}$$

From the requirement that Euclidean action functional is finite, we require

$$F_{a\mu\nu}(x) \longrightarrow 0 \quad \text{as} \quad |x| \longrightarrow \infty.$$

In another word, the gauge field $A_{\mu}(x)$ approaches to the configuration which is equivalent to the vacuum,

$$A_{\mu}(x) \longrightarrow -ig(x)\partial_{\mu}g(x)^{\dagger} \quad \text{as} \quad |x| \longrightarrow \infty. \tag{10.16.52}$$

Explicitly the instanton solution is given by

$$A_{\mu}(x) \equiv -\frac{r^2}{r^2 + \rho^2} ig(x)\partial_{\mu}g(x)^{\dagger}, \tag{10.16.53a}$$

$$r^2 = (x^4)^2 + (\vec{x})^2 = |x|^2, \quad g(x) \equiv \frac{x^4 + i\vec{x}\vec{\tau}}{r}. \tag{10.16.53b}$$

This solution appears and disappears instantly, and so we call this solution as an instanton.

In order to gain a proper understanding of the instanton, we consider QCD vacuum carefully. We shall consider QCD in the temporal gauge

$$A_{\alpha 0}(x) = 0, \tag{10.16.54a}$$

and concentrate on the $SU(2)$ subgroup. In this gauge, the spatial gauge fields are time independent and, under a gauge transformation, transform as

$$A_i(\vec{x}) \equiv \frac{\tau_\alpha}{2} A_{\alpha i}(\vec{x}) \longrightarrow \Omega(\vec{x}) A_i(\vec{x}) \Omega^{-1}(\vec{x}) - \frac{i}{g_3} \Omega(\vec{x}) \nabla_i \Omega^{-1}(\vec{x}). \tag{10.16.54b}$$

The rich structure of QCD vacuum results from the requirement that the gauge transformation matrices $\Omega(\vec{x})$ go to unity at spatial infinity. Such requirement maps the physical space onto the group space and this $S_3 \to S_3$ map splits $\Omega(\vec{x})$ into different homotopy classes $\{\Omega_n(\vec{x})\}$, characterized by an integer winding number n which specifies how precisely $\Omega(\vec{x})$ behaves as $\vec{x} \to \infty$,

$$\Omega_n(\vec{x}) \longrightarrow \exp[2\pi i n], \quad \text{as} \quad \vec{x} \longrightarrow \infty. \tag{10.16.55}$$

Because we can construct the n-gauge transformation matrix $\Omega_n(\vec{x})$ by compounding n times $\Omega_1(\vec{x})$, an n-vacuum state corresponding to $A_{n\alpha\mu}(\vec{x})$ is not really gauge invariant. Indeed, the action of the gauge transformation matrix $\Omega_1(\vec{x})$ on an n-vacuum state gives an $(n+1)$-vacuum state,

$$\Omega_1(\vec{x}) \left| n \right\rangle = \left| n+1 \right\rangle. \tag{10.16.56}$$

We should construct the gauge invariant vacuum state, the θ-vacuum, by superposing these n-vacuum states,

$$\left|\theta\right\rangle = \sum_n \exp[-in\theta] \left| n \right\rangle \quad \text{with} \quad \Omega_1(\vec{x}) \left|\theta\right\rangle = \exp[i\theta] \left|\theta\right\rangle. \tag{10.16.57}$$

With the ν instanton solutions in the theory, we can show the fact that the winding number ν, which is also called Pontryagin number has the following gauge invariant expression after a little algebra:

$$\nu = \frac{g_3^2}{32\pi^2} \int d^4x \varepsilon^{\mu\nu\rho\sigma} F_{\alpha\mu\nu} F_{\alpha\rho\sigma}. \tag{10.16.58}$$

The vacuum-to-vacuum transition amplitude assumes the following form:

$$\begin{aligned} {}_+\left\langle \theta \middle| \theta \right\rangle_- &= \sum_\nu \exp[i\nu\theta] \sum_n \left({}_+\left\langle n+\nu \middle| n \right\rangle_- \right) \\ &= \sum_\nu \int \mathcal{D}[A_{\alpha\mu}] \exp\left[i \int d^4x \mathcal{L}_{QCD} + i\nu\theta \right] \\ &\quad \times \delta\left(\nu - \frac{g_3^2}{32\pi^2} \int d^4x \varepsilon^{\mu\nu\rho\sigma} F_{\alpha\mu\nu} F_{\alpha\rho\sigma} \right). \end{aligned} \tag{10.16.59}$$

We can interpret this new θ-term in terms of the new effective Lagrangian density for *QCD*, whereby all of the quarks acquire the masses from the spontaneous breakdown of the electro-weak gauge symmetry

$$\mathcal{L}'_{QCD} = -\frac{1}{4}F_{\alpha\mu\nu}F^{\mu\nu}_{\alpha} + \theta\frac{g_3^2}{32\pi^2}\varepsilon^{\mu\nu\rho\sigma}F_{\alpha\mu\nu}F_{\alpha\rho\sigma} + \sum_{q=u_j,d_j,j=1,2,3}\overline{q}(x)i\gamma^{\mu}D_{\mu}q(x). \tag{10.16.60}$$

Perturbation theory is connected with the $\nu = 0$ sectors. The effect of the $\nu \neq 0$ sectors are nonperturbative. These contribution are naturally related to the chiral anomaly. For n_f flavors, the axial $U(1)_{\text{A}}$ current J_5^{μ} has a chiral anomaly,

$$\partial_{\mu}J_5^{\mu} = n_f\frac{g_3^2}{32\pi^2}\varepsilon^{\mu\nu\rho\sigma}F_{\alpha\mu\nu}F_{\alpha\rho\sigma}, \quad J_5^{\mu} = \sum_{i=1}^{n_f}\overline{q}_i\gamma^{\mu}\gamma_5 q_i. \tag{10.16.61a}$$

Since the right-hand side of the above equation is a four-divergence,

$$\varepsilon^{\mu\nu\rho\sigma}F_{\alpha\mu\nu}F_{\alpha\rho\sigma} = \partial_{\mu}K^{\mu}, \quad K^{\mu} = \varepsilon^{\mu\nu\rho\sigma}A_{\alpha\nu}\left[F_{\alpha\rho\sigma} - \frac{g_3}{3}f_{\alpha\beta\gamma}A_{\beta\rho}A_{\gamma\sigma}\right], \tag{10.16.61b}$$

we can construct a conserved axial $U(1)_{\text{A}}$ current as

$$\tilde{J}_5^{\mu} = J_5^{\mu} - n_f\frac{g_3^2}{32\pi^2}K^{\mu}. \tag{10.16.61c}$$

Thus a chirality change ΔQ_5 is related to the winding number ν as

$$\Delta Q_5 = \int d^4x\partial_{\mu}J_5^{\mu} = n_f\frac{g_3^2}{32\pi^2}\int d^4x\varepsilon^{\mu\nu\rho\sigma}F_{\alpha\mu\nu}F_{\alpha\rho\sigma} = n_f\nu.$$

The conserved current $\tilde{J}_5^{\mu}$ is not gauge invariant for transformations with nontrivial winding number. Since we can show that

$$\Omega_1(\frac{g_3^2}{32\pi^2}\int d^3\vec{x}K^0)\Omega_1^{-1} = \frac{g_3^2}{32\pi^2}\int d^3\vec{x}K^0 - 1,$$

we obtain the following gauge transformation of the associated charge $\tilde{Q}_5$:

$$\Omega_1\tilde{Q}_5\Omega_1^{-1} = \tilde{Q}_5 + n_f \quad \text{with} \quad \tilde{Q}_5 = \int d^3x\tilde{J}_5^0. \tag{10.16.62}$$

The new interaction term, the θ-term in Eq. (10.16.60), is not the only new source of CP-violation arising from the complex structure of the *QCD* vacuum. It is augmented by an analogous term coming from the electro-weak sector of the theory. In general, the mass matrix of the quarks which emerges from spontaneous breakdown of the electro-weak gauge symmetry is neither Hermitian nor diagonal. In general, we have the quark mass term as

$$\mathcal{L}_{\text{quark mass}} = -\overline{q}_{\mathrm{R}_i} M_{ij} q_{\mathrm{L}_j} - \overline{q}_{\mathrm{L}_i} (M^{\dagger})_{ij} q_{\mathrm{R}_j}. \tag{10.16.63a}$$

This mass matrix can be diagonalized by separate unitary transformations of the chiral quark fields. These transformations constitute a chiral $U(1)_{\mathrm{A}}$ rotation,

$$q_{\mathrm{R}} \longrightarrow \exp[i\alpha] q_{\mathrm{R}}, \quad q_{\mathrm{L}} \longrightarrow \exp[-i\alpha] q_{\mathrm{L}}, \tag{10.16.63b}$$

with

$$\alpha = (1/2n_f) \arg\det M.$$

This chiral $U(1)_{\mathrm{A}}$ rotation changes the vacuum angle. Using Eq. (10.16.62), we can show that the rotations (10.16.63b) on the θ-vacuum shift the vacuum angle by $2n_f\alpha$, since the action of $\exp[i2\alpha\tilde{Q}_5]$ on $|\theta\rangle$ produces $|\theta + 2n_f\alpha\rangle$ as

$$\begin{aligned}
\Omega_1 \exp\left[i2\alpha\tilde{Q}_5\right]\left|\theta\right\rangle &= \Omega_1 \exp\left[i2\alpha\tilde{Q}_5\right]\Omega_1^{-1}\Omega_1\left|\theta\right\rangle \\
&= \exp\left[i2\alpha(\tilde{Q}_5 + n_f)\right]\exp[i\theta]\left|\theta\right\rangle \\
&= \exp\left[i(\theta + 2n_f\alpha)\right]\exp\left[i2\alpha\tilde{Q}_5\right]\left|\theta\right\rangle \\
\Rightarrow \quad \exp\left[i2\alpha\tilde{Q}_5\right]\left|\theta\right\rangle &= \left|\theta + 2n_f\alpha\right\rangle.
\end{aligned}$$

Hence, in the full theory, the effective CP-violating interaction arising from the more complex structure of QCD vacuum is the one given in Eq. (10.16.60), where the θ-parameter is replaced with $\overline{\theta}$ defined by

$$\overline{\theta} \equiv \theta + 2n_f\alpha = \theta + \arg\det M. \tag{10.16.64}$$

The strong CP problem is really why the combination of QCD and electro-weak parameters which make up $\overline{\theta}$ should be so small. The θ-term in Eq. (10.16.60) already gives an immediate contribution to the electric dipole moment of the neutron and the stringent experimental bound on the electric dipole moment of the neutron requires that we should have $\overline{\theta} \leq 10^{-9}$. In principle, since $\overline{\theta}$ is a free parameter of the theory, any value of $\overline{\theta}$ is equally likely. We shall explore why this number is so small, or, even we can make it disappear.

What Peccei and Quinn did is the following. They replaced the strong CP-violating static θ-parameter by the dynamical CP-conserving interactions of the axion field $a(x)$ with the assumption that full Lagrangian density of the standard model is invariant under the additional global chiral $U(1)_{\mathrm{P.Q.}}$ symmetry, where $\delta_{\mathrm{P.Q.}} a(x) = v\alpha$, with the introduction of the $SU(2)_{\text{weak isospin}}$ doublets of the Higgs scalar fields, ϕ_u and ϕ_d, $(y = \mp g')$, replacing the Higgs scalar field ϕ, with ϕ_d also coupled to the right-handed charged leptons,

$$\mathcal{L}^{\text{quark}}_{\text{Yukawa}} = -\left\{G_{u_j}\overline{(u_j, d_j)_{\text{L}}}\phi_u u_{j\text{R}} + G_{d_j}\overline{(u_j, d_j)_{\text{L}}}\phi_d d_{j\text{R}} + \text{h.c.}\right\}.$$

Isolating the axion as the common phase field of the doublets, ϕ_u and ϕ_d,

$$\phi_u = v_u \begin{pmatrix} 1 \\ 0 \end{pmatrix} \exp\left[\frac{iax}{v}\right], \; \phi_d = v_d \begin{pmatrix} 0 \\ 1 \end{pmatrix} \exp\left[\frac{ia}{xv}\right], \; x = \frac{v_d}{v_u}, \; v = (v_u^2 + v_d^2)^{1/2},$$

the $U(1)_{\text{P.Q.}}$ symmetry which guarantees the invariance of $\mathcal{L}^{\text{quark}}_{\text{Yukawa}}$ is under

$$\left\{\begin{array}{lcl} u_{j\text{R}} & \longrightarrow & \exp\left[-i\alpha x\right] u_{j\text{R}}, \\ d_{j\text{R}} & \longrightarrow & \exp\left[-i(\alpha/x)\right] d_{j\text{R}}, \end{array}\right. \qquad \left\{\begin{array}{lcl} \phi_u & \longrightarrow & \exp\left[i\alpha x\right] \phi_u, \\ \phi_d & \longrightarrow & \exp\left[i(\alpha/x)\right] \phi_d. \end{array}\right.$$

We write the effective standard model Lagrangian density as

$$\mathcal{L}^{\text{eff}}_{\text{SM}} = \mathcal{L}_{\text{SM}} + \overline{\theta}\frac{g_3^2}{32\pi^2}\varepsilon^{\mu\nu\rho\sigma}F_{\alpha\mu\nu}F_{\alpha\rho\sigma} + \frac{1}{2}\partial_\mu a^\dagger \partial^\mu a + \mathcal{L}_{\text{int}}\left(\frac{\partial_\mu a}{v}; \psi\right)$$
$$+ \frac{a}{v}\xi\frac{g_3^2}{32\pi^2}\varepsilon^{\mu\nu\rho\sigma}F_{\alpha\mu\nu}F_{\alpha\rho\sigma}. \tag{10.16.65a}$$

The $U(1)_{\text{P.Q.}}$ symmetry is spontaneously broken by the doublets, ψ stands for any field in the theory and ξ is a model-dependent parameter associated with the global chiral anomaly of the $U(1)_{\text{P.Q.}}$ current $J^\mu_{\text{P.Q.}}$,

$$\partial_\mu J^\mu_{\text{P.Q.}} = \xi\frac{g_3^2}{32\pi^2}\varepsilon^{\mu\nu\rho\sigma}F_{\alpha\mu\nu}F_{\alpha\rho\sigma},$$

with

$$J^\mu_{\text{P.Q.}} = -v\partial^\mu a + x\overline{u_{j\text{R}}}\gamma^\mu u_{j\text{R}} + \frac{1}{x}\overline{d_{j\text{R}}}\gamma^\mu d_{j\text{R}}.$$

Under the chiral $U(1)_{\text{A}}$ rotation (10.16.63b), the CP-violating term, $\mathcal{L}_{\text{CP}} \propto \overline{\theta}\varepsilon^{\mu\nu\rho\sigma}F_{\alpha\mu\nu}F_{\alpha\rho\sigma}$, in $\mathcal{L}^{\text{eff}}_{\text{SM}}$ in the presence of the n_{g} generations of the fermions assumes the following form:

$$\mathcal{L}_{\text{CP}} = \frac{g_3^2}{32\pi^2}\left(\overline{\theta} + 4n_{\text{g}}\alpha\right)\varepsilon^{\mu\nu\rho\sigma}F_{\alpha\mu\nu}F_{\alpha\rho\sigma}.$$

Since $\overline{\theta}$ is specified only up to mod(2π), $\mathcal{L}^{\text{eff}}_{\text{SM}}$ remains invariant under the transformation of α with the discrete $Z_{4n_{\text{g}}}$ group specified by $\alpha \to \alpha + (2\pi/4n_{\text{g}})n$ with n integer.

By the reduction of the $U(1)_{\text{P.Q.}}$ symmetry group to the discrete Z_{4n_g} group at the center of the spontaneously broken gauge group, we can resolve the domain wall problem in cosmology in the context of the $SO(10)$ grand unified model.

The presence of the last term in Eq. (10.16.65a) provides an effective potential for the axion field. Its vacuum expectation value is no longer arbitrary. The minimum of this potential determines the vacuum expectation value of the axion,

$$\left\langle \frac{\partial V_{\text{eff}}}{\partial a} \right\rangle = -\frac{\xi}{v} \frac{g_3^2}{32\pi^2} \left\langle \varepsilon^{\mu\nu\rho\sigma} F_{\alpha\mu\nu} F_{\alpha\rho\sigma} \right\rangle \Bigg|_{\langle a \rangle} = 0. \tag{10.16.66a}$$

The periodicity of the pseudoscalar expectation value $\left\langle \varepsilon^{\mu\nu\rho\sigma} F_{\alpha\mu\nu} F_{\alpha\rho\sigma} \right\rangle$ in the $\overline{\theta} + \xi \left\langle a \right\rangle / v$ parameter forces the axion vacuum expectation value to settle down at

$$\overline{\theta} + \frac{\xi \left\langle a \right\rangle}{v} = 0, \quad \text{mod}(2\pi), \quad \text{or} \quad \left\langle a \right\rangle = -\left(\frac{v}{\xi}\right)\overline{\theta}, \quad \left(\frac{v}{\xi}\right) \text{mod}(2\pi). \tag{10.16.66b}$$

This solves the strong CP-violation problem, since $\mathcal{L}_{\text{SM}}^{\text{eff}}$, when expressed in terms of the physical axion field

$$a_{\text{phys.}} = a - \left\langle a \right\rangle,$$

no longer contains the CP-violating term, $\mathcal{L}_{\text{CP}} \propto \overline{\theta} \varepsilon^{\mu\nu\rho\sigma} F_{\alpha\mu\nu} F_{\alpha\rho\sigma}$. Also the axion itself acquires the mass by expanding V_{eff} at its minimum,

$$m_a^2 = \left\langle \frac{\partial^2 V_{\text{eff}}}{\partial a^2} \right\rangle = -\frac{\xi}{v} \frac{g_3^2}{32\pi^2} \frac{\partial}{\partial a} \left\langle \varepsilon^{\mu\nu\rho\sigma} F_{\alpha\mu\nu} F_{\alpha\rho\sigma} \right\rangle \Bigg|_{\langle a \rangle}. \tag{10.16.67}$$

Therefore, the standard model with the global chiral $U(1)_{\text{P.Q.}}$ symmetry no longer has the dangerous CP-violating interaction. Instead it contains additional interactions of a massive axion field both with matter fields and gluon field as

$$\mathcal{L}_{\text{SM}}^{\text{eff}} = \mathcal{L}_{\text{SM}} + \mathcal{L}_{\text{int}} \left(\frac{\partial_\mu a_{\text{phys.}}}{v}; \psi \right) + \frac{1}{2} \partial^\mu a_{\text{phys.}}^\dagger \partial_\mu a_{\text{phys.}} - \frac{1}{2} m_a^2 a_{\text{phys.}}^\dagger a_{\text{phys.}}$$
$$+ \frac{a_{\text{phys.}}}{v} \xi \frac{g_3^2}{32\pi^2} \varepsilon^{\mu\nu\rho\sigma} F_{\alpha\mu\nu} F_{\alpha\rho\sigma}. \tag{10.16.65b}$$

Replacing the last term of the above axion Lagrangian density with effective interactions of axion with the light pseudoscalar mesons (π and η), we can estimate the standard axion mass as

$$m_a^{\text{st}} = \frac{f_\pi m_\pi}{v} \frac{\sqrt{m_u m_d}}{m_u + m_d} \simeq 25\text{KeV} \quad \text{and} \quad m_a^{\text{P.Q.}} = n_g \left(x + \frac{1}{x} \right) m_a^{\text{st}}. \tag{10.16.68a}$$

With the inclusion of the coupling of ϕ_d with the right-handed charged leptons $l_{j\mathrm{R}}$ in $\mathcal{L}^{\mathrm{quark}}_{\mathrm{Yukawa}}$, we have

$$\mathcal{L}_{\mathrm{Yukawa}} = -\left\{G_{u_j}\overline{(u_j, d_j)_{\mathrm{L}}}\phi_u u_{j\mathrm{R}} + G_{d_j}\overline{(u_j, d_j)_{\mathrm{L}}}\phi_d d_{j\mathrm{R}} + G_{l_j}\overline{l_{j\mathrm{L}}}\phi_d l_{j\mathrm{R}} + \mathrm{h.c.}\right\}.$$

The $U(1)_{\mathrm{P.Q.}}$ symmetry that guarantees the invariance of $\mathcal{L}_{\mathrm{Yukawa}}$ is under

$$\left\{\begin{array}{lcl} u_{j\mathrm{R}} & \longrightarrow & \exp\left[-i\alpha x\right] u_{j\mathrm{R}}, \\ d_{j\mathrm{R}} & \longrightarrow & \exp\left[-i(\alpha/x)\right] d_{j\mathrm{R}}, \\ l_{j\mathrm{R}} & \longrightarrow & \exp\left[-i(\alpha/x)\right] l_{j\mathrm{R}}, \end{array}\right. \qquad \left\{\begin{array}{lcl} \phi_u & \longrightarrow & \exp\left[i\alpha x\right] \phi_u, \\ \phi_d & \longrightarrow & \exp\left[i(\alpha/x)\right] \phi_d. \end{array}\right.$$

This transformation generates the $U(1)_{\mathrm{P.Q.}}$ current given by

$$J^{\mu}_{\mathrm{P.Q.}} = -v\partial^{\mu}a + x\overline{u_{j\mathrm{R}}}\gamma^{\mu}u_{j\mathrm{R}} + \frac{1}{x}\overline{d_{j\mathrm{R}}}\gamma^{\mu}d_{j\mathrm{R}} + \frac{1}{x}\overline{l_{j\mathrm{R}}}\gamma^{\mu}l_{j\mathrm{R}},$$

which has the global chiral anomaly,

$$\partial_{\mu}J^{\mu}_{\mathrm{P.Q.}} = \xi\frac{g_3^2}{32\pi^2}\varepsilon^{\mu\nu\rho\sigma}F_{\alpha\mu\nu}F_{\alpha\rho\sigma} + \xi_{\gamma}\frac{e^2}{16\pi^2}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma},$$

with ξ and ξ_{γ}, respectively, given by

$$\xi = n_{\mathrm{g}}\left(x + \frac{1}{x}\right) \quad \text{and} \quad \xi_{\gamma} = n_{\mathrm{g}}\frac{4}{3}\left(x + \frac{1}{x}\right).$$

The last term in Eq. (10.16.65a) is replaced with the following expression:

$$\frac{a}{v}\left(\xi\frac{g_3^2}{32\pi^2}\varepsilon^{\mu\nu\rho\sigma}F_{\alpha\mu\nu}F_{\alpha\rho\sigma} + \xi_{\gamma}\frac{e^2}{16\pi^2}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right).$$

To compute the effective interaction of the axion with the light hadrons, π and η, we employ the effective Lagrangian method. Since π and η are the Nambu–Goldstone bosons associated with the approximate $SU(2)_{\mathrm{L}} \times SU(2)_{\mathrm{R}}$ symmetry of QCD, their interactions are described by an effective chiral Lagrangian density given by

$$\mathcal{L}_{\mathrm{chiral}} = \frac{1}{4}f_{\pi}^2\mathrm{tr}\partial_{\mu}\Sigma^{\dagger}\partial^{\mu}\Sigma \quad \text{with} \quad \Sigma = \exp\left[\frac{i}{f_{\pi}}(\vec{\tau}\cdot\vec{\pi} + \eta)\right].$$

This chiral Lagrangian density must be augmented by the axion Lagrangian density given by

$$\mathcal{L}_{\mathrm{axion}} = -\frac{1}{2}f_{\pi}^2 m_{\pi}^2\,\mathrm{tr}\left[\Sigma A M + M^{\dagger}A^{\dagger}\Sigma^{\dagger}\right] + \frac{1}{2}\partial_{\mu}a^{\dagger}\partial^{\mu}a,$$

where

$$A = \begin{pmatrix} \exp[-iax/v] & 0 \\ 0 & \exp[-ia/xv] \end{pmatrix},$$

and

$$M = \begin{pmatrix} m_u/(m_u + m_d) & 0 \\ 0 & m_d/(m_u + m_d) \end{pmatrix}.$$

We are guaranteed the $U(1)_{\text{P.Q.}}$ invariance of $\mathcal{L}_{\text{axion}}$ since, under the $U(1)_{\text{P.Q.}}$ transformation, we have

$$\Sigma \longrightarrow \Sigma \begin{pmatrix} \exp[i\alpha x] & 0 \\ 0 & \exp[i(\alpha/x)] \end{pmatrix}.$$

The anomaly, which breaks the $SU(2)_{\text{L}} \times SU(2)_{\text{R}} \times U(1)_{\text{P.Q.}}$ symmetry through the coupling of the gluon, the axion and the η, serves to give an effective mass term to the field combination which couples to

$$\varepsilon^{\mu\nu\rho\sigma} F_{\alpha\mu\nu} F_{\alpha\rho\sigma}$$

as

$$\mathcal{L}_{\text{anomaly}} = -\frac{1}{2} m^2 \left[\eta + \frac{f_\pi}{v} \frac{n_{\text{g}} - 1}{2} \left(x + \frac{1}{x} \right) a \right]^2,$$

with

$$m^2 \simeq m_\eta^2 \gg m_\pi^2.$$

The quadratic terms in the axion in

$$\mathcal{L}_{\text{axion}} + \mathcal{L}_{\text{anomaly}},$$

when diagonalized, immediately yields the axion mixing with π^0 and η as

$$a \simeq a_{\text{phys.}} - \xi_{a\pi} \pi^0_{\text{phys.}} - \xi_{a\eta} \eta_{\text{phys.}}.$$

The electromagnetic anomaly of the π^0 and η fields,

$$\mathcal{L}_{\pi,\eta} = \frac{e^2}{16\pi^2} \left[\frac{\pi^0}{f_\pi} + \frac{5}{3} \frac{\eta}{f_\pi} \right] \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma},$$

through the mixing with the axion gives an effective $a\gamma\gamma$ coupling with the effective coupling constant,

$$K_{a\gamma\gamma} = n_{\text{g}} \left(x + \frac{1}{x} \right) \frac{m_u}{m_u + m_d}.$$

Experimental search for the axion around the predicted mass value revealed no evidence for its existence.

Undauntedly Dine, Fischler, and Srednicki further proposed the invisible axion scenario. They repeated the same analysis with the inclusion of the

$SU(2)_{\text{weak isospin}} \times U(1)_{\text{weak hypercharge}}$ singlet Higgs scalar field Φ besides the $SU(2)_{\text{weak isospin}}$ doublets, ϕ_u and ϕ_d, with ϕ_d also coupled to the right-handed charged leptons, where the Higgs scalar field Φ has the vacuum expectation value V. The Higgs scalar field Φ is not coupled to the quarks and the leptons but to the $SU(2)_{\text{weak isospin}}$ doublets in the following Higgs potential:

$$V(\phi_u, \phi_d, \Phi) = \sum_{q=u,d} \lambda_q \left(\phi_q^\dagger \phi_q - v_q^2\right)^2 + \lambda\left(\Phi^\dagger\Phi - V^2\right)^2$$
$$+ \left(a\phi_u^\dagger\phi_u + b\phi_d^\dagger\phi_d\right)\Phi^\dagger\Phi$$
$$+ c\left(\phi_u^i\varepsilon_{ij}\phi_d^j\Phi^\dagger\Phi + \text{h.c.}\right) + d\left(\phi_u^i\varepsilon_{ij}\phi_d^j\right)^\dagger \phi_u^i\varepsilon_{ij}\phi_d^j + e\left(\phi_u^\dagger\phi_d\right)^\dagger\phi_u^\dagger\phi_d. \tag{10.16.69}$$

We estimate the mass of the axion in this case by isolating the axion,

$$\phi_u = v_u \begin{pmatrix} 1 \\ 0 \end{pmatrix} \exp\left[i\frac{X_1 a}{V}\right], \quad \phi_d = v_d \begin{pmatrix} 0 \\ 1 \end{pmatrix} \exp\left[i\frac{X_2 a}{V}\right], \quad \Phi = V\exp\left[i\frac{a}{V}\right],$$

where

$$X_1 = \frac{2v_d^2}{v^2} \quad \text{and} \quad X_2 = \frac{2v_u^2}{v^2},$$

resulting in the estimate

$$m_a^{\text{inv.}} = \left(\frac{v}{V}\right) 2n_g m_a^{\text{st}}. \tag{10.16.68b}$$

These formulas correspond to the previous formula for ϕ_u and ϕ_d with the replacements,

$$x \longleftrightarrow X_1 \quad \text{and} \quad x^{-1} \longleftrightarrow X_2.$$

The invisible axion has a coupling to electrons given by

$$\mathcal{L}_{aee} = -i\frac{X_2 m_e}{V} a\bar{e}\gamma_5 e.$$

By taking V large, we can make m_a as well as the strength of the couplings with the other particles small, with the former by the order of magnitude v/V. Question is how small we should make the mass of the axion.

We shall now consider the cosmological constraint on V. In the early universe where the temperature T is $T > V$, the Higgs potential for the Higgs scalar field Φ is parabolic and the Higgs scalar field Φ behaves as a free field. After the phase transition $T < V$, the Higgs potential for the Higgs scalar field Φ is of the sombrero shape and the Higgs scalar field Φ settles down at $\Phi = \langle\Phi\rangle$. Since the

Higgs potential for the Higgs scalar field Φ depends on Φ in the form $\Phi^\dagger\Phi$, the vacuum expectation value $\langle\Phi\rangle$ has a phase degree of freedom by an amount $\alpha(x)$,

$$\left\langle\Phi\right\rangle = V\exp[i\alpha(x)]. \tag{10.16.70}$$

Since the true minimum occurs at $\alpha(x) = 0$, by power expanding $\langle\Phi\rangle$, we can consider that the Higgs scalar field Φ at each space–time point x has the axion field as

$$a(x) \cong V\alpha(x). \tag{10.16.71}$$

After the phase transition,

$$T < V,$$

we still have

$$T > f_\pi \text{ or } QCD \text{ scale } \Lambda_{QCD}, \tag{10.16.72}$$

so that the axion remains massless ($m_a = 0$). Namely at the temperature

$$f_\pi \text{ or } QCD \text{ scale } \Lambda_{QCD} < T < V, \tag{10.16.73}$$

the universe is filled with the static axion ($p_a = 0$). The field with $p_a = 0$ and $m_a = 0$ has no energy density. After further expansion of the universe, when

$$T < f_\pi \text{ or } QCD \text{ scale } \Lambda_{QCD}, \tag{10.16.74}$$

we start experiencing the instanton effect and the axion field acquires the mass. The energy density of the axion is given by the mass term of the Lagrangian density,

$$\rho_a = m_a^2 a^\dagger a \cong m_a^2 V^2 \alpha(x)^2 \cong 1.50 \times 10^{42}\alpha(x)^2 \ \text{KeV}/\text{cm}^3, \quad \alpha(x) = O(1). \tag{10.16.75}$$

From this point on, the amplitude $\alpha(x)$ of the axion decreases as a result of the expansion of the universe. We have the Boltzmann equation for the expansion of the universe,

$$\ddot{\alpha} + 3\frac{\dot{R}}{R}\dot{\alpha} + m_a(t)^2\alpha = 0,$$

where R is the scale factor of the universe and $\dot{R}/R$ is the Hubble constant H at time t.

Omitting the detailed analysis, we merely state the energy density ρ_a^0 of the axion at the present universe as

$$\rho_a^0 \cong 10\rho_c^0 \left(\frac{V}{10^{12}\ \text{GeV}}\right)^{7/6}, \tag{10.16.76}$$

where ρ_c^0 is the critical density of the present universe. We shall consider the two cases, $\rho > \rho_c^0$ and $\rho < \rho_c^0$.

If

$$\rho > \rho_c^0,$$

the universe keeps expanding permanently. If

$$\rho < \rho_c^0,$$

the universe returns to compress. At

$$\rho = \rho_c^0,$$

the curvature of the universe is 0 and hence Minkowskian. The value of ρ_c^0 is given by

$$\rho_c^0 = \frac{3}{8\pi}\frac{H_0^2}{G_N} = 11h_0^2\ \text{KeV}/\text{cm}^3.$$

The superscript and subscript "0" designate the present value. Writing

$$H_0 = h_0 \times \left(100\ \text{Kms}^{-1}\text{Mpc}^{-1}\right),$$

according to the observation, we have

$$\frac{1}{2} < h_0 < 1.$$

Invoking the inflationary scenario, from the equation for the current axion density ρ_a^0, we find the stringent cosmological constraint for V,

$$10^{10} < V < 10^{12}\ \text{GeV}, \tag{10.16.77a}$$

or

$$10^{-5} < m_a < 10^{-3}\ \text{eV}. \tag{10.16.77b}$$

If we have the axion density sufficient to close the universe,

$$\rho_a^0 > \rho_c^0,$$

the axion is the good candidate for the dark matter of the universe.

If the axion indeed serve to close the universe, it is possible to search for their traces experimentally. The axions in the dark matter of the universe can be converted in an external electromagnetic field into photons. Since the axions move with the nonrelativistic velocity, the converted photons will have a very sharp frequency distribution centered around the axion mass, m_a. Thus we can try to detect the signal of these axions by means of a variable frequency resonant cavity placed in an external electromagnetic field. Exploring the mass range of the axions of the order of

$$4.5 \times 10^{-4} < m_a < 5 \times 10^{-4} \text{ eV},$$

preliminary experimental result provided the nontrivial upper bound on the axion −photon−photon coupling constant as

$$K_{a\gamma\gamma} < 1.6.$$

There are compelling astrophysical and cosmological reasons for wanting some dark matter in the universe. The ''invisible'' axions have never been in thermal equilibrium and hence they are cold. The ''invisible'' axions are the most sensible candidate for this dark matter of the universe.

Extensive worldwide attempts to detect the axion from the universe with the variable frequency resonant cavity placed in an external electromagnetic field are underway at the moment.

Peccei−Quinn axion hypotheses and the invisible axion scenario are the extensions of the standard model.

10.17
Lattice Gauge Field Theory and Quark Confinement

Gauge Field Sector of Lattice Gauge Field Theory

In this section, we discuss the gauge sector of the lattice gauge field theory, its classical continuum limit, the path integral quantization of the lattice gauge field theory and the strong coupling limit of the lattice gauge field theory to realize the linearly rising potential for the quark confinement in the form of Wilson's area law.

The lattice gauge field theory is defined on the discrete Euclidean space−time by cutting the continuum Euclidean space−time into the hyper-square lattice of side a. We associate the link variables U's which connect the neighboring lattice sites with

$$U_\mu(\mathbf{n}) \equiv \exp[iaA_\mu] \equiv \exp[iaA_{\alpha\mu}T_\alpha]. \tag{10.17.1}$$

Letting $\hat{\mu}$ to be the unit vector in the μ-direction with $\mu = 1, 2, 3, 4$, we define

$$U_{-\mu}(\mathbf{n} + a\hat{\mu}) \equiv U_{\mu}(\mathbf{n})^{-1}. \tag{10.17.2}$$

The gauge transformation is defined by

$$U_{\mu}(\mathbf{n}) \longrightarrow U(\omega(\mathbf{n}))\, U_{\mu}(\mathbf{n})\, U(\omega(\mathbf{n} + \hat{\mu}))^{\dagger}. \tag{10.17.3}$$

The minimum lattice square, the plaquet $P_{\mu\nu}$, in the $\hat{\mu}$–$\hat{\nu}$ plane is defined by

$$\mathbf{n} \overset{U_{\mu}(\mathbf{n})}{\Rightarrow} \mathbf{n} + a\hat{\mu} \overset{U_{\nu}(\mathbf{n}+a\hat{\mu})}{\Rightarrow} \mathbf{n} + a\hat{\mu} + a\hat{\nu} \overset{U_{-\mu}(\mathbf{n}+a\hat{\mu}+a\hat{\nu})}{\Rightarrow} \mathbf{n} + a\hat{\nu} \overset{U_{-\nu}(\mathbf{n}+a\hat{\nu})}{\Rightarrow} \mathbf{n}. \tag{10.17.4}$$

The Euclidean action functional of the lattice gauge field theory is defined by

$$I_{\mathrm{E}}^{\mathrm{gauge}} \equiv \frac{1}{2g^2} \sum_{P_{\mu\nu}} \left\{ \mathrm{Tr}\left[U_{\mu}(\mathbf{n})\, U_{\nu}(\mathbf{n} + a\hat{\mu})\, U_{-\mu}(\mathbf{n} + a\hat{\mu} + a\hat{\nu})\, U_{-\nu}(\mathbf{n} + a\hat{\nu}) \right] + \mathrm{h.c.} - 2 \right\}. \tag{10.17.5}$$

The action functional for the gauge field, (10.17.5), is given by the sum over all plaquets $P_{\mu\nu}$ of the products of the four link variables, U's, of each plaquet $P_{\mu\nu}$. The action functional for the gauge field, (10.17.5), is clearly gauge invariant under the discrete gauge transformation, (10.17.3).

We observe that

$$\begin{aligned} U_{\nu}(\mathbf{n} + a\hat{\mu}) &= \exp\left[iaA_{\nu}(\mathbf{n} + a\hat{\mu}) \right] \approx \exp\left[iaA_{\nu}(\mathbf{n}) + ia^2 \partial_{\mu} A_{\nu}(\mathbf{n}) \right], \\ U_{-\mu}(\mathbf{n} + a\hat{\mu} + a\hat{\nu}) &= U_{\mu}(\mathbf{n} + a\hat{\nu})^{-1} \approx \exp\left[-iaA_{\mu}(\mathbf{n}) - ia^2 \partial_{\nu} A_{\mu}(\mathbf{n}) \right], \\ U_{-\nu}(\mathbf{n} + a\hat{\nu}) &= U_{\nu}(\mathbf{n})^{-1} = \exp\left[-iaA_{\nu}(\mathbf{n}) \right]. \end{aligned}$$

The trace term in the action functional, (10.17.5), in the classical continuum limit $a \to 0$ is given by

$$\begin{aligned} &\mathrm{Tr}\left[U_{\mu}(\mathbf{n})\, U_{\nu}(\mathbf{n} + a\hat{\mu})\, U_{-\mu}(\mathbf{n} + a\hat{\mu} + a\hat{\nu})\, U_{-\nu}(\mathbf{n} + a\hat{\nu}) \right] \\ &\approx \mathrm{Tr}\left[e^{iaA_{\mu}(\mathbf{n})} e^{iaA_{\nu}(\mathbf{n}) + ia^2 \partial_{\mu} A_{\nu}(\mathbf{n})} e^{-iaA_{\mu}(\mathbf{n}) - ia^2 \partial_{\nu} A_{\mu}(\mathbf{n})} e^{-iaA_{\nu}(\mathbf{n})} \right] \\ &\approx \mathrm{Tr} \exp\left[ia^2 \{ \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} + i[A_{\mu}, A_{\nu}] \} \right], \end{aligned} \tag{10.17.6}$$

where the use has been made of the Baker–Campbell–Hausdorff formula whose iterated commutator is truncated of the order a^2,

$$\exp[iaA_{\mu}(\mathbf{n})] \exp\left[iaA_{\nu}(\mathbf{n}) \right] \approx \exp\left[ia(A_{\mu}(\mathbf{n}) + A_{\nu}(\mathbf{n})) - \frac{a^2}{2} \left[A_{\mu}(\mathbf{n}), A_{\nu}(\mathbf{n}) \right] \right]. \tag{10.17.7}$$

The action functional, $I_{\mathrm{E}}^{\text{gauge}}$, in the classical continuum limit $a \to 0$, is given by

$$I_{\mathrm{E}}^{\text{gauge}} \approx -\frac{1}{2g^2} \sum_{P_{\mu\nu}} a^4 \mathrm{Tr}\left(F_{\mu\nu}(\mathbf{n}) F^{\mu\nu}(\mathbf{n})\right) \approx -\frac{1}{2g^2} \int d^4x \mathrm{Tr}\left(F_{\mu\nu}(x) F^{\mu\nu}(x)\right). \tag{10.17.8}$$

Hence, we recover the action functional of the continuum non-Abelian gauge field theory. In (10.17.6), the terms of the order $O(a^3)$ and the higher order terms do not contribute to the action functional, (10.17.8), due to the fact that the generators are traceless,

$$\mathrm{Tr} T_\alpha = 0.$$

Path integral quantization of the lattice gauge field theory is carried out by

$$\int \prod dU_\mu \exp[I_{\mathrm{E}}^{\text{gauge}}],$$

with the invariant Haar measure,

$$\int dU = 1, \quad \int dU f(U_0 U) = \int d(U_0 U) f(U_0 U) = \int dU f(U). \tag{10.17.9}$$

Wilson operator P_C is defined as the operator which takes the products of the link variables, Us, starting from some starting point and back to the starting point along the closed contour C. Its vacuum expectation value is given by

$$\langle P_C UU \cdots U \rangle = \int \prod dU_\mu (P_C UU \cdots U) \exp\left[I_{\mathrm{E}}^{\text{gauge}}\right], \tag{10.17.10}$$

where we take the contour C to be the rectangular contour with the sides $R \gg a$ in the space-like direction and $T \gg a$ in the time-like direction. We define $\beta \equiv 1/g^2$ in the action functional, (10.17.5), and consider the strong coupling limit $g \to \infty$. We expand $\exp[I_{\mathrm{E}}^{\text{gauge}}(10.17.5)]$ in powers of β. The dominant contribution to (10) comes from the plaquets, numbering RT/a^2, which are the contribution from the action functional and densely cover the interior of the closed contour C. Then, in the strong coupling limit $g \to \infty$, we have

$$\langle P_C UU \cdots U \rangle \approx \left(\frac{1}{2g^2}\right)^{RT/a^2} = \exp\left[-RT\left(\frac{1}{a^2}\right) \ln\left(2g^2\right)\right]. \tag{10.17.11}$$

The term multiplying $-T$ is the potential between the heavy quark and the heavy antiquark in the static limit,

$$V(R) = \alpha R \quad \text{with} \quad \alpha = \left(\frac{1}{a^2}\right) \ln\left(2g^2\right). \tag{10.17.12}$$

We have proven that *QCD* provides the quark confinement potential, if we can find the linearly rising potential in an appropriate continuum limit at some finite g.

Fermion Field Sector of Lattice Gauge Field Theory

In this section, we discuss the fermion sector of the lattice gauge field theory.

The action functional, $I_{\mathrm{E}}^{\mathrm{F}}$, for the fermion in the lattice gauge field theory is defined by

$$\begin{aligned} I_{\mathrm{E}}^{\mathrm{F}} \equiv a^4 \sum_x \Bigg\{ & \sum_\mu \frac{i}{2a}\Big[\overline{\psi}(x)\gamma^\mu U_\mu(x)\psi(x+a^\mu) - \overline{\psi}(x+a^\mu)\gamma^\mu U_\mu(x)^\dagger \psi(x)\Big] \\ & -m_0\overline{\psi}(x)\psi(x) + \sum_\mu \frac{r}{2a}\Big[\overline{\psi}(x)\gamma^\mu U_\mu(x)\psi(x+a^\mu) \\ & +\overline{\psi}(x+a^\mu)\gamma^\mu U_\mu(x)^\dagger\psi(x)\Big] - 2\overline{\psi}(x)\psi(x) \Bigg\}, \end{aligned} \tag{10.17.13}$$

where the summation over x is taken over each lattice site and $a^\mu \equiv a\hat{\mu}$. This fermion action functional, $I_{\mathrm{E}}^{\mathrm{F}}$, is locally gauge invariant under

$$\begin{aligned} U_\mu(x) &\longrightarrow U(\omega(x))\,U_\mu(x)\,U(\omega(x+a^\mu))^\dagger, \\ \psi(x) &\longrightarrow U(\omega(x))\psi(x), \\ \overline{\psi}(x) &\longrightarrow \overline{\psi}(x)\,U(\omega(x))^\dagger. \end{aligned} \tag{10.17.14}$$

Observing that, up to the second order, we have the following expansions of $U_\mu(x)$ and $\psi(x+a^\mu)$:

$$\begin{aligned} U_\mu(x) &= \exp\left[iaA_{\alpha\mu}(x)T_\alpha\right] \approx 1 + iaA_\mu(x) + \tfrac{1}{2}\left(iaA_\mu(x)\right)^2, \\ \psi(x+a^\mu) &\approx \psi(x) + a\partial_\mu\psi(x) + \tfrac{1}{2}a^2\partial_\mu^2\psi(x), \end{aligned} \tag{10.17.15}$$

the limit $a \to 0$ of the fermion action functional, $I_{\mathrm{E}}^{\mathrm{F}}$, is given by

$$\begin{aligned} I_{\mathrm{E}}^{\mathrm{F}} \approx \int d^4x \Bigg\{ & \frac{i}{2}\left[\overline{\psi}\gamma^\mu D_\mu\psi - D_\mu\overline{\psi}\gamma^\mu\psi\right] - m_0\overline{\psi}\psi \\ & + \frac{ar}{4}\left[\overline{\psi}D_\mu D^\mu\psi + \overline{\psi}\overleftarrow{D}_\mu\overleftarrow{D}^\mu\psi\right]\Bigg\}. \end{aligned} \tag{10.17.16}$$

The term proportional to r in the action functional, $I_{\mathrm{E}}^{\mathrm{F}}$, for the fermion in the lattice gauge field theory is called Wilson term. In order to understand its relevance, we shall consider the model with

$$U_\mu = 1 \quad \text{and} \quad r = 0. \tag{10.17.17}$$

Regarding p_μ as the momentum operator in the coordinate representation, we have

$$\exp[iap_\mu]\psi(x) = \psi(x+a^\mu).$$

Letting $\psi(x) \sim \exp[-ik_\mu x^\mu]$ and $\overline{\psi}(x) \sim \exp[ik_\mu x^\mu]$, we obtain the kernel of the quadratic part of the fermion action functional, $I_{\mathrm{E}}^{\mathrm{F}}$, as

$$\sum_\mu \frac{i}{2a}\left[\gamma^\mu \exp[-iak_\mu] - \exp[iak_\mu]\gamma^\mu\right] - m_0. \tag{10.17.18}$$

The two-point Green's function is given by

$$\left[\sum_\mu \frac{1}{a}\gamma^\mu \sin ak_\mu - m_0\right]^{-1}. \tag{10.17.19}$$

The momentum is defined in the fundamental Brillouin zone and we choose

$$-\frac{\pi}{2a} \leq k_\mu < \frac{3\pi}{2a}, \quad \mu = 1, 2, 3, 4. \tag{10.17.20}$$

While keeping the momentum in the region $k_\mu \approx 0$, we take the limit $a \to 0$ to obtain the "free" two-point Green's function for the fermion as

$$\left[\sum_\mu \gamma^\mu k_\mu - m_0\right]^{-1}. \tag{10.17.21}$$

However, if we let $k_1 \to k_1 + \pi/a$ and take the limit $a \to 0$, we obtain

$$\left[\sum_{\mu=2}^{4} \gamma^\mu k_\mu - \gamma^1 k_1 - m_0\right]^{-1}, \tag{10.17.22}$$

which is also the "free" two-point Green's function for the fermion with the momentum $k_1 \to -k_1$ and provides the fermion of mass m_0. We have two fermion poles in each direction, and end up with $2^4 = 16$ fermion poles for the "free" two-point Green's function.

This situation is called the species doubling. On the other hand, if we set $r \neq 0$, we obtain the two-point Green's function as

$$\left[\sum_\mu \frac{1}{a}\gamma^\mu \sin ak_\mu - m_0 - \sum_\mu \frac{r}{a}\left(1 - \cos ak_\mu\right)\right]^{-1}, \tag{10.17.23}$$

which removes the redundant fermion poles other than $k_\mu \approx 0$.

We consider the species doubling from the symmetry property. We define the operators T_μ by

$$T_\mu \equiv \gamma^\mu \gamma_5 \exp\left[i\pi \frac{x^\mu}{a}\right], \tag{10.17.24}$$

with the property $T_\mu T_\nu + T_\nu T_\mu = \delta_{\mu\nu}$. We define the 16 independent operators by

$$\begin{array}{cccccccc} 1, & T_1T_2, & T_1T_3, & T_1T_4, & T_2T_3, & T_2T_4, & T_3T_4, & T_1T_2T_3T_4, \\ T_1, & T_2, & T_3, & T_4, & T_1T_2T_3, & T_2T_3T_4, & T_3T_4T_1, & T_4T_1T_2. \end{array} \tag{10.17.25}$$

Designating one of these operators as T, the action functional, I_E^F, for the fermion in the lattice gauge field theory, (10.17.13), with $r = 0$, is invariant under the transformation specified by

$$\begin{array}{ccc} \psi(x) & \to & T\psi(x), \\ \overline{\psi}(x) & \to & \overline{\psi}(x)T^{-1}. \end{array} \tag{10.17.26}$$

Since T_μ adds the momentum by π/a in the μ direction, the 15 Ts other than $T = 1$ produce the 15 poles in the fundamental Brillouin zone, (10.17.20), from the pole with $k_\mu \approx 0$ for the "free" two-point Green's function for the fermion.

The 8 Ts in the first line of (10.17.25) commute with γ_5 while the 8 Ts in the second line of (10.17.25) anticommute with γ_5. Thus when we try to introduce the left-handed Weyl-type fermion into the theory, we end up with the equal number of the left-handed Weyl-type fermions and the right-handed Weyl-type fermions. Hence we face the technical difficulty for the implementation of the lattice regularization to the electro-weak gauge field theory. Success of the lattice gauge field theory approach to the infrared regime is limited to the strong coupling limit of QCD.

10.18 WKB Approximation in Path Integral Formalism

Customary WKB method in quantum mechanics is the short wavelength approximation to wave mechanics. We leave the details of the standard approach as an exercise to the reader. WKB method in quantum theory in path integral formalism consists of the replacement of general Lagrangian (density) with a quadratic Lagrangian (density). We begin with Euclidean quantum theory. As a preliminary, we first discuss the method of steepest descent.

Method of Steepest Descent

Consider the integral,

$$Z \equiv \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi g}} \exp\left[-\frac{I(x)}{g}\right]. \tag{10.18.1}$$

In the neighborhood of the stationary point of $I(x)$, $dI(x)/dx\big|_{x=x_0} = 0$, we Taylor-expand $I(x)$ around $x = x_0$,

$$I(x) = I_0 + \frac{I_0^{(2)}}{2}(x - x_0)^2 + \frac{I_0^{(3)}}{3!}(x - x_0)^3 + \frac{I_0^{(4)}}{4!}(x - x_0)^4 + \cdots .$$

Substituting this into Z, and replacing $x - x_0$ with x, we have

$$Z \simeq \exp\left[-\frac{I_0}{g}\right]\int_{-\infty}^{\infty}\frac{dx}{\sqrt{2\pi g}}\exp\left[-\frac{I_0^{(2)}}{2g}x^2 - \frac{I_0^{(3)}}{3!g}x^3 - \frac{I_0^{(4)}}{4!g}x^4 - \cdots\right]$$

$$\underset{x\to\sqrt{g}x}{=} \exp\left[-\frac{I_0}{g}\right]\int_{-\infty}^{\infty}\frac{dx}{\sqrt{2\pi g}}\exp\left[-\frac{I_0^{(2)}}{2}x^2 - \sqrt{g}\frac{I_0^{(3)}}{3!}x^3 - g\frac{I_0^{(4)}}{4!}x^4 - \cdots .\right]$$

$$\simeq \exp\left[-\frac{I_0}{g}\right]\int_{-\infty}^{\infty}\frac{dx}{\sqrt{2\pi g}}\exp\left[-\frac{I_0^{(2)}}{2}x^2\right]\left(1 - g\frac{I_0^{(4)}}{4!}x^4 + \frac{g}{2}(\frac{I_0^{(3)}}{3!})^2x^6 + O(g^2)\right).$$

We apply the Gaussian integral formula to obtain

$$Z = \exp\left[-\frac{I_0}{g}\right]\frac{1}{\sqrt{I_0^{(2)}}}\left(1 - \frac{g}{8}\frac{I_0^{(4)}}{(I_0^{(2)})^2} + \frac{5g}{24}\frac{(I_0^{(3)})^2}{(I_0^{(2)})^3} + O(g^2)\right). \tag{10.18.2}$$

WKB Approximation in Quantum Mechanics

Consider the Euclidean generating functional of (the connected parts of) Green's functions,

$$Z[\vec{J}] = \exp\left[W[\vec{J}]/\hbar\right] = \int \mathcal{D}\vec{q}\exp\left[\{I[\vec{q}] - (\vec{q}\vec{J})\}/\hbar\right]. \tag{10.18.3}$$

In the neighborhood of the stationary point of $I[\vec{q}]$,

$$\left.\frac{\delta I[\vec{q}]}{\delta\vec{q}(t)}\right|_{\vec{q}=\vec{q}_0} - \vec{J}(t) = 0,$$

we Taylor-expand $I[\vec{q}] - (\vec{q}\vec{J})$ around $\vec{q} = \vec{q}_0$:

$$I[\vec{q}] - (\vec{q}\vec{J}) = I_0 - (\vec{q}_0\vec{J}) + \frac{1}{2}\left(I_0^{(2)}(\vec{q} - \vec{q}_0)^2\right) + \frac{1}{3!}\left(I_0^{(3)}(\vec{q} - \vec{q}_0)^3\right) + \cdots,$$

$$(I_0^{(n)}\vec{q}^n) \equiv \int dt_1\cdots dt_n \left.\frac{\delta^n I}{\delta q_{r_1}(t_1)\cdots\delta q_{r_n}(t_n)}\right|_{\vec{q}=\vec{q}_0} q_{r_1}(t_1)\cdots q_{r_n}(t_n).$$

Substituting this into $Z[\vec{J}]$ and replacing $\vec{q}-\vec{q}_0$ with $\vec{q}$, and $\vec{q}$ with $\sqrt{\hbar}\vec{q}$,

$$\begin{aligned}
Z[\vec{J}] &\simeq \exp\left[\left\{I_0-(\vec{q}_0\vec{J})\right\}/\hbar\right]\int\mathcal{D}\vec{q}\exp\left[+\frac{1}{2\hbar}(I_0^{(2)}\vec{q}^2)+\frac{1}{3!\hbar}(I_0^{(3)}\vec{q}^3)-\cdots\right]\\
&=\exp\left[\left\{I_0-(\vec{q}_0\vec{J})\right\}/\hbar\right]\left(\mathrm{Det}(-I_0^{(2)})\right)^{-1/2}(1+O(\hbar))\\
&=\exp\left[\frac{I_0-(\vec{q}_0\vec{J})}{\hbar}-\frac{1}{2}\mathrm{Tr}\ln\left(-I_0^{(2)}\right)+O(\hbar)\right].
\end{aligned}$$

From this, it follows that

$$\begin{aligned}
&W[\vec{J}]=I_0-(\vec{q}_0\vec{J})-\frac{\hbar}{2}\mathrm{Tr}\ln\mathrm{Det}\Delta^{-1}[\vec{q}_0]+O(\hbar^2),\\
&I_0\equiv I[\vec{q}_0],\quad \Delta^{-1}_{r_1,r_2}[\vec{q}_0](t_1,t_2)\equiv-\left.\frac{\delta^2 I}{\delta q_{r_1}(t_1)\delta q_{r_2}(t_2)}\right|_{\vec{q}=\vec{q}_0}.
\end{aligned}$$

This is the path integral version of WKB approximation .

WKB Approximation in Quantum Field Theory

Consider now the Euclidean generating functional of (the connected parts of) Green'sfunctions,

$$Z[J]=\exp\left[W[J]/\hbar\right]=\int\mathcal{D}\phi\exp\left[\left\{I[\phi]-(\phi J)\right\}/\hbar\right].\tag{10.18.4}$$

In the neighborhood of the stationary point of $I[\phi]$,

$$\left.\frac{\delta I[\phi]}{\delta\phi(x)}\right|_{\phi=\phi_0}-J(x)=0,$$

we functional Taylor-expand $I[\phi]-(\phi J)$ around $\phi=\phi_0$,

$$\begin{aligned}
&I[\phi]-(\phi J)=I_0-(\phi_0 J)+\frac{1}{2}\left(I_0^{(2)}(\phi-\phi_0)^2\right)+\frac{1}{3!}\left(I_0^{(3)}(\phi-\phi_0)^3\right)+\cdots,\\
&\left(I_0^{(n)}\phi^n\right)\equiv\int d^4x_1\cdots d^4x_n\left.\frac{\delta^n I}{\delta\phi(x_1)\cdots\delta\phi(x_n)}\right|_{\phi=\phi_0}\phi(x_1)\cdots\phi(x_n).
\end{aligned}$$

Substituting this into $Z[J]$ and replacing $\phi-\phi_0$ with ϕ, and ϕ with $\sqrt{\hbar}\phi$, we get

$$\begin{aligned}
Z[J] &\simeq \exp\left[\left\{I_0-(\phi_0 J)\right\}/\hbar\right]\int\mathcal{D}\phi\exp\left[+\frac{1}{2\hbar}\left(I_0^{(2)}\phi^2\right)+\frac{1}{3!\hbar}\left(I_0^{(3)}\phi^3\right)-\cdots\right]\\
&=\exp\left[\left\{I_0-(\phi_0 J)\right\}/\hbar\right]\left(\mathrm{Det}(-I_0^{(2)})\right)^{-1/2}(1+O(\hbar))\\
&=\exp\left[\frac{I_0-(\phi_0 J)}{\hbar}-\frac{1}{2}\mathrm{Tr}\ln\left(-I_0^{(2)}\right)+O(\hbar)\right].
\end{aligned}$$

From this, it follows that

$$W[J] = I_0 - (\phi_0 J) - \frac{\hbar}{2}\mathrm{Tr}\ln \mathrm{Det}\Delta^{-1}[\phi_0] + O(\hbar^2),$$

$$I_0 \equiv I[\phi_0], \quad \Delta^{-1}[\phi_0](x_1, x_2) \equiv - \left.\frac{\delta^2 I}{\delta\phi(x_1)\delta\phi(x_2)}\right|_{\phi=\phi_0}.$$

This is the path integral version of WKB approximation .

Establishing the bounds on the error in the WKB approximation in Euclidean quantum theory is tedious but possible. When it comes to the Minkowskian quantum theory, the approximation method is the method of a stationary phase and then establishing the bounds on the error is almost impossible.

10.19
Hartree–Fock Equation

We consider the computation of the *ground state energy E of the system of A identical fermions* (*electrons or nucleons*).

Density Matrix Approach: By defining the density matrices for the one-body operators (for the kinetic energy term) and the two-body operators (for the potential energy term), it appears that the variational program based on the density matrices is straightforward. But it is not easy.

Hartree–Fock Program: Hartree introduced the product wavefunction of the one particle wavefunctions (orbitals) as the trial ground state wavefunction. Fock antisymmetrized the product wavefunction introduced by Hartree.

Slater employed the Slater determinant ϕ,

$$\phi(1, 2, ..., A) \equiv \frac{1}{\sqrt{A!}} \begin{vmatrix} \varphi_1(1) & \varphi_2(1) & \cdot & \varphi_A(1) \\ \varphi_1(2) & \varphi_2(2) & \cdot & \varphi_A(2) \\ \cdot & \cdot & \cdot & \cdot \\ \varphi_1(A) & \varphi_2(A) & \cdot & \varphi_A(A) \end{vmatrix}, \tag{10.19.1}$$

as the trial ground state wavefunction. In the Slater determinant, each orbital φ_α has the single particle energy ε_α. Hartree–Fock Program consists of choosing the Slater determinant $\phi(1, 2, \ldots, A)$ as the ground state wavefunction $\langle 1, 2, 3, \ldots, A \,|\Psi\rangle$ and finding the best determinantal wavefunction by demanding

$$\delta \langle\phi|\, H \,|\phi\rangle = 0, \tag{10.19.2}$$

where the Hamiltonian operator is known,

$$H = \sum_{i=1}^{A} t_i + \sum_{i<j}^{A} v_{ij}.$$

If the Slater determinant ϕ is chosen, we have

$$E = \sum_{\mu=1}^{A} t_{\mu\mu} + \frac{1}{2} \sum_{\mu,\nu=1}^{A} (v_{\mu\nu,\mu\nu} - v_{\mu\nu,\nu\mu}). \tag{10.19.3}$$

Here the matrix elements are defined by

$$t_{\alpha\beta} \equiv \langle\alpha| t |\beta\rangle = \int d\tau \varphi_\alpha^* t \varphi_\beta, \tag{10.19.4a}$$

$$v_{\alpha\beta,\gamma\delta} \equiv \langle\alpha\beta| v |\gamma\delta\rangle = \int d\tau_1 \cdot d\tau_{2'} \varphi_\alpha^*(1)\varphi_\beta^*(2) v_{12} \varphi_\gamma(1')\varphi_\delta(2'). \tag{10.19.4b}$$

Note that v_{12} is the matrix in the spin-space and isospin-space.

In implementing the variational program,

$$\delta \langle\phi| H |\phi\rangle = 0,$$

we consider the variation of the orbitals specified as

$$\varphi_\mu \to \varphi_\mu + \delta\varphi_\mu \quad \text{for all} \quad \mu \quad \text{with} \quad \delta\varphi_\mu = \sum_{\sigma>A} c_{\sigma\mu}\varphi_\sigma, \tag{10.19.5}$$

to induce $\phi' = \phi + \delta\phi$. In $\delta\varphi_\mu$, the inclusion of the term, $\sum_{\nu\le A} c_{\nu\mu}\varphi_\nu$, is excluded because such term does not contribute to the Slater determinant (recall the property of the determinant).

Now we have

$$\langle\phi'| H |\phi'\rangle = \langle\phi| H |\phi\rangle + \text{Series in } (c_{\sigma\mu}, c_{\sigma\mu}^*). \tag{10.19.6}$$

Defining $v_{\mu\nu,\mu\nu}^{(a)}$ as

$$v_{\mu\nu,\mu\nu}^{(a)} \equiv v_{\mu\nu,\mu\nu} - v_{\mu\nu,\nu\mu}, \tag{10.19.7}$$

we explicitly have

$$\langle\phi'| H |\phi'\rangle - \langle\phi| H |\phi\rangle = \sum_{\sigma\mu} \left(c_{\sigma\mu}^* t_{\sigma\mu} + t_{\mu\sigma} c_{\sigma\mu}\right) + \frac{1}{2} \sum_{\sigma\mu\nu} \Big(c_{\sigma\mu}^* v_{\sigma\nu,\mu\nu}^{(a)} + c_{\sigma\nu}^* v_{\mu\sigma,\mu\nu}^{(a)} + v_{\mu\nu,\sigma\nu}^{(a)} c_{\sigma\mu} + v_{\mu\nu,\mu\sigma}^{(a)} c_{\sigma\nu}\Big). \tag{10.19.8}$$

Recall symmetry properties,

$$v_{\alpha\beta,\gamma\delta} = v_{\beta\alpha,\delta\gamma}, \quad \sum_{\mu\nu} v_{\mu\nu,\mu\nu} = \sum_{\mu\nu} v_{\mu\nu,\nu\mu}. \tag{10.19.9}$$

We then have

$$\sum_{\sigma\mu\nu} c^*_{\sigma\nu} v_{\mu\sigma,\mu\nu} = \sum_{\sigma\mu\nu} c^*_{\sigma\nu} v_{\sigma\mu,\nu\mu} = \sum_{\sigma\mu\nu} c^*_{\sigma\mu} v_{\sigma\nu,\mu\nu}, \tag{10.19.10a}$$

$$\sum_{\sigma\mu\nu} v_{\mu\nu,\mu\sigma} c_{\sigma\nu} = \sum_{\sigma\mu\nu} v_{\nu\mu,\sigma\mu} c_{\sigma\nu} = \sum_{\sigma\mu\nu} v_{\mu\nu,\sigma\nu} c_{\sigma\mu}. \tag{10.19.10b}$$

Now

$$\langle\phi'|\,H\,|\phi'\rangle - \langle\phi|\,H|\,\phi\rangle = \sum_{\sigma\mu} c^*_{\sigma\mu}\left(t_{\sigma\mu} + \sum_{\nu=1}^{A} v^{(a)}_{\sigma\nu,\mu\nu}\right) + \sum_{\sigma\mu}\left(t_{\mu\sigma} + \sum_{\nu=1}^{A} v^{(a)}_{\mu\nu,\sigma\nu}\right) c_{\sigma\mu} + \text{quadratic term in } (c, c^*). \tag{10.19.9}$$

The vanishing of the variational derivatives with respect to c and c^*,

$$\frac{\delta}{\delta c_{\sigma\mu}}\left(\langle\phi'|\,H\,|\phi'\rangle - \langle\phi|\,H|\,\phi\rangle\right) = \frac{\delta}{\delta c^*_{\sigma\mu}}\left(\langle\phi'|\,H\,|\phi'\rangle - \langle\phi|\,H|\,\phi\rangle\right) = 0,$$

gives

$$t_{\sigma\mu} + U_{\sigma\mu} = 0, \quad t_{\mu\sigma} + U_{\mu\sigma} = 0, \tag{10.19.12}$$

or

$$h_{\sigma\mu} = 0, \quad h_{\mu\sigma} = 0, \tag{10.19.13}$$

where we define $U_{\sigma\mu}$, $U_{\mu\sigma}$, $h_{\sigma\mu}$, and $h_{\mu\sigma}$ as

$$U_{\sigma\mu} \equiv \sum_{\nu=1}^{A} v^{(a)}_{\sigma\nu,\mu\nu}, \quad U_{\mu\sigma} \equiv \sum_{\nu=1}^{A} v^{(a)}_{\mu\nu,\sigma\nu}, \tag{10.19.14}$$

$$h_{\sigma\mu} \equiv t_{\sigma\mu} + U_{\sigma\mu}, \quad h_{\mu\sigma} \equiv t_{\mu\sigma} + U_{\mu\sigma}. \tag{10.19.15}$$

We define

$$h_{\alpha\beta} \equiv t_{\alpha\beta} + U_{\alpha\beta}, \tag{10.19.16}$$

$$U_{\alpha\beta} \equiv \sum_{\nu=1}^{A} v^{(a)}_{\alpha\nu,\beta\nu} = \iint \sum_{\nu=1}^{A} \varphi^*_\alpha(1)\varphi^*_\nu(2) v_{12} \varphi_\beta(1)\varphi_\nu(2), \tag{10.19.17}$$

for all indices, α and β, irrespective of $\alpha \le A$, $\alpha > A$, $\beta \le A$ or $\beta > A$.

Table of $h_{\alpha\beta}$.

	$\beta = \nu \le A$	$\beta = \tau > A$
$\alpha = \mu \le A$	$h_{\mu\nu}$,	$h_{\mu\tau} \equiv 0$,
$\alpha = \sigma > A$	$h_{\sigma\nu} \equiv 0$,	$h_{\sigma\tau}$.

The upper right entry and the lower left entry of the table are what the variational principle tells us. This implementation of the variational principle can be sharpened to $h_{\alpha\beta}$ diagonal. Diagonalization is carried out by

$$\varphi_\mu \longrightarrow \varphi'_\mu = \sum_\nu R_{\mu\nu}\varphi_\nu, \quad R_{\mu\nu}\text{: unitary.} \tag{10.19.18}$$

Then the Slater determinant gets changed into

$$\phi \rightarrow \phi' = (\text{const})\phi,$$

and the result of the variation is unaltered. Then the orbital φ_α is the eigenfunction of the operator h with the eigenvalue ε_α. Thus we obtain Hartree–Fock equation,

$$t_{\alpha\beta} + U_{\alpha\beta} = \varepsilon_\alpha \delta_{\alpha\beta}. \tag{10.19.19}$$

Hartree–Fock equation is a nonlinear equation. Its solution is effected by iteration, starting from some initial guess $\varphi_\alpha^{(0)}$.

There remains two important questions. One question has to do with the stability of the iterative solutions. Namely, do they provide the true minimum? The second variation of $\langle\phi|\, H\,|\phi\rangle$ should be examined. Hence Legendre test and Jacobi test are in order. Another question has to do with the degeneracy of the Hartree–Fock solution.

10.20 Problems for Chapter 10

10.1. (due to H. C.). Find the solution or solutions $q(t)$ which extremize

$$I \equiv \int_0^T \left[\frac{m}{2}\dot{q}^2 - \frac{1}{6}q^6\right] dt,$$

subject to

$$q(0) = q(T) = 0.$$

10.2. (due to H. C.). Find the solution $q(t)$ which extremizes

$$I \equiv \int_0^T \left[\frac{m}{2}\dot{q}^2 - \frac{\lambda}{3}q^3\right] dt,$$

subject to

$$q(0) = q(T) = 0.$$

10.3. The action for a particle in a gravitational field is given by

$$I \equiv -m \int \sqrt{g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}} dt,$$

where $g_{\mu\nu}$ is the metric tensor. Show that the motion of this particle is governed by

$$\frac{d^2 x^\rho}{ds^2} = -\Gamma^\rho_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds},$$

with

$$\Gamma^\rho_{\mu\nu} \equiv \frac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}),$$

which is called the Christoffel symbol.

Hint: This problem corresponds to a free fall of the particle in the curved space–time.

10.4. (due to H. C.). The space–time structure in the presence of a black hole is given by

$$(ds)^2 = \left(1 - \frac{1}{r}\right)(dt)^2 - (dr)^2 \Big/ \left(1 - \frac{1}{r}\right) - r^2 \left[\sin^2\theta (d\phi)^2 + (d\theta)^2\right],$$

and the motion of a particle is such that $\int ds$ is minimized. Let the particle move in the x–y plane and hence $\theta = \frac{\pi}{2}$. Then the equations of motion of this particle subject to the gravitational pull of the black hole are obtained by extremizing

$$s \equiv \int_{t_i}^{t_f} \sqrt{\left(1 - \frac{1}{r}\right) - \left(\frac{dr}{dt}\right)^2 \Big/ \left(1 - \frac{1}{r}\right) - r^2 \left(\frac{d\phi}{dt}\right)^2}\, dt,$$

with initial and final coordinates fixed.

(a) Derive the equation of motion obtained by varying ϕ. Integrate this equation once to obtain an equation with one integration constant.

(b) Derive the equation of motion obtained by varying r. Find a way to obtain an equation involving $\frac{dr}{dt}$ and $\frac{d\phi}{dt}$ and a second integration constant.

(c) Let the particle be at $r = 2$, $\phi = 0$, with $\frac{dr}{dt} = \frac{d\phi}{dt} = 0$ at the initial time $t = 0$. Determine the motion of this particle as best you can. How long does it take for this particle to reach $r = 1$?

10.5. (due to H. C.). The invariant distance ds in the neigborhood of a black hole is given by

$$(ds)^2 = \left(1 - \frac{2M}{r}\right)(dt)^2 - (dr)^2 \Big/ \left(1 - \frac{2M}{r}\right),$$

where r, θ, ϕ (we set $\theta = \phi =$ constant) are the spherical polar coordinates and M is the mass of the black hole. The motion of a particle extremizes the invariant distance.
(a) Write down the integral which should be extremized. From the expression of this integral, find a first-order equation satisfied by $r(t)$.
(b) If $(r - 2M)$ is small and positive, solve the first-order equation. How much time does it take for the particle to fall to the critical distance $r = 2M$?

10.6. (due to H. C.). Find the kink solution by extremizing

$$I \equiv \int_{-\infty}^{+\infty} dt \left(\frac{1}{2}\phi_t^2 - \frac{m^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 + \frac{1}{4}\frac{m^4}{\lambda} \right).$$

10.7. Extremize

$$I \equiv \int_0^{x_0} dx \int d\Omega \tilde{f}(x,\theta) \left[\cos\theta \frac{\partial f(x,\theta)}{\partial x} + f(x,\theta) \right.$$

$$\left. - \frac{\kappa}{4\pi} \int w(\vec{n} - \vec{n}_0) f(x,\theta_0) d\Omega_0 \right],$$

treating f and $\tilde{f}$ as independent. Here κ is a constant, the unit vectors, $\vec{n}$ and $\vec{n}_0$, are pointing in the direction specified by spherical angles, (θ,φ) and (θ_0,φ_0), and $d\Omega_0$ is the differential solid angle at $\vec{n}_0$. Obtain the steady-state transport equation for anisotropic scattering from the very heavy scatterers,

$$\cos\theta \frac{\partial f(x,\theta)}{\partial x} = -f(x,\theta) + \frac{\kappa}{4\pi} \int w(\vec{n} - \vec{n}_0) f(x,\theta_0) d\Omega_0,$$

$$-\cos\theta \frac{\partial \tilde{f}(x,\theta)}{\partial x} = -\tilde{f}(x,\theta) + \frac{\kappa}{4\pi} \int w(\vec{n}_0 - \vec{n}) \tilde{f}(x,\theta_0) d\Omega_0.$$

Interpret the result for $\tilde{f}(x,\theta)$.

10.8. Extremize

$$I \equiv \int dt d^3\vec{x} \mathcal{L} \left(\psi, \frac{\partial}{\partial t}\psi, \vec{\nabla}\psi, \varphi, \frac{\partial}{\partial t}\varphi, \vec{\nabla}\varphi \right),$$

where

$$\mathcal{L} = -\vec{\nabla}\varphi\vec{\nabla}\psi - \frac{a^2}{2}\left(\varphi\frac{\partial}{\partial t}\psi - \psi\frac{\partial}{\partial t}\varphi \right),$$

treating ψ and φ as independent. Obtain the diffusion equation

$$\vec{\nabla}^2\psi(t,\vec{x}) = a^2\frac{\partial\psi(t,\vec{x})}{\partial t},$$

$$\vec{\nabla}^2\varphi(t,\vec{x}) = -a^2\frac{\partial\varphi(t,\vec{x})}{\partial t}.$$

Interpret the result for φ.

10.9. Extremize

$$I \equiv \int dt d^3\vec{x} \left\{ -\frac{1}{2m} \left(\left(\frac{\hbar}{i}\vec{\nabla} - \frac{e}{c}\vec{A} \right) \psi \right)^* \left(\left(\frac{\hbar}{i}\vec{\nabla} - \frac{e}{c}\vec{A} \right) \psi \right) + \frac{1}{2} \left[\psi^* \left(i\hbar\frac{\partial}{\partial t} - e\phi \right) \psi + \left(\left(i\hbar\frac{\partial}{\partial t} - e\phi \right) \psi \right)^* \psi \right] - \psi^* V \psi \right\},$$

$$\vec{\nabla}\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} = 0,$$

treating ψ and ψ^* as independent. Obtain the Schrödinger equation

$$\left(i\hbar\frac{\partial}{\partial t} - e\phi \right) \psi = \frac{1}{2m} \left(\frac{\hbar}{i}\vec{\nabla} - \frac{e}{c}\vec{A} \right)^2 \psi + V\psi,$$

$$-\left(i\hbar\frac{\partial}{\partial t} + e\phi \right) \psi^* = \frac{1}{2m} \left(\frac{\hbar}{i}\vec{\nabla} + \frac{e}{c}\vec{A} \right)^2 \psi^* + V\psi^*.$$

Demonstrate that the Schrödinger equation is invariant under the gauge transformation

$$\begin{array}{lllll} \vec{A} & \rightarrow & \vec{A}' & = & \vec{A} + \vec{\nabla}\Lambda, \\ \phi & \rightarrow & \phi' & = & \phi - (1/c)(\partial\Lambda/\partial t), \\ \psi & \rightarrow & \psi' & = & \exp[(ie/\hbar c)\Lambda]\psi, \end{array} \quad \text{where} \quad \left(\vec{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \right)\Lambda = 0.$$

10.10. Extremize

$$I \equiv \int dt d^3\vec{x} \left\{ -\left| \left(\frac{1}{i}\vec{\nabla} - e\vec{A} \right) \psi \right|^2 + \left| \left(i\frac{\partial}{\partial t} - e\phi \right) \psi \right|^2 - m^2 |\psi|^2 \right\},$$

$$\vec{\nabla}\vec{A} + \frac{\partial\phi}{\partial t} = 0,$$

treating ψ and ψ^* as independent. Obtain the Klein–Gordon equation

$$\left(i\frac{\partial}{\partial t} - e\phi \right)^2 \psi - \left(\frac{1}{i}\vec{\nabla} - e\vec{A} \right)^2 \psi = m^2\psi,$$

$$\left(i\frac{\partial}{\partial t} + e\phi \right)^2 \psi^* - \left(\frac{1}{i}\vec{\nabla} + e\vec{A} \right)^2 \psi^* = m^2\psi^*.$$

10.11. Extremize

$$I \equiv \int d^4x \mathcal{L}_{\text{tot}},$$

where $\mathcal{L}_{\text{tot}}$ is given by

$$\mathcal{L}_{\text{tot}} = \frac{1}{4}\left[\overline{\psi}_\alpha(x), D_{\alpha\beta}(x)\psi_\beta(x) \right] + \frac{1}{4}\left[D^{\text{T}}_{\beta\alpha}(-x)\overline{\psi}_\alpha(x), \psi_\beta(x) \right] + \frac{1}{2}\phi(x)K(x)\phi(x) + \mathcal{L}_{\text{int}}(\phi(x), \psi(x), \overline{\psi}(x)),$$

with $D_{\alpha\beta}(x)$, $D^{\mathrm{T}}_{\beta\alpha}(-x)$ and $K(x)$ given by

$$D_{\alpha\beta}(x) = (i\gamma_\mu\partial^\mu - m + i\varepsilon)_{\alpha\beta},$$
$$D^{\mathrm{T}}_{\beta\alpha}(-x) = (-i\gamma^{\mathrm{T}}_\mu\partial^\mu - m + i\varepsilon)_{\beta\alpha},$$
$$K(x) = -\partial^2 - \kappa^2 + i\varepsilon,$$

and $\mathcal{L}_{\mathrm{int}}$ is given by the Yukawa coupling specified by

$$\mathcal{L}_{\mathrm{int}}(\phi(x), \psi(x), \overline{\psi}(x)) = -G_0\overline{\psi}_\alpha(x)\gamma_{\alpha\beta}(x)\psi_\beta(x)\phi(x).$$

The $\gamma^{\mu'}$s are the Dirac γ matrices with the property specified by

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu},$$
$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0.$$

The $\psi(x)$ is the four-component Dirac spinor and the $\overline{\psi}(x)$ is the Dirac adjoint of $\psi(x)$ defined by

$$\overline{\psi}(x) \equiv \psi^\dagger(x)\gamma^0.$$

(a) Obtain the Euler–Lagrange equations of motion for the ψ field, the $\overline{\psi}$ field and the ϕ field.

(b) Causal Green's function $\Delta_{\mathrm{F}}(x - x')$ for Klein–Gordon field $\phi(x)$ is defined by

$$i\Delta_{\mathrm{F}}(x - x') = \int \frac{d^3k}{(2\pi)^3 2\sqrt{k^2 + m^2}} \{\theta(t - t')\exp\left[-ik(x - x')\right] + \theta(t' - t)\exp\left[ik(x - x')\right]\}$$

$$= i\int \frac{d^4k}{(2\pi)^4}\exp\left[-ik(x - x')\right]\frac{1}{k^2 - m^2 + i\varepsilon}.$$

Show that $\Delta_{\mathrm{F}}(x - x')$ satisfies the following differential equation:

$$(-\partial^2 - m^2)\Delta_{\mathrm{F}}(x - x') = \delta^4(x - x').$$

(c) Causal Green's function $S_{\mathrm{F}}(x - x')$ for Dirac field $\psi(x)$ is defined by

$$iS_{\mathrm{F}}(x - x') = \int \frac{d^3p}{(2\pi)^3}\frac{m}{E}\{\theta(t - t')\Lambda_+(p)\exp\left[-ip(x - x')\right] + \theta(t' - t)\Lambda_-(p)\exp\left[ip(x - x')\right]\}$$
$$= i\int \frac{d^4p}{(2\pi)^4}\exp\left[-ip(x - x')\right]\frac{\gamma_\mu p^\mu + m}{p^2 - m^2 + i\varepsilon},$$

with the positive (negative) energy projection operator, $\Lambda_{\pm}(p)$, defined by

$$\Lambda_{+}(p) = \frac{\gamma_\mu p^\mu + m}{2m} \quad \text{and} \quad \Lambda_{-}(p) = \frac{-\gamma_\mu p^\mu + m}{2m}.$$

Show that $S_{\mathrm{F}}(x - x')$ satisfies the following differential equation:

$$(i\gamma^\mu \partial_\mu - m) S_{\mathrm{F}}(x - x') = \delta^4(x - x').$$

(d) Show that $\Delta_{\mathrm{F}}(x - x')$ and $S_{\mathrm{F}}(x - x')$ are related by

$$S_{\mathrm{F}}(x - x') = (i\gamma^\mu \partial_\mu + m)\Delta_{\mathrm{F}}(x - x').$$

10.12. Derive Lagrange equations of motion for quantum mechanics,

$$\frac{d}{dt}\left(\frac{\partial L(\hat{q}_s(t), \dot{\hat{q}}_s(t))}{\partial \dot{\hat{q}}_r(t)}\right) - \frac{\partial L(\hat{q}_s(t), \dot{\hat{q}}_s(t))}{\partial \hat{q}_r(t)} = 0,$$

from the Heisenberg equations of motion, the definition of the Hamiltonian, and the equal time canonical commutators,

$$\frac{d\hat{q}_r(t)}{dt} = i\left[H(\hat{q}_s(t), \hat{p}_s(t)), \hat{q}_r(t)\right], \quad \frac{d\hat{p}_r(t)}{dt} = i\left[H(\hat{q}_s(t), \hat{p}_s(t)), \hat{p}_r(t)\right],$$

$$H(\hat{q}_s(t), \hat{p}_s(t)) = \sum_r \hat{p}_r(t)\dot{\hat{q}}_r(t) - L(\hat{q}_s(t), \dot{\hat{q}}_s(t)),$$

$$[\hat{q}_r(t), \hat{p}_s(t)] = i\hbar\delta_{rs}, \quad [\hat{q}_r(t), \hat{q}_s(t)] = [\hat{p}_r(t), \hat{p}_s(t)] = 0,$$

where the canonical momentum is defined by

$$\hat{p}_r(t) = \frac{\partial L(\hat{q}_s(t), \dot{\hat{q}}_s(t))}{\partial \dot{\hat{q}}_r(t)}.$$

Hint for Problem 10.12:
Nishijima, K.: *Theory of Fields*, Kinokuniya-shoten, 1986, Tokyo. Chapter 1. (In Japanese).

10.13. Derive Euler–Lagrange equations of motion for quantum field theory,

$$\partial_\mu\left(\frac{\partial \mathcal{L}(\hat{\phi}(x), \partial_\mu \hat{\phi}(x))}{\partial(\partial_\mu \hat{\phi}(x))}\right) - \frac{\partial \mathcal{L}(\hat{\phi}(x), \partial_\mu \hat{\phi}(x))}{\partial \hat{\phi}(x)} = 0,$$

from the Heisenberg equations of motion, the definition of the Hamiltonian, the equal time canonical commutators, and the definition of the canonical momentum.

Hint for Problem 10.13:
Nishijima, K.: *Fields and Particles: Field Theory and Dispersion Relations*, Benjamin Cummings, Massachusetts, 1969 and 1974.

10.14. Consider the Stueckelberg Lagrangian density for vector meson theory,

$$\mathcal{L}(A^{\rho}) = -\frac{1}{4}F_{\rho\sigma}F^{\rho\sigma} + \frac{1}{2}\mu^2 A_{\rho}A^{\rho} - \frac{1}{2}\xi(\partial_{\rho}A^{\rho})^2 - J_{\rho}A^{\rho}; \quad F_{\rho\sigma} \equiv \partial_{\rho}A_{\sigma} - \partial_{\sigma}A_{\rho},$$

with J^{ρ} the conserved current, $\partial_{\rho}J^{\rho} = 0$.

(a) Extremize the action functional, $I[A^{\rho}] = \int d^4x\mathcal{L}(A^{\rho})$. Obtain the field equation,

$$(\partial^2 + \mu^2)A^{\rho} - (1-\xi)\partial^{\rho}(\partial_{\sigma}A^{\sigma}) = J^{\rho}.$$

Taking the divergence of both sides of the above field equation, we obtain

$$\xi\left[\partial^2 + \frac{\mu^2}{\xi}\right](\partial_{\rho}A^{\rho}) = 0.$$

We let $m^2 \equiv \mu^2/\xi$ with $\xi > 0$. Show that the "transverse" field defined by

$$A_{\rho}^{\mathrm{T}} = A_{\rho} + \frac{1}{m^2}\partial_{\rho}(\partial_{\sigma}A^{\sigma}) = A_{\rho} + \frac{\xi}{\mu^2}\partial_{\rho}(\partial_{\sigma}A^{\sigma})$$

is divergenceless.

(b) By canonical quantization with operators satisfying

$$\left[a^{(\lambda)}(k), a^{(\lambda')\dagger}(k')\right] = \delta_{\lambda\lambda'}(2\pi)^3 2\sqrt{\vec{k}^2 + \mu^2}\,\delta^3(\vec{k} - \vec{k}'), \quad 1 \le \lambda, \lambda' \le 3,$$

$$\left[a^{(0)}(k), a^{(0)\dagger}(k')\right] = -(2\pi)^3 2\sqrt{\vec{k}^2 + m^2}\,\delta^3(\vec{k} - \vec{k}'),$$

and all other commutators vanishing, show that the field has the Fourier decomposition on two hyperboloids,

$$A_{\rho}(x) = \int \frac{d^3\vec{k}}{2(2\pi)^3\sqrt{\vec{k}^2 + \mu^2}} \times \sum_{\lambda=1}^{3}\left[a^{(\lambda)}(k)\varepsilon_{\rho}^{(\lambda)}(k)\exp[-ikx] + a^{(\lambda)\dagger}(k)\varepsilon_{\rho}^{(\lambda)*}(k)\exp[ikx]\right]$$
$$+ \int \frac{d^3\vec{k}}{2(2\pi)^3\sqrt{\vec{k}^2 + m^2}}\frac{k_{\rho}}{\mu}\left[a^{(0)}(k)\exp[-ikx] + a^{(0)\dagger}(k)\exp[ikx]\right].$$

The polarizations, $\varepsilon_\rho^{(\lambda)}(k)$, for $\lambda = 1, 2, 3$ are three orthonormal space-like four-vectors orthogonal to k_ρ $(k^2 = \mu^2)$. Show that the two-point "free" Green's function is given by

$$\frac{1}{i}\langle 0|\, \mathrm{T}(A_\rho(x)A_\sigma(y))\,|0\rangle = -\int \frac{d^4k}{(2\pi)^4}\exp\left[-ik(x-y)\right]\left(\frac{\eta_{\rho\sigma}-k_\rho k_\sigma/\mu^2}{k^2-\mu^2+i\varepsilon}+\frac{k_\rho k_\sigma/\mu^2}{k^2-\mu^2/\xi+i\varepsilon}\right).$$

(c) When $\xi \to 0$ and $\mu \neq 0$, show that we recover Proca formalism of massive vector meson theory.

(d) Prove the identity,

$$\frac{\eta_{\rho\sigma}-k_\rho k_\sigma/\mu^2}{k^2-\mu^2+i\varepsilon}+\frac{k_\rho k_\sigma/\mu^2}{k^2-\mu^2/\xi+i\varepsilon} = \frac{\eta_{\rho\sigma}}{k^2-\mu^2+i\varepsilon}+\frac{1-\xi}{\xi}\frac{k_\rho k_\sigma}{(k^2-\mu^2/\xi+i\varepsilon)(k^2-\mu^2+i\varepsilon)}.$$

In the limit $\mu \to 0$ and $\xi \neq 0$, (m goes to zero), we obtain the two-point "free" Green's function as

$$\frac{1}{i}\langle 0|\, \mathrm{T}(A_\rho(x)A_\sigma(y))\,|0\rangle = -\int \frac{d^4k}{(2\pi)^4}\exp\left[-ik(x-y)\right]\left(\frac{\eta_{\rho\sigma}}{k^2+i\varepsilon}+\frac{1-\xi}{\xi}\frac{k_\rho k_\sigma}{(k^2+i\varepsilon)^2}\right),$$

which is the covariant gauge "free" Green's function for the electromagnetic field.

Hint for Problem 10.14:

Stueckelberg, E.C.G.; Helv. Phys. Acta. **11**, (1938), 225.

Nishijima, K.; *Fields and Particles: Field Theory and Dispersion Relations,* Benjamin Cummings, Massachusetts, 1969 and 1974, Chapter 2.

Itzykson, C., and Zuber, J.B. ; *Quantum Field Theory*, McGraw-Hill, New York, 1985, Chapter 3.

The analysis in this problem is the intuitive motivation for the Nakanishi–Lautrup B field.

10.15. Extremize the action functional for the electromagnetic field A_μ,

$$I = \int d^4x \left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + B\partial^\mu A_\mu + \frac{1}{2}\alpha B^2\right),$$
$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Obtain the Euler–Lagrange equations of motion for A_μ field and B field. Can you perform the q-number gauge transformation after canonical quantization?

10.16. Extremize the action functional for the neutral massive vector field U_μ,

$$I = \int d^4x \left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}m_0^2 U^\mu U_\mu\right),$$
$$F_{\mu\nu} \equiv \partial_\mu U_\nu - \partial_\nu U_\mu.$$

Obtain the Euler–Lagrange equation of motion for U_μ field. Examine the massless limit $m_0 \to 0$ after canonical quantization.

10.17. Extremize the action functional for the neutral massive vector field A_μ,

$$I = \int d^4x \left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}m_0^2 A^\mu A_\mu + B\partial^\mu A_\mu + \frac{1}{2}\alpha B^2\right),$$
$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Obtain the Euler–Lagrange equations of motion for A_μ field and B field. Examine the massless limit $m_0 \to 0$ after canonical quantization.

Hint for Problems 10.15, 10.16, and 10.17:
Lautrup, B.: Mat. Fys. Medd. Dan. Vid. Selsk. **35(11)**. 29. (1967).
Nakanishi, N.: Prog. Theor. Phys. Suppl. **51**. 1. (1972).
Yokoyama, K.: Prog. Theor. Phys. **51**. 1956. (1974), **52**. 1669. (1974).

10.18. Derive the Schwinger–Dyson equation for the self-interacting scalar field $\hat{\phi}(x)$, whose Lagrangian density is given by

$$\mathcal{L}\left(\hat{\phi}(x), \partial_\mu\hat{\phi}(x)\right) = \frac{1}{2}\partial_\mu\hat{\phi}(x)\partial^\mu\hat{\phi}(x) - \frac{1}{2}m^2\hat{\phi}^2(x) - \frac{\lambda_4}{4!}\hat{\phi}^4(x).$$

Hint: Introduce the proper self-energy part $\mathbf{\Pi}^*(x, y)$ and the vertex operator $\mathbf{\Lambda}_4(x, y, z, w)$, and mimic the discussion in Section 10.4.

10.19. Derive the Schwinger–Dyson equation for the self-interacting scalar field $\hat{\phi}(x)$ whose Lagrangian density is given by

$$\mathcal{L}\left(\hat{\phi}(x), \partial_\mu\hat{\phi}(x)\right) = \frac{1}{2}\partial_\mu\hat{\phi}(x)\partial^\mu\hat{\phi}(x) - \frac{1}{2}m^2\hat{\phi}^2(x) - \frac{\lambda_3}{3!}\hat{\phi}^3(x) - \frac{\lambda_4}{4!}\hat{\phi}^4(x).$$

Hint: Introduce the proper self-energy part $\Pi^*(x, y)$ and the vertex operators $\Lambda_3(x, y, z)$ and $\Lambda_4(x, y, z, w)$, and mimic the discussion in Section 10.4.

10.20. Derive the Schwinger–Dyson equation for the ps–ps meson theory whose Lagrangian density is given by

$$\mathcal{L} = \frac{1}{4}\left[\widehat{\overline{\psi}}_\alpha(x), D_{\alpha\beta}(x)\hat{\psi}_\beta(x)\right] + \frac{1}{4}\left[D^{\mathrm{T}}_{\beta\alpha}(-x)\widehat{\overline{\psi}}_\alpha(x), \hat{\psi}_\beta(x)\right]$$
$$+ \frac{1}{2}\hat{\phi}(x)K(x)\hat{\phi}(x) - ig_0\widehat{\overline{\psi}}_\alpha(x)(\gamma_5)_{\alpha\beta}\hat{\psi}_\beta(x)\hat{\phi}(x),$$

where the kernels of the quadratic part of the Lagrangian density are given by

$$D_{\alpha\beta}(x) = (i\gamma_\mu\partial^\mu - m + i\varepsilon)_{\alpha\beta}, \quad D^{\mathrm{T}}_{\beta\alpha}(-x) = (-i\gamma_\mu^{\mathrm{T}}\partial^\mu - m + i\varepsilon)_{\beta\alpha},$$

$$K(x) = -\partial^2 - \kappa^2 + i\varepsilon,$$

with

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \quad (\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0, \quad \widehat{\overline{\psi}}_\alpha(x) = (\hat{\psi}^\dagger(x)\gamma^0)_\alpha.$$

Hint: Introduce the proper self-energy parts, $\mathbf{\Pi}^*(x, y)$ and $\mathbf{\Sigma}^*(x, y)$, and the vertex operator $\mathbf{\Lambda}(x, y, z)$, and mimic the discussion in Section 10.4.

10.21. Consider the bound state problem of zero total momentum $\vec{P} = 0$ for the system of identical two fermions of the mass m exchanging a spinless and massless boson whose Lagrangian density is given by

$$\mathcal{L} = \widehat{\overline{\psi}}(x)(i\gamma_\mu\partial^\mu - m + i\varepsilon)\hat{\psi}(x) \\ + \frac{1}{2}\partial_\mu\hat{\phi}(x)\partial^\mu\hat{\phi}(x) - g\widehat{\overline{\psi}}(x)\hat{\psi}(x)\hat{\phi}(x).$$

Define the bound state wavefunction of the two fermions in configuration space by

$$[U_{\vec{P}}(x)]_{\alpha\beta} = \left\langle 0 \left| \mathrm{T}[\hat{\psi}_\alpha\left(\frac{x}{2}\right)\hat{\psi}_\beta\left(-\frac{x}{2}\right)] \right| B \right\rangle,$$

and the bound state wavefunction in momentum space by

$$[u_{\vec{P}}(p)]_{\alpha\beta} = \int d^4x \exp[ipx][U_{\vec{P}}(x)]_{\alpha\beta}.$$

Also define the zero total momentum $\vec{P} = 0$ bound state wavefunction $\chi(p)$ in momentum space by

$$[u_{\vec{P}=0}(p)]_{\alpha\beta} = \frac{\delta_{\alpha\beta}\chi(p)}{p^2 - m^2 + i\varepsilon}.$$

(a) Show that the Bethe–Salpeter equation for the bound state in the ladder approximation is given by

$$\chi(p) = ig^2 \int \frac{d^4q}{(2\pi)^4}\left[\frac{1}{(p-q)^2 + i\varepsilon} - \frac{1}{(p+q)^2 + i\varepsilon}\right]\frac{\chi(q)}{q^2 - m^2 + i\varepsilon}.$$

(b) Solve this eigenvalue problem by dropping the antisymmetrizing term in the kernel above. The antisymmetrizing term originates from the spin-statistics relation for the fermions. This approximation provides the solution in terms of the Gauss hypergeometric function.

(c) Our discussion so far is covariant. Discuss the covariant normalization of the bound state wavefunction.

Hint: This problem is discussed in the following article.

Goldstein, J.; Phys. Rev. **91**, 1516, (1953).

10.22. In Wick–Cutkosky model, we consider the bound state problem for the system of distinguishable two spinless bosons of equal mass m exchanging a spinless and massless boson. The Lagrangian density of this model is given by

$$\mathcal{L} = \sum_{i=1}^{2} \left\{ \frac{1}{2}\partial_\mu\hat{\phi}_i(x)\partial^\mu\hat{\phi}_i(x) - \frac{1}{2}m^2\hat{\phi}_i^2(x) \right\} + \frac{1}{2}\partial_\mu\hat{\phi}(x)\partial^\mu\hat{\phi}(x)$$
$$-g\hat{\phi}_1^\dagger(x)\hat{\phi}_1(x)\hat{\phi}(x) - g\hat{\phi}_2^\dagger(x)\hat{\phi}_2(x)\hat{\phi}(x).$$

(a) Show that the Bethe–Salpeter equation for the bound state of the two bosons $\hat{\phi}_1(x_1)$ and $\hat{\phi}_2(x_2)$ is given by

$$S'_{\mathrm{F}}(x_1, x_2;\, B) = \int d^4x_3 d^4x_4 \Delta_{\mathrm{F}}(x_1 - x_3)\Delta_{\mathrm{F}}(x_2 - x_4)$$
$$\times(-g^2)D_{\mathrm{F}}(x_3 - x_4)S'_{\mathrm{F}}(x_3, x_4;\, B),$$

where $\Delta_{\mathrm{F}}(x)$ and $D_{\mathrm{F}}(x)$ are given by

$$\Delta_{\mathrm{F}}(x) = \int \frac{d^4k}{(2\pi)^4}\frac{\exp[ikx]}{k^2 - m^2 + i\varepsilon},$$

and

$$D_{\mathrm{F}}(x) = \int \frac{d^4k}{(2\pi)^4}\frac{\exp[ikx]}{k^2 + i\varepsilon}.$$

(b) Transform the coordinates x_1 and x_2 to the center-of-mass coordinate X and the relative coordinate x by

$$X = \frac{1}{2}(x_1 + x_2), \quad x = x_1 - x_2,$$

and the momentum p_1, and p_2 to the center-of-mass momentum P and the relative momentum p by

$$P = p_1 + p_2, \quad p = \frac{1}{2}(p_1 - p_2).$$

Define the Fourier transform $\Psi(p)$ of $S'_{\mathrm{F}}(x_1, x_2;\, B)$ by

$$S'_{\mathrm{F}}(x_1, x_2;\, B) = \exp[-iPX]\int d^4p \exp[-ipx]\Psi(p).$$

Show that the above Bethe–Salpeter equation assumes the following form in momentum space:

$$\left[\left(\frac{P}{2} + p\right)^2 - m^2\right]\left[\left(\frac{P}{2} - p\right)^2 - m^2\right]\Psi(p) = ig^2\int \frac{d^4q}{(2\pi)^4}\frac{\Psi(q)}{(p - q)^2 + i\varepsilon}.$$

(c) Assume that $\Psi(p)$ can be expressed as

$$\Psi(p) = -\int_{-1}^{1} \frac{g(z)dz}{[p^2 + zpP - m^2 + (P^2/4) + i\varepsilon]^3}.$$

Substitute this expression into the Bethe–Salpeter equation in momentum space above. Carry out the q integration using the formula,

$$\int d^4q \frac{1}{(p-q)^2 + i\varepsilon} \cdot \frac{1}{[q^2 + zqP - m^2 + (P^2/4) + i\varepsilon]^3}$$
$$= \frac{i\pi^2}{2[-m^2 + (P^2/4) - z^2(P^2/4)]} \cdot \frac{1}{[p^2 + zpP - m^2 + (P^2/4) + i\varepsilon]},$$

and compare the result with the original expression for $\Psi(p)$ to obtain the integral equation for $g(z)$ as

$$g(z) = \int_0^1 \varsigma d\varsigma \int_{-1}^{1} dy \int_{-1}^{1} dx \frac{\lambda g(x)}{2(1-\eta^2+\eta^2x^2)} \delta\left(z - \left\{\varsigma y + (1-\varsigma)x\right\}\right).$$

The dimensionless coupling constant λ is given by $\lambda = (g/4\pi m)^2$ and the squared mass of the bound state is given by $M^2 = P^2 = 4m^2\eta^2$, $0 < \eta < 1$.

(d) Performing the ς integration, we obtain the integral equation for $g(z)$ as

$$g(z) = \lambda \int_z^1 dx \frac{1+z}{1+x} \frac{g(x)}{2(1-\eta^2+\eta^2x^2)} + \lambda \int_{-1}^{z} dx \frac{1-z}{1-x} \frac{g(x)}{2(1-\eta^2+\eta^2x^2)}.$$

Observe that $g(z)$ satisfies the boundary conditions

$$g(\pm 1) = 0.$$

(e) Differentiating the integral equation for $g(z)$ obtained above twice, reduce it to the second-order ordinary differential equation for $g(z)$,

$$\frac{d^2}{dz^2} g(z) = -\frac{\lambda}{1-z^2} \frac{g(z)}{1-\eta^2+\eta^2z^2} \quad \text{with} \quad g(\pm 1) = 0.$$

This is the eigenvalue problem.

(f) In the limit, $1 \gg 1 - \eta > 0$, the function

$$\frac{1}{1-\eta^2+\eta^2z^2},$$

has a sharp peak at $z = 0$. Approximated this function by

$$\frac{1}{1-\eta^2+\eta^2z^2} \approx \delta(z) \int_{-1}^{1} \frac{dz}{1-\eta^2+\eta^2z^2} \approx \frac{\pi}{(1-\eta^2)^{1/2}} \delta(z),$$

and obtain the approximate differential equation for $g(z)$ as

$$\frac{d^2}{dz^2}g(z) = -\frac{\lambda\pi}{(1-\eta^2)^{1/2}}g(0)\delta(z).$$

Show that the solution for this differential equation satisfying the boundary condition $g(\pm 1) = 0$ is given by

$$g(z) = \frac{\pi}{2}\frac{1}{(1-\eta^2)^{1/2}}g(0)\lambda(1-|z|),$$

which is nodeless and hence represents the lowest eigenfunction. Show that the lowest approximate eigenvalue is given by

$$\lambda \approx (2/\pi)\sqrt{1-\eta^2}.$$

Hint: We cite the following articles and the following books for the Wick–Cutkosky model (scalar meson theory), which is a solvable case of the Bethe–Salpeter equation.
Wick, G.C.; Phys. Rev. **96**, 1124, (1954).
Cutkosky, R.E.; Phys. Rev. **96**, 1135, (1954).
Nishijima, K.; *Fields and Particles: Field Theory and Dispersion Relations*, Benjamin Cummings, Massachusetts, 1969 and 1974, Chapter 7.
Itzykson, C., and Zuber, J.B. ; *Quantum Field Theory*, McGraw-Hill, New York, 1985, Chapter 10.

10.23. Solve the integro-differential equation,

$$\frac{d^2X(t)}{dt^2} = 2C\int_0^\beta [X(t)-X(s)]\exp[-w|t-s|]\,ds - f(t),$$

with the boundary conditions

$$X(0) = X(\beta) = 0 \quad \text{and} \quad f(t) = ik\left[\delta(t-\tau) - \delta(t-\sigma)\right].$$

(a) Setting

$$Z(t) \equiv \frac{w}{2}\int_0^\beta \exp[-w|t-s|]\,X(s)ds,$$

show that

$$\frac{d^2Z(t)}{dt^2} = w^2[Z(t) - X(t)],$$

and that the original integro-differential equation becomes

$$\frac{d^2X(t)}{dt^2} = \frac{4C}{w}[X(t) - Z(t)] - f(t).$$

(b) Derive the second-order ordinary differential equation for $X(t) - Z(t)$ and solve for $X(t) - Z(t)$.

(c) Determine $X(t)$ in the limit of $\beta \to \infty$.

10.24. Repeat the analysis of the polaron problem discussed in Section 10.6 with the following trial action functional:

$$I_1 = -\frac{1}{2}\int \left(\frac{d\vec{q}(\tau)}{d\tau}\right)^2 d\tau - \frac{1}{2}C\iint \left(\vec{q}(\tau) - \vec{q}(\sigma)\right)^2 \exp[-w\,|\tau - \sigma|]d\tau\, d\sigma,$$

with w and v defined below as the variational parameters,

$$v^2 = w^2 + \frac{4C}{w}.$$

Hint for Problems 10.23 and 10.24: the polaron problems are discussed in the following article:
Feynman, R.P.: Phys. Rev. **97**., 660, (1955).

10.25. Solve a system of nonlinear differential equations in *QED*,

$$\frac{1}{d_{A^2}}\frac{d}{d\varsigma}(d_{A^2}) = \frac{4}{3}\frac{g^2}{16\pi^2}\Gamma^2 h^2_{\psi\overline{\psi}} d_{A^2}, \quad \frac{1}{h_{\psi\overline{\psi}}}\frac{d}{d\varsigma}(h_{\psi\overline{\psi}}) = -\frac{g^2}{16\pi^2}\Gamma^2 h^2_{\psi\overline{\psi}} d_{A^2},$$
$$\frac{1}{\Gamma}\frac{d}{d\varsigma}(\Gamma) = -\frac{g^2}{16\pi^2}\Gamma^2 h^2_{\psi\overline{\psi}} d_{A^2} \quad \text{with} \quad d_{A^2}(0) = h_{\psi\overline{\psi}}(0) = \Gamma(0) = 1.$$

10.26. Solve a system of nonlinear differential equations in *QCD*,

$$\begin{aligned}
\frac{1}{d_{A^2}}\frac{d}{d\varsigma}(d_{A^2}) &= -\frac{19}{4}\frac{C_2(G)}{3}\frac{g^2}{16\pi^2}\Gamma^2_{A^3} d^3_{A^2} - \frac{C_2(G)}{12}\frac{g^2}{16\pi^2}\Gamma^2_{c\overline{c}A} h^2_{c\overline{c}} d_{A^2} \\
&\quad + \frac{8T(R)}{6}\frac{g^2}{16\pi^2}\Gamma^2_{\psi\overline{\psi}A} h^2_{\psi\overline{\psi}} d_{A^2}, \\
\frac{1}{\Gamma_{c\overline{c}A}}\frac{d}{d\varsigma}(\Gamma_{c\overline{c}A}) &= -\frac{C_2(G)}{8}\frac{g^2}{16\pi^2}\Gamma^2_{c\overline{c}A} h^2_{c\overline{c}} d_{A^2} - \frac{3C_2(G)}{8}\frac{g^2}{16\pi^2}\Gamma_{c\overline{c}A}\Gamma_{A^3} d^2_{A^2} h_{c\overline{c}}, \\
\frac{1}{h_{c\overline{c}}}\frac{d}{d\varsigma}(h_{c\overline{c}}) &= -\frac{C_2(G)}{2}\frac{g^2}{16\pi^2}\Gamma^2_{c\overline{c}A} h^2_{c\overline{c}} d_{A^2}, \\
\frac{1}{h_{\psi\overline{\psi}}}\frac{d}{d\varsigma}(h_{\psi\overline{\psi}}) &= -\frac{C_2(G)}{2}\frac{g^2}{16\pi^2}\Gamma^2_{\psi\overline{\psi}A} h^2_{\psi\overline{\psi}} d_{A^2}, \\
\Gamma_{A^3} d_{A^2} &= \Gamma_{c\overline{c}A} h_{c\overline{c}}, \quad \Gamma_{A^3} d_{A^2} = \Gamma_{\psi\overline{\psi}A} h_{\psi\overline{\psi}}, \\
d_{A^2}(0) &= h_{c\overline{c}}(0) = h_{\psi\overline{\psi}}(0) = \Gamma_{c\overline{c}A}(0) = 1.
\end{aligned} \tag{10.20.10}$$

Hint for Problems 10.25 and 10.26: Many unknowns satisfy the identical nonlinear differential equation.

10.27. In the Glashow–Weinberg–Salam model, there exist two options to introduce the heavy leptons to eliminate $Z\overline{\nu}\nu$ coupling.

(a) We introduce the charged heavy lepton E^+ and form the left-handed triplet given by

$$\text{left-handed} : \vec{L} = \begin{pmatrix} E^+ \\ \nu \\ e^- \end{pmatrix}_{\mathrm{L}}, \quad Y_{\text{weak hypercharge}} = 0,$$

and the right-handed $SU(2)_{\text{weak isospin}}$ singlets,

$$e_{\mathrm{R}}^-, \quad Y_{\text{weak hypercharge}} = -2 \quad \text{and} \quad E_{\mathrm{R}}^+, \quad Y_{\text{weak hypercharge}} = +2.$$

Show that the neutral current is given by

$$j_\mu^3 = \overline{E_{\mathrm{L}}^+}\gamma_\mu E_{\mathrm{L}}^+ - \overline{e_{\mathrm{L}}^-}\gamma_\mu e_{\mathrm{L}}^-,$$

which contains no $\overline{\nu}\gamma_\mu\nu$ term and hence neither A_μ nor Z_μ couple to the neutrinos.

(b) We can further introduce the neutral heavy lepton E^0. The right-handed singlet E_{R}^0 has $Y_{\text{weak hypercharge}} = 0$. The two left-handed doublets are given by

$$\begin{pmatrix} (\nu + E^0)/\sqrt{2} \\ e^- \end{pmatrix}_{\mathrm{L}}, \quad Y_{\text{weak hypercharge}} = -1,$$

and

$$\begin{pmatrix} E^+ \\ (\nu - E^0)/\sqrt{2} \end{pmatrix}_{\mathrm{L}}, \quad Y_{\text{weak hypercharge}} = +1.$$

Show that neither the hypercharge current $j_\mu^{\text{weak hypercharge}}$ nor the neutral current j_μ^3 contains $\overline{\nu}\gamma_\mu\nu$ term.

(c) Incorporate the hadrons with the $SU(3)_{\text{color}}$ quark model in the models in parts (a) and (b).

We note that the model in part (a) is known as LPZ model and the model in part (b) is known as PZ II model.

Hint for Problem 10.27: This problem is discussed in the following article. Prentki, J. and Zumino, B. : Nucl. Phys. **B47**, 99, (1972).

10.28. One model of the elementary particle interactions which avoids the neutral current entirely aside from the electromagnetic current was proposed by Georgi and Glashow. The gauge group is $O(3)$.

(a) Show that with a triplet Higgs scalar field $\vec{\phi}$, the Higgs–Kibble mechanism gives a mass to the charged bosons, but leaves the neutral boson massless.

(b) In order to incorporate the leptons, the following scheme is suggested:

$$\text{left-handed} : \vec{L} = \begin{pmatrix} E^+ \\ \nu \sin\beta + E^0 \cos\beta \\ e^- \end{pmatrix}_{\mathrm{L}}, \quad \text{right-handed} : \vec{R} = \begin{pmatrix} E^+ \\ E^0 \\ e^- \end{pmatrix}_{\mathrm{R}},$$

$(E^0 \sin\beta - \nu\cos\beta)_{\mathrm{L}}$ is a singlet.

Show that the neutral vector boson couples in a parity-conserving manner. Calculate the effective weak coupling constant G_{F} in terms of e, β, and M_{W}. Derive an upper bound for M_{W},

$$M_{\mathrm{W}} \leq \sqrt{\frac{e^2\sqrt{2}}{4G_{\mathrm{F}}}}.$$

(c) Write down the most general mass matrix that can arise from the explicit mass terms as well as from the Yukawa couplings to the Higgs scalar field of the form

$$-\mathcal{L}_{\text{mass}} = m_0(\overline{\vec{L}}\vec{R} + \overline{\vec{R}}\vec{L}) + G_1[\overline{\vec{L}}\vec{T}\left\langle\vec{\phi}\right\rangle\vec{R} + \text{h.c.}]$$
$$+ G_2[\overline{(E^0\sin\beta - \nu\cos\beta)}_{\mathrm{L}}\left\langle\vec{\phi}\right\rangle\vec{R} + \text{h.c.}].$$

By diagonalizing the mass matrix, derive the relation

$$M_{E^+} + M_{e^-} = 2\cos\beta M_{E^0}.$$

Hint for Problem 10.28: This problem is discussed in the following article.
Georgi, H. and Glashow, S.L. : Phys. Rev. Lett. **28**, 1494, (1972).

The Georgi–Glashow $O(3)$ model was ruled out experimentally due to the discovery of the neutral current. There exist many models of the unification of weak and electromagnetic interactions which were ruled out.

10.29. One important feature of the Glashow–Weinberg–Salam model is the existence of the weak neutral leptonic current. To expose some striking consequences of the existence of weak neutral leptonic currents, consider the elastic $e + \nu \rightarrow e + \nu$ cross sections to the lowest order in the Glashow–Weinberg–Salam model. In the limit where the incident neutrino energy E_ν is small as compared with the masses of the W and Z bosons, $m_{W^\pm}$ and m_{Z^0}, show that the effective interaction Lagrangian density is given by

$$\mathcal{L}_{\text{int}}^{\text{eff}} = -\frac{1}{\sqrt{2}}G_{\mathrm{F}}\left\{\overline{\nu}_e\gamma^\rho(1-\gamma_5)e][\overline{e}\gamma_\rho(1-\gamma_5)\nu_e] + (\overline{\nu}_\mu\gamma^\rho\nu_\mu\right.$$
$$\left.+\overline{\nu}_e\gamma^\rho\nu_e)(2\sin^2\theta_{\mathrm{W}}\overline{e}_R\gamma_\rho e_R - \cos 2\theta_{\mathrm{W}}\overline{e}_L\gamma_\rho e_L)\right\},$$

where the first term and the second term represent the $W^{\pm}$ contribution and the Z^0 contribution, respectively.

After a Fierz transformation on the first term, show that the effective interaction Lagrangian density is

$$\mathcal{L}_{\text{int}}^{\text{eff}} = -\sqrt{2}G_{\text{F}} \left\{ \left[\overline{\nu}_e \gamma^\rho \frac{1-\gamma_5}{2} \nu_e \right] \left[\overline{e} \gamma_\rho \left(\frac{1-\gamma_5}{2} + 2\sin^2\theta_{\text{W}} \right) e \right] \right.$$
$$\left. + \left[\overline{\nu}_\mu \gamma^\rho \frac{1-\gamma_5}{2} \nu_\mu \right] [\overline{e} \gamma_\rho \left(-\frac{1-\gamma_5}{2} + 2\sin^2\theta_{\text{W}} \right) e] \right\}.$$

Show that a general effective interaction Lagrangian density,

$$\mathcal{L}_{\text{int}}^{\text{eff}} = -\sqrt{2}G_{\text{F}} \left\{ \left[\overline{\nu} \gamma^\rho \frac{1-\gamma_5}{2} \nu \right] \left[\overline{e} \gamma_\rho \left(C_{\text{L}} \frac{1-\gamma_5}{2} + C_{\text{R}} \frac{1+\gamma_5}{2} \right) e \right] \right.,$$

where C_{L} and C_{R} are real coefficients, leads to a differential cross-section formula in the center of mass frame,

$$\frac{d\sigma}{d\Omega} = \frac{G_{\text{F}}^2 E_\nu^2}{(2\pi)^2} \left[C_{\text{L}}^2 (p \cdot q)^2 + C_{\text{R}}^2 (p \cdot q')^2 - C_{\text{L}} C_{\text{R}} m_e^2 (q \cdot q')^2 \right],$$

for the elastic scattering process, $e(p) + \overline{\nu}(q) \to e(p') + \overline{\nu}(q')$. Show that, for the elastic scattering process, $e(p) + \nu(q) \to e(p') + \nu(q')$, the expression for the differential cross section is obtained by interchanging the coefficients, C_{L} and C_{R}.

When the incident neutrino energy is much larger than the electron mass in the center of mass frame, show that we have

$$(p \cdot q)^2 = m_e^2 E_\nu^2, \quad (p \cdot q')^2 = m_e^2 E_\nu^2 (1 - \frac{E_e'}{E_\nu})^2,$$

$$m_e^2 q \cdot q' \simeq m_e^2 p \cdot p' = m_e^2 E_\nu^2 \frac{E_e'}{E_\nu} \frac{m_e}{E_\nu},$$

and the last term above may be neglected. Obtain the differential cross sections,

$$\frac{d\sigma}{d\Omega}(\overline{\nu}_e e \to \overline{\nu}_e e) = \frac{G_{\text{F}}^2 E_\nu^2}{(2\pi)^2} \left[4\sin^4\theta_{\text{W}} + (1 + 2\sin^2\theta_{\text{W}})^2 \left(\frac{1+\cos\theta}{2} \right)^2 \right],$$
$$\frac{d\sigma}{d\Omega}(\nu_e e \to \nu_e e) = \frac{G_{\text{F}}^2 E_\nu^2}{(2\pi)^2} \left[(1 + 2\sin^2\theta_{\text{W}})^2 + 4\sin^4\theta_{\text{W}} \left(\frac{1+\cos\theta}{2} \right)^2 \right],$$
$$\frac{d\sigma}{d\Omega}(\overline{\nu}_\mu e \to \overline{\nu}_\mu e) = \frac{G_{\text{F}}^2 E_\nu^2}{(2\pi)^2} \left[4\sin^4\theta_{\text{W}} + (1 - 2\sin^2\theta_{\text{W}})^2 \left(\frac{1+\cos\theta}{2} \right)^2 \right],$$
$$\frac{d\sigma}{d\Omega}(\nu_\mu e \to \nu_\mu e) = \frac{G_{\text{F}}^2 E_\nu^2}{(2\pi)^2} \left[(1 - 2\sin^2\theta_{\text{W}})^2 + 4\sin^4\theta_{\text{W}} \left(\frac{1+\cos\theta}{2} \right)^2 \right],$$

and the total cross sections,

$$\sigma(\overline{\nu}_e e \to \overline{\nu}_e e) = \frac{G_F^2 E_\nu^2}{\pi}\left[4\sin^4\theta_W + \frac{1}{3}(1+2\sin^2\theta_W)^2\right],$$
$$\sigma(\nu_e e \to \nu_e e) = \frac{G_F^2 E_\nu^2}{\pi}\left[(1+2\sin^2\theta_W)^2 + \frac{4}{3}\sin^4\theta_W\right],$$
$$\sigma(\overline{\nu}_\mu e \to \overline{\nu}_\mu e) = \frac{G_F^2 E_\nu^2}{\pi}\left[4\sin^4\theta_W + \frac{1}{3}(1-2\sin^2\theta_W)^2\right],$$
$$\sigma(\nu_\mu e \to \nu_\mu e) = \frac{G_F^2 E_\nu^2}{\pi}\left[(1-2\sin^2\theta_W)^2 + \frac{4}{3}\sin^4\theta_W\right].$$

10.30. Let $\langle 1,2,\ldots,A|\Psi\rangle$ be an antisymmetric wavefunction for A identical particles. "1" stands for all coordinates (space, spin, isospin) of particle 1.

One then defines the density matrices ρ_1 and ρ_{12} as

$$\rho_1 : (1'|\rho_1|1'') = A\int d\tau_2 \cdot d\tau_A \ \langle\Psi|1',2,\ldots,A\rangle\langle 1'',2,\ldots,A|\Psi\rangle,$$

ρ_{12} :

$$(1',2'|\rho_{12}|1'',2'') = A(A-1)\int d\tau_3 \cdot d\tau_A \langle\Psi|1',2',3,\ldots,A\rangle$$
$$\langle 1'',2'',3,\ldots,A|\Psi\rangle.$$

(a) Show that for any one-body operator

$$F = \sum_{i=1}^{A} f_i,$$

one has

$$\langle\Psi|F|\Psi\rangle = \mathrm{tr}(\rho_1 f_1) = \int d\tau_1 d\tau_1' \,(1|\rho|1')\ (1'|f|1)\,.$$

(b) Show that for any two-body operator

$$F = \sum_{i<j1}^{A} f_{ij},$$

one has

$$\langle\Psi|F|\Psi\rangle = \frac{1}{2}\mathrm{tr}(\rho_{12} f_{12}) = \frac{1}{2}\int d\tau_1..d\tau_{2'}\,(1,2|\rho|1',2')\ (1',2'|f|1,2)\,.$$

(c) The Hamiltonian of A identical particles is

$$H = T + V = \sum_i t_i + \sum_{i<j} v_{ij}.$$

In terms of ρ_1 and ρ_{12}, we get the expression for

$$E = \langle\Psi|\, H\, |\Psi\rangle\,.$$

10.31. Consider a complete and orthogonal set of one particle wavefunction

$$\langle i\,|\alpha\rangle \;= \varphi_\alpha(\vec{x}_i, s_i, t_i),$$

$$\int d\tau_i\; \langle\alpha|\, i\rangle\; \langle i\,|\beta\rangle \;= \delta_{\alpha\beta}, \quad \text{orthogonality of the state,}$$

$$\sum_\alpha \langle i\,|\alpha\rangle\;\; \langle\alpha\,|\,j\rangle = \delta(i,j), \quad \text{completeness of the state.}$$

The simplest A-particle wavefunction is the Slater determinant,

$$\langle 1, 2, \ldots, A\,|\phi\rangle \;= \frac{1}{\sqrt{A!}} \det \|\langle i\,|\mu\rangle\,\|\,, \qquad \begin{cases} i = 1, 2, \ldots, a, \\ \mu = \alpha_1, \alpha_2, \ldots, \alpha_A. \end{cases}$$

Prove the following statements, which are valid for ϕ:

(a)

$$\big(1\,\big|\rho\,\big|1'\big) \;= \sum_{\mu=1}^{A} \langle 1\,|\mu\,\rangle\!\langle \mu\,|1'\rangle\,,$$

(b)

$$\big(1, 2\,\big|\rho\,\big|1', 2'\big) \;= \big(1\,\big|\rho\,\big|1'\big)\;\; \big(2\,\big|\rho\,\big|2'\big) \;-\big(1\,\big|\rho\,\big|2'\big)\;\; \big(2\,\big|\rho\,\big|1'\big)\,,$$

(c)

$$(\rho_1)^2 = \rho_1,$$

(d)

$$\mathrm{tr}(\rho_1) = A,$$

(e)

$$\mathrm{tr}_2(\rho_{12}) = (A-1)\rho_1.$$

10.32. Consider the stability of Hartree–Fock solution. The variation of the determinantal trial function ϕ is achieved by varying the individual "orbitals" ($\equiv$ single-particle functions), φ_μ ($\mu = 1, 2, \ldots, A$);

$$\varphi_\mu \to \varphi'_\mu = \varphi_\mu + \sum_{\sigma>A}^{\infty} \varphi_\sigma c_{\sigma\mu}.$$

(a) In order to keep the φ'_μ orthogonal and normalized to second order in the parameters $c_{\sigma\mu}$, one must write

$$\varphi'_\mu = \varphi_\mu + \sum_{\sigma > A}^{\infty} \varphi_\sigma c_{\sigma\mu} - \frac{1}{2} \sum_{\nu \le A} \varphi_\nu \left(\sum_{\tau > A} c^*_{\tau\nu} c_{\tau\mu} \right).$$

Show that this assures that

$$\int d\tau \varphi'^*_\mu \varphi'_\nu = \delta_{\mu\nu} \quad \text{to order } c^2.$$

(b) We had the result that

$$\overline{E} \equiv \langle\phi| H |\phi\rangle = \sum_{\mu \le A} t_{\mu\mu} + \frac{1}{2} \sum_{\mu,\nu \le A} v^{(a)}_{\mu\nu,\mu\nu}.$$

If ϕ is a solution of the Hartree–Fock variational problem, then

$$\overline{E}' \equiv \langle\phi'| H |\phi'\rangle = \overline{E} + \delta^{(2)} \langle H \rangle ,$$

$\delta^{(2)} \langle H\rangle$ being a quadratic form in the coefficients, $c_{\sigma\mu}$, $c^*_{\sigma\mu}$.

Find an expression for $\delta^{(2)} \langle H\rangle$. Use the fact that if in $t_{\mu\mu}$ for example, φ_μ is replaced with φ'_μ, one gets

$$t_{\mu\mu} \to t_{\mu\mu} + \sum_\sigma t_{\mu\sigma} c_{\sigma\mu} - \frac{1}{2} \sum t_{\mu\lambda} c^*_{\tau\lambda} c_{\tau\mu}.$$

With this recipe, you can easily collect all quadratic terms. The result is simplified by using the Hartree–Fock equation,

$$t_{\alpha\beta} + U_{\alpha\beta} \equiv \left(t_{\alpha\beta} + \sum_{\nu \le A} v^{(a)}_{\alpha\nu,\beta\nu} \right) = \varepsilon_\alpha \delta_{\alpha\beta}.$$

Show that

$$\delta^{(2)} \langle H\rangle = \sum_{\sigma\mu} (\varepsilon_\sigma - \varepsilon_\mu) \left| c_{\sigma\mu} \right|^2 + \frac{1}{2} \sum c^*_{\sigma\mu} c^*_{\tau\nu} v^{(a)}_{\sigma\tau,\mu\nu}$$
$$+ \sum c^*_{\sigma\mu} v^{(a)}_{\sigma\nu,\mu\tau} c_{\tau\nu} + \frac{1}{2} \sum v^{(a)}_{\mu\nu,\sigma\tau} c_{\sigma\mu} c_{\tau\nu}.$$

This shows that for the lowest energy state, one wants all $(\varepsilon_\sigma - \varepsilon_\mu) > 0$. The requirement $\delta^{(2)} \langle H\rangle \ge 0$ is called the stability condition.

10.33. Consider the degeneracy of Hartree–Fock energy. One can show that there are now trivial variations ϕ' of ϕ, for which $\delta^{(2)}\langle H\rangle = 0$.

(a) This can be seen most easily, if you recognize that, given a determinant ϕ, the function,

$$\phi' \equiv \exp[i\lambda F]\phi \quad \text{with} \quad F = \sum_{i=1}^{A} f_i,$$

(f_i being a "one-body operator") is also a determinant where λ is a small parameter. Prove this. Show furthermore that if F is Hermitian, ϕ' remains normalized.

(b) With the above ϕ', find an expression for $\delta\langle H\rangle$ by a power series expansion in λ. You get an alternative form for the condition $\delta\langle H\rangle = 0$. Use it to show for example that the mean value of the total momentum is zero for a Hartree–Fock wavefunction ϕ.

(c) Show that if F is a constant of the motion,

$$[H, F] = 0,$$

then all functions $\phi' = \exp[i\lambda F]\phi$ are solutions of the Hartree–Fock problem, with the same mean energy. Describe in which way, for $F = P$, ϕ' differs from ϕ.

10.34. Consider a (not very realistic) crude model of a "nucleus," represented by A particles of mass m occupying the A lowest levels of a one-dimensional harmonic oscillator (no spin, no isospin), with the angular frequency, ω.

(a) Let F be the total momentum,

$$F = P = \sum_{j=1}^{A} p_j.$$

Using the result of the mean square value of a one particle operator, calculate $\langle P^2\rangle$ which is also $\langle(\Delta P)^2\rangle$ in terms of m, ω, and A. (Note that $\langle P\rangle = 0$.)

(b) What is the total kinetic energy of the center of mass motion of this system? Compare this to its total kinetic energy.

(c) Define the center of mass position as

$$X = \frac{1}{A}\sum_{j=1}^{A} x_j.$$

Find $\langle(\Delta X)^2\rangle$ in terms of m, ω, and A. (Note that $\langle X\rangle = 0$.)

(d) Show that

$$\sqrt{\langle(\Delta P)^2\rangle}\sqrt{\langle(\Delta X)^2\rangle} = \frac{\hbar}{2}.$$

10.35. Consider the quantization of self-interacting scalar field theory with Lagrangian density,

$$\mathcal{L} = \frac{1}{2}\hat{\phi}(x)K(x)\hat{\phi}(x) + \mathcal{L}_{\text{int}}(\hat{\phi}) = \frac{1}{2}\hat{\phi}(x)K(x)\hat{\phi}(x) + \frac{1}{4!}\hat{\phi}^4.$$

(a) Following the canonical procedure, construct the S operator as

$$S = \mathrm{T}\exp[i\int d^4x\mathcal{L}_{\text{int}}(\hat{\phi})].$$

(b) Following the normal ordering prescription, show that the normal ordered S operator is given by

$$: S :=: U\exp\left[i\int d^4x\mathcal{L}_{\text{int}}(\hat{\phi})\right] : .$$

Here we have

$$U = \exp\left[\frac{i}{2}\int\int d^4xd^4y\frac{\delta}{\delta\hat{\phi}(x)}D_0^{\mathrm{F}}(x-y)\frac{\delta}{\delta\hat{\phi}(y)}\right],$$

with

$$D_0^{\mathrm{F}}(x-y) = \frac{1}{i}\left\langle 0\left|\mathrm{T}(\hat{\phi}(x)\hat{\phi}(y))\right|0\right\rangle_{J=0}.$$

(c) Introduce the external hook coupling, $\mathcal{L}_{\text{external}}(\hat{\phi}) = J(x)\hat{\phi}(x)$. Making use of the formula,

$$\exp\left[\frac{i}{2}\int\int d^4xd^4y\frac{\delta}{\delta\hat{\phi}(x)}D_0^{\mathrm{F}}(x-y)\frac{\delta}{\delta\hat{\phi}(y)}\right]\exp\left[i\int d^4zJ(z)\hat{\phi}(z)\right]\Bigg|_{\hat{\phi}=0}$$
$$= \exp\left[-\frac{i}{2}\int\int d^4xd^4yJ(x)D_0^{\mathrm{F}}(x-y)J(y)\right],$$

obtain the generating functional of Green's functions in functional integral form.

10.36. Consider the quantization of the neutral ps-ps meson theory with the external hook coupling, $\mathcal{L}_{\text{external}}(\hat{\phi}, \hat{\psi}, \hat{\overline{\psi}}) = J(x)\hat{\phi}(x) + \overline{\eta}(x)\hat{\psi}(x) + \hat{\overline{\psi}}(x)\eta(x)$. Show that U operator assumes the following form:

$$U = \exp\left[\frac{i}{2}\int\int d^4x d^4y \frac{\delta}{\delta\hat{\phi}(x)} D_0^{\text{F}}(x-y)\frac{\delta}{\delta\hat{\phi}(y)} \right.$$
$$\left. + i\int\int d^4x d^4y \frac{\delta}{\delta\hat{\psi}_\alpha(x)} S_{0,\alpha\beta}^{\text{F}}(x-y)\frac{\delta}{\delta\hat{\overline{\psi}}_\beta(y)}\right],$$

with

$$S_{0,\alpha\beta}^{\text{F}}(x-y) = \frac{1}{i}\left\langle 0\left|\text{T}(\hat{\psi}_\alpha(x)\hat{\overline{\psi}}_\beta(y))\right|0\right\rangle_{J=\eta=\overline{\eta}=0}.$$

Obtain the generating functional of Green's functions in functional integral form.

10.37. Extend Problem 10.36 to the most general renormalizable Lagrangian density given by

$$\mathcal{L}_{\text{tot}} = \hat{\overline{\psi}}_n(x)(i\gamma^\mu D_\mu - m)_{n,m}\hat{\psi}_m(x) + \frac{1}{2}\left(D_\mu\hat{\phi}(x)\right)_i\left(D^\mu\hat{\phi}(x)\right)_i$$
$$-\frac{1}{4}\hat{F}_{\alpha\mu\nu}(x)\hat{F}_\alpha^{\mu\nu}(x) + \hat{\overline{\psi}}_n(x)(\Gamma_i)_{n,m}\hat{\psi}_m(x)\hat{\phi}_i(x) + V\left(\hat{\phi}_i(x)\right),$$

where the potential $V(\phi(x))$ is a locally G invariant quartic polynomial and

$$\left(D_\mu\hat{\psi}(x)\right)_n \equiv \left(\partial_\mu\delta_{n,m} + i(t_\gamma)_{n,m}\hat{A}_{\gamma\mu}(x)\right)\hat{\psi}_m(x),$$
$$\left(D_\mu\hat{\phi}(x)\right)_i \equiv \left(\partial_\mu\delta_{i,j} + i(\theta_\gamma)_{i,j}\hat{A}_{\gamma\mu}(x)\right)\hat{\phi}_j(x),$$
$$\theta_\alpha^* = \theta_\alpha^{\text{T}} = -\theta_\alpha,\ \frac{\partial V\left(\phi(x)\right)}{\partial\phi_i(x)}(\theta_\alpha)_{i,j}\phi_j(x) = 0,\ \left[t_\alpha, \gamma^0 m\right] = 0,$$
$$\left[t_\alpha, \gamma^0\Gamma_i\right] = \gamma^0\Gamma_j(\theta_\alpha)_{j,i} = -(\theta_\alpha)_{i,j}\gamma^0\Gamma_j,\ V^* = V,\ m^\dagger = \gamma^0 m\gamma^0,\ \Gamma_i^\dagger = \gamma^0\Gamma_i\gamma^0.$$

Naive application of the result of Problem 10.36 to non-Abelian gauge field theory gives a divergent result for the generating functional (of the connected part) of Green's functions. Introduce the Faddeev–Popov determinant in R_ξ-gauge in the presence of spontaneous symmetry breaking,

$$\left.\frac{\partial V\left(\tilde{\phi}(x)\right)}{\partial\phi_i(x)}\right|_{\tilde{\phi}(x)=\tilde{v}} = 0, \quad i = 1, \ldots, n,$$

$$F_\alpha\left(\{\phi_a(x)\}\right) = \sqrt{\xi}\left(\partial_\mu A_\alpha^{\prime\mu}(x) - \frac{1}{\xi}\tilde{\eta}(x)i(\theta_\alpha\tilde{v})\right), \quad \xi > 0,$$

$$\Delta_F\left[\{\phi_a\}\right]\int\prod_x dg(x)\prod_{\alpha,x}\delta\left(F_\alpha\left(\{\phi_a^g\}\right) - a_\alpha(x)\right) = 1.$$

Obtain the generating functional (of the connected part) of Green's function in the functional integral form.

10.38. In quantum mechanics, the operator ordering prescription becomes an important issue in making the transition from canonical formalism to path integral formalism. We examine the connection among Weyl correspondence, α-ordering, and Well-ordered operator in some details. The parameter α connects various operator ordering prescription continuously.

(a) Define the α-ordered Hamiltonian $H^{(\alpha)}(q,p)$ of the quantum Hamiltonian $H_{\mathrm{q}}(\hat{q},\hat{p})$ by

$$H^{(\alpha)}(q,p) \equiv \int \left\langle p + \left(\frac{1}{2} - \alpha\right) u \right| H_{\mathrm{q}}(\hat{q},\hat{p}) \left| p - \left(\frac{1}{2} + \alpha\right) u \right\rangle \exp\left[\frac{iqu}{\hbar}\right] du.$$

Show that the quantum Hamiltonian $H_{\mathrm{q}}(\hat{q},\hat{p})$ is given by

$$H_{\mathrm{q}}(\hat{q},\hat{p}) = \int\int\int \left| q + \left(\frac{1}{2} + \alpha\right) v \right\rangle \left\langle q - \left(\frac{1}{2} - \alpha\right) v \right| \\ H^{(\alpha)}(q,p) \exp\left[\frac{ipv}{\hbar}\right] \frac{dp}{2\pi\hbar} dq dv.$$

(b) Consider the operator ordering prescription specified by

$$E(\hat{p},\hat{q};a,b) \equiv \exp\left[\frac{i}{\hbar}(a\hat{p} + b\hat{q})\right] \to \exp\left[\frac{i}{\hbar}(ap + qb)\right].$$

With the use of Baker–Campbell–Hausdorff formula,

$$\exp\left[\hat{A} + \hat{B}\right] = \exp\left[\hat{A}\right] \exp\left[\hat{B}\right] \exp\left[-\frac{1}{2}\left[\hat{A},\hat{B}\right]\right], \quad \left[\hat{A},\hat{B}\right] = c\text{-number},$$

as applied to $E(\hat{p},\hat{q};a,b)$, show that

$$E\left(\hat{p},\hat{q};a,b\right) = \begin{cases} \exp\left[ia\hat{p}/\hbar\right] \exp\left[ib\hat{q}/\hbar\right] \exp\left[-iab/2\hbar\right], \\ \exp\left[ib\hat{q}/\hbar\right] \exp\left[ia\hat{p}/\hbar\right] \exp\left[iab/2\hbar\right]. \end{cases}$$

(c) Compute the c-number function $E^{(\alpha)}(p,q;a,b)$ from the definition of the α-ordering prescription specified in part (a), and using the results of part (b). Show that $E^{(\alpha)}(p,q;a,b)$ is given by

$$E^{(\alpha)}(p,q;a,b) = \exp[i(ap + bq)/\hbar - i\alpha ab/\hbar].$$

(d) Establish the following correspondence:

$$\begin{cases} \exp[ia\hat{p}/\hbar]\exp[ib\hat{q}/\hbar] & \to & \exp[\frac{i}{\hbar}(ap + qb)], & (\alpha = +1/2), \\ \exp[\frac{i}{\hbar}(a\hat{p} + b\hat{q})] & \to & \exp[\frac{i}{\hbar}(ap + qb)], & (\alpha = 0), \\ \exp[ib\hat{q}/\hbar]\exp[ia\hat{p}/\hbar] & \to & \exp[\frac{i}{\hbar}(ap + qb)], & (\alpha = -1/2). \end{cases}$$

(e) Show that, in order to obtain Weyl correspondence $(\hat{p}^m \hat{q}^n)_{\mathrm{W}}$, we only have to differentiate $E(\hat{p}, \hat{q}; a, b)$ with respect to a and b, m times and n times, respectively, and set $a = b = 0$. For example,

$$
\begin{aligned}
(\hat{p}\hat{q})_{\mathrm{W}} &= \frac{1}{2}\left(\hat{p}\hat{q} + \hat{q}\hat{p}\right), \\
(\hat{p}^2\hat{q})_{\mathrm{W}} &= \frac{1}{3}\left(\hat{p}^2\hat{q} + \hat{p}\hat{q}\hat{p} + \hat{q}\hat{p}^2\right).
\end{aligned}
$$

(f) Show that, by setting $\alpha = 0$, we have the Weyl correspondence.

(g) Show that $\alpha = 1/2$ gives the well-ordered operator, ($\hat{p}\hat{q}$-ordering), and that $\alpha = -1/2$ gives the anti-well-ordered operator, ($\hat{q}\hat{p}$-ordering).

(h) Show that, in the α-ordering prescription, the transformation function $\left\langle q_{t_b}, t_b \middle| q_{t_a}, t_a \right\rangle$ in discrete representation is given by

$$
\begin{aligned}
\left\langle q_{t_b}, t_b \middle| q_{t_a}, t_a \right\rangle = & \int_{q_{t_0}=q_{t_a}}^{q_{t_n}=q_{t_b}} \prod_{k=1}^{n-1} \frac{d^f q_{t_k}}{\sqrt{(2\pi\hbar)^f}} \int \prod_{k=0}^{n-1} \frac{d^f p_{t_k}}{\sqrt{(2\pi\hbar)^f}} \\
& \times \exp\left[\frac{i}{\hbar}\sum_{k=0}^{n-1}\left\{\sum_{r=1}^{f} p_{r,t_k}(q_{r,t_k+\delta t} - q_{r,t_k}) - H^{(\alpha)}(q^{(\alpha)}_{t_k+\delta t}, p_{t_k})\delta t\right\}\right],
\end{aligned}
$$

with $q^{(\alpha)}_{t_k+\delta t}$ defined by

$$
q^{(\alpha)}_{t_k+\delta t} \equiv \left(\frac{1}{2} - \alpha\right) q_{t_k+\delta t} + \left(\frac{1}{2} + \alpha\right) q_{t_k}.
$$

Note that $\alpha = 0$ corresponds to the mid-point rule.

10.39. Consider a simple derivation of the one pion-exchange potential from the symmetric ps–ps meson theory. The Lagrangian density is given by

$$
\begin{aligned}
\mathcal{L} = & \frac{1}{4}\left[\hat{\overline{\psi}}_\alpha(x), D_{\alpha\beta}(x)\hat{\psi}_\beta(x)\right] + \frac{1}{4}\left[D^{\mathrm{T}}_{\beta\alpha}(-x)\hat{\overline{\psi}}_\alpha(x), \hat{\psi}_\beta(x)\right] \\
& + \frac{1}{2}\hat{\vec{\phi}}(x)K(x)\hat{\vec{\phi}}(x) - iG\hat{\overline{\psi}}_\alpha(x)\vec{\tau}(\gamma_5)_{\alpha\beta}\hat{\psi}_\beta(x)\hat{\vec{\phi}}(x),
\end{aligned}
$$

where $\vec{\tau}$ is the isospin matrix,

$$
\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

Here $\hat{\psi}(x)$ is the iso-doublet nucleon and $\hat{\vec{\phi}}(x)$ is the iso-triplet pion. The kernels of the quadratic part of the Lagrangian density are given by

$$\begin{aligned} D_{\alpha\beta}(x) &= (i\gamma_\mu \partial^\mu - m + i\varepsilon)_{\alpha\beta}, \\ D^{\mathrm{T}}_{\beta\alpha}(-x) &= (-i\gamma_\mu^{\mathrm{T}} \partial^\mu - m + i\varepsilon)_{\beta\alpha}, \\ K(x) &= -\partial^2 - \mu^2 + i\varepsilon. \end{aligned}$$

The covariant scattering matrix element $\mathcal{M}$ in the static limit and the potential $V(\vec{r})$ are related to each other in the following manner:

$$\mathcal{M}_{fi}(\vec{q}) = \int V(\vec{r}) \exp\left[-i\vec{q}\cdot\vec{r}\right] d^3\vec{r}, \quad V(\vec{r}) = \frac{1}{(2\pi)^3} \int \mathcal{M}_{fi}(\vec{q}) \exp[i\vec{q}\cdot\vec{r}] d^3\vec{q}.$$

Note that the potential is nothing more than the three-dimensional Fourier transform of the lowest-order covariant scattering matrix element.

(a) Show that the covariant scattering matrix element for the direct scattering part is given by

$$\mathcal{M}_{fi}^{(\text{direct})} = -\frac{G^2\left(\vec{\tau}^{(1)}\cdot\vec{\tau}^{(2)}\right)\left(\overline{u}_1^f\gamma_5 u_1^i\right)\left(\overline{u}_2^f\gamma_5 u_2^i\right)}{\left(p_1^f - p_1^i\right)^2 - \mu^2}.$$

(b) In the standard representation of the Dirac γ matrices, the γ_5 is given by

$$\gamma_5 = \gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

In the static limit (or nonrelativistic limit), show that

$$\begin{aligned} \overline{u}_1^f\gamma_5 u_1^i &= \begin{pmatrix} \chi^{(s_1^f)\dagger}, & -\chi^{(s_1^f)\dagger}\frac{\vec{\sigma}^{(1)}\cdot\vec{p}_1^f}{2m} \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} \chi^{(s_1^i)} \\ \frac{\vec{\sigma}^{(1)}\cdot\vec{p}_1^i}{2m}\chi^{(s_1^i)} \end{pmatrix} \\ &= \chi^{(s_1^f)\dagger}\frac{\vec{\sigma}^{(1)}\cdot\vec{q}}{2m}\chi^{(s_1^i)}, \\ \overline{u}_2^f\gamma_5 u_2^i &= -\chi^{(s_2^f)\dagger}\frac{\vec{\sigma}^{(2)}\cdot\vec{q}}{2m}\chi^{(s_2^i)}. \end{aligned}$$

Here $\vec{q}$ is the three-momentum transfer,

$$\vec{q} = \vec{p}_2^f - \vec{p}_2^i,$$

and χs are the two-component spinors.

(c) Show that the covariant scattering matrix element for the direct scattering part is reduced to the following form in the static limit:

$$\mathcal{M}_{fi}^{(\text{direct})} \to -\frac{G^2\left(\vec{\tau}^{(1)}\cdot\vec{\tau}^{(2)}\right)\left(\vec{\sigma}^{(1)}\cdot\vec{q}\right)\left(\vec{\sigma}^{(2)}\cdot\vec{q}\right)}{(2m)^2\left(\vec{q}^2+\mu^2\right)}.$$

(d) By applying the inverse Fourier transform to the reduced scattering matrix element, we obtain the one pion-exchange potential in the static limit as

$$V(\vec{r}) = g^2\left(\vec{\tau}^{(1)}\cdot\vec{\tau}^{(2)}\right)\frac{1}{\mu^2}\left(\vec{\sigma}^{(1)}\cdot\vec{\nabla}\right)\left(\vec{\sigma}^{(2)}\cdot\vec{\nabla}\right)\frac{\exp[-\mu r]}{r},$$

where the coupling constant is redefined as

$$g^2 = \frac{G^2}{4\pi}\frac{\mu^2}{(2m)^2}.$$

10.40. Consider WKB approximation to Schrödinger equation in one spatial dimension with the use of matrix method.

(a) Schrödinger equation in one spatial dimension is given by

$$\frac{d^2}{dx^2}\psi(x) = w(x)\psi(x) \quad \text{with} \quad w(x) = \frac{2m}{\hbar^2}(V(x)-E),$$

where $V(x)$ is the potential, E is the total energy, and $\hbar$ is the Planck constant divided by 2π. Show that this can be expressed as the first-order ordinary differential equation with the two-component wavefunction $\Psi(x)$, $\psi(x)$, and $d\psi(x)/dx$ as its two components, in Dirac equation form

$$\frac{d}{dx}\Psi(x) = Q(x)\Psi(x) \quad \text{with} \quad Q(x) = \begin{pmatrix} 0 & 1 \\ w(x) & 0 \end{pmatrix}. \qquad \text{(A)}$$

(b) Writing the two component wavefunction $\Psi(x)$ as

$$\Psi(x) = S(x, x_0)\Psi(x_0),$$

the transformation function $S(x, x_0)$ satisfies the following composition law:

$$S(x, x')S(x', x_0) = S(x, x_0).$$

From the differential equation (A), show that the transformation function $S(x, x_0)$ satisfies

$$S(x+\Delta x, x) = 1 + \Delta x Q(x) = \exp[Q(x)dx]. \qquad \text{(B)}$$

Hence show that $S(x, x_0)$ is given by

$$S(x, x_0) = \lim_{\Delta x = 0} \prod_{x_0}^{x} \exp[Q(x)dx] = \exp\left[\int_{x_0}^{x} Q(x)dx\right].$$

(c) We first assume that $Q(x)$ is constant in the interval $[x, x+a]$. From Eq. (B), we have

$$S(x+a, x) = \exp[aQ].$$

We further assume that

$$w(x) = -s^2, \quad Q = \begin{pmatrix} 0 & 1 \\ -s^2 & 0 \end{pmatrix}.$$

Establish the identity,

$$\begin{pmatrix} 0 & s^{-1} \\ -s & 0 \end{pmatrix} = \begin{pmatrix} s^{-1/2} & 0 \\ 0 & s^{1/2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} s^{1/2} & 0 \\ 0 & s^{-1/2} \end{pmatrix},$$

and show that Q can be expressed as

$$Q = s \begin{pmatrix} s^{-1/2} & 0 \\ 0 & s^{1/2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} s^{1/2} & 0 \\ 0 & s^{-1/2} \end{pmatrix}.$$

Show further that

$$\begin{aligned} S(x+a, x) &= \begin{pmatrix} s^{-1/2} & 0 \\ 0 & s^{1/2} \end{pmatrix} \exp\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} as\right] \begin{pmatrix} s^{1/2} & 0 \\ 0 & s^{-1/2} \end{pmatrix} \\ &= \begin{pmatrix} \cos as, & (1/s)\sin as \\ -s\sin as, & \cos as \end{pmatrix}. \end{aligned}$$

(d) We next divide the interval $[x_0, x]$ into the subintervals $[x_0, x_1], [x_1, x_2], \ldots$, let the values of s as s_0 in $[x_0, x_1]$, s_1 in $[x_1, x_2], \ldots$ and further let the phases $(x_1 - x_0)s_0, (x_2 - x_1)s_1, \ldots$ as $F_0, F_1, \ldots$. Show that $S(x, x_0)$ can be expressed as

$$\begin{aligned} S(x, x_0) = \prod_{x_0}^{x} S(x_{n+1}, x_n) &= \begin{pmatrix} s^{1/2} & 0 \\ 0 & s^{-1/2} \end{pmatrix} \cdots \exp\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} F_1\right] \\ &\times \begin{pmatrix} (s_0/s_1)^{1/2} & 0 \\ 0 & (s_1/s_0)^{1/2} \end{pmatrix} \end{aligned}$$

$$\times \exp\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} F_0\right]\begin{pmatrix} s_0^{-1/2} & 0 \\ 0 & s_0^{1/2} \end{pmatrix}. \qquad \text{(C)}$$

We now consider the general factor in the product (C),

$$\begin{pmatrix} \left(\frac{s_{n-1}}{s_n}\right)^{1/4} & 0 \\ 0 & \left(\frac{s_n}{s_{n-1}}\right)^{1/4} \end{pmatrix} \exp\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} F_n\right]\begin{pmatrix} \left(\frac{s_n}{s_{n+1}}\right)^{1/4} & 0 \\ 0 & \left(\frac{s_{n+1}}{s_n}\right)^{1/4} \end{pmatrix}.$$

We let Δx be infinitesimal, and represent the coordinate x_n by x and s_n by s so that

$$s_{n-1}/s_n = 1 - \Delta x(s'/s), \quad s_{n+1}/s_n = 1 + \Delta x(s'/s), \quad F_n = s\Delta x.$$

Here s is regarded as the continuous and differentiable function $s(x)$ of x with $s' = ds/dx$. With this representation, show that the general factor written out above becomes

$$1 + \Delta x \begin{pmatrix} -s'/2s & s \\ -s & s'/2s \end{pmatrix}. \qquad \text{(D)}$$

WKB approximation consists of regarding $|s'/2s| \ll |s|$. Introducing the local de Broglie wavelength $\lambda(x) = 2\pi/s(x)$, show that the condition $|s'/2s| \ll |s|$ reads as

$$\frac{\lambda}{4\pi}\left|\frac{ds}{dx}\right| \ll s.$$

Equation (D) becomes

$$1 + \Delta x \begin{pmatrix} 0 & s \\ -s & 0 \end{pmatrix}.$$

With this approximation, show that the transformation function $S(x, x_0)$ becomes

$$S(x, x_0) = \begin{pmatrix} s^{1/2} & 0 \\ 0 & s^{-1/2} \end{pmatrix} \exp\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \int_{x_0}^{x} s dx\right]\begin{pmatrix} s_0^{-1/2} & 0 \\ 0 & s_0^{1/2} \end{pmatrix}$$

$$= \begin{pmatrix} (s_0/s)^{1/2}\cos F & (ss_0)^{-1/2}\sin F \\ -(ss_0)^{1/2}\sin F & (s/s_0)^{1/2}\cos F \end{pmatrix} \quad \text{with} \quad F = \int_{x_0}^{x} s dx.$$

(e) Next we consider the case,

$$w(x) = r^2, \quad Q = \begin{pmatrix} 0 & 1 \\ r^2 & 0 \end{pmatrix}.$$

Assuming that $Q(x)$ is constant in the interval $[x, x+a]$, show that $S(x+a, x)$ can be expressed as

$$S(x+a,x) = \begin{pmatrix} r^{-1/2} & 0 \\ 0 & r^{1/2} \end{pmatrix} \exp\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ar \right] \begin{pmatrix} r^{1/2} & 0 \\ 0 & r^{-1/2} \end{pmatrix}$$

$$= \begin{pmatrix} \cosh ar, & (1/r)\sinh ar \\ r\sinh ar, & \cosh ar \end{pmatrix}.$$

By regarding r as the continuous and differentiable function $r(x)$ of x, and assuming that $|r'/2r|$ is small compared with $|r|$, we obtain

$$S(x, x_0) = \begin{pmatrix} (r_0/r)^{1/2}\cosh F & (rr_0)^{-1/2}\sinh F \\ (rr_0)^{1/2}\sinh F & (r/r_0)^{1/2}\cosh F \end{pmatrix} \quad \text{with} \quad F = \int_{x_0}^{x} r dx.$$

(f) Near the turning point, say $x = a$ where $s(a) = r(a) = 0$ with the assumption that $E > V(x)$ for $x < a$ (classically allowed zone) and $E < V(x)$ for $x > a$ (classically forbidden zone), show that the wavefunction $\psi(x)$ can be written generally as

$$\psi(x) = \frac{C_1}{\sqrt{s}} \exp\left[\frac{i}{\hbar} \int_a^x s dx \right] + \frac{C_2}{\sqrt{s}} \exp\left[-\frac{i}{\hbar} \int_a^x s dx \right], \quad x < a, \tag{E}$$

$$\psi(x) = \frac{C_3}{\sqrt{r}} \exp\left[-\frac{1}{\hbar} \int_a^x r dx \right] + \frac{C_4}{\sqrt{r}} \exp\left[\frac{1}{\hbar} \int_a^x r dx \right], \quad x > a. \tag{F}$$

The wavefunction should be dumped in the classically forbidden zone, $E < V(x)$ $(x > a)$. Show that we must choose $C_4 = 0$ in Eq. (F),

$$\psi(x) = \frac{C}{2\sqrt{r}} \exp\left[-\frac{1}{\hbar} \int_a^x r dx \right], \quad x > a. \tag{G}$$

(g) Near the turning point where the WKB approximation is not valid, we assume that

$$E - V(x) \approx F_0(x-a), \quad F_0 = -\left.\frac{dV(x)}{dx}\right|_{x=a} < 0.$$

In a region where Eq. (G) is valid, show that we have

$$\psi(x) = \frac{C}{2\sqrt{2m|F_0|}} \frac{1}{(x-a)^{1/4}} \exp\left[-\frac{1}{\hbar}\int_a^x \sqrt{2m|F_0|(x-a)}dx\right]. \qquad \text{(H)}$$

We first let x pass $x = a$ from the classically forbidden zone ($x > a$) to the classically allowed zone ($x < a$) along a semicircle of radius ρ in the upper half plane. On this semicircle, show that we have

$$x - a = \rho \exp[i\theta], \quad \theta : 0 \to \pi,$$

$$\int_a^x \sqrt{(x-a)}dx = \frac{2}{3}\rho^{3/2}\left(\cos\frac{3}{2}\theta + i\sin\frac{3}{2}\theta\right).$$

At the end of the semicircle, we have the pure imaginary exponent,

$$-\frac{i}{\hbar}\int_a^x \sqrt{2m|F_0|(a-x)}dx = -\frac{i}{\hbar}\int_a^x sdx.$$

The change of the prefactor of the exponential in Eq. (H) is given by

$$(x-a)^{-1/4} \to (a-x)^{-1/4}\exp\left[-i\frac{\pi}{4}\right].$$

Show that the connection formula for C_2 is given by

$$C_2 = \frac{C}{2}\exp\left[-i\frac{\pi}{4}\right].$$

Likewise, the connection formula for C_1 is obtained by letting x pass $x = a$ from the classically forbidden zone ($x > a$) to the classically allowed zone ($x < a$) along a semicircle of radius ρ in the lower half plane. Show that

$$C_1 = \frac{C}{2}\exp\left[i\frac{\pi}{4}\right].$$

Show that the wavefunction in the forbidden zone ($x > a$) corresponds to the wavefunction in the allowed zone ($x < a$),

$$\psi(x) = \frac{C}{\sqrt{s}}\cos\left(\frac{1}{\hbar}\int_a^x sdx + \frac{\pi}{4}\right).$$

Show also that, independent of the number of and the location of the turning points, the connection formula from the classically forbidden zone to the classically allowed zone is given by

$$\underbrace{\frac{C}{2\sqrt{r}}\exp\left[-\frac{1}{\hbar}\left|\int_a^x rdx\right|\right]}_{\text{for the region } V(x)>E} \quad \to \quad \underbrace{\frac{C}{\sqrt{s}}\cos\left(\frac{1}{\hbar}\left|\int_a^x sdx\right| - \frac{\pi}{4}\right)}_{\text{for the region } V(x)<E}.$$

Hint: We cite the following article and the books for the discussion of WKB approximation.
Klein, O.: Z. Physik. **80**, (1932), 792.
Bender, Carl M., and Orszag, Steven A.: *Advanced Mathematical Methods For Scientists and Engineers*, Springer, New York, 1999.
Pauli, W.: *General Principles of Quantum Mechanics*, Springer, Heidelberg, 1980; Helv. Phys. Acta. **5**, 179, (1932).

10.41. (a) Show that the Lagrangian density appropriate for the Laplace equation is

$$\mathcal{L} = (\nabla\psi)^2.$$

(b) Show that, for two conductors with a fixed potential difference V_0, the electrostatic energy is given by $(1/2)CV_0^2$, where C is the capacity.

(c) Show that the variational principle for the capacity is

$$\delta[C] = \delta\left[\left(1/4\pi\, V_0^2\right)\int(\nabla\psi)^2\, dV\right] = 0.$$

10.42. Consider the scalar Helmholtz equation

$$\nabla^2\psi + k^2\psi = 0,$$

where ψ satisfies either *homogeneous Dirichlet* or *homogeneous Neumann* conditions on a surface S. Show that the variational principle for k^2 is

$$\delta\left[k^2\right] = \delta\left[-\int\psi\nabla^2\psi dV/\int\psi^2 dV\right].$$

Upon application of Green's theorem, show that the variational principle for k^2 becomes

$$\delta\left[k^2\right] = \delta\left[\int(\nabla\psi)^2 dV/\int\psi^2 dV\right].$$

10.43. Consider the vibration of a clamped circular membrane. Let the radius of the membrane be a. The boundary conditions for ψ are $\psi(a) = 0$ and the overall requirement that ψ be continuous in value and gradient inside the boundary. Consider only the circularly symmetric modes so that ψ is a function of r only. Then, show that the variational principle is

$$\delta\left[k^2\right] = \delta\left[\int_0^a (d\psi/dr)^2 r dr/\int_0^a \psi^2 r dr\right].$$

Introduce the dimensionless independent variable $x = r/a$. Then

$$\delta\left[(ka)^2\right] = \delta\left[\int_0^1 (d\psi/dx)^2 x dx \Big/ \int_0^1 \psi^2 x dx\right].$$

Choose the possible trial functions as $(1 - x^2)$, which vanishes at $x = 1$ and at the same time has a continuous gradient at $x = 0$. Obtain

$$\left[(ka)^2\right] = 6.$$

10.44. Consider the reaction matrix, satisfying the integral equation

$$R = V + VG_0^0 R. \tag{A}$$

A related wavefunction ψ_a may be defined by the integral equation

$$\psi_a = \chi_a + G_0^0 V\psi_a, \tag{B}$$

where χ_a is an unperturbed wavefunction satisfying $K\chi_a = \varepsilon_a \chi_a$.

(a) Show that a matrix element of R is written as

$$R_{ba} = (\chi_b, R\chi_a) = (\chi_b, V\psi_a) = (\psi_b, V\chi_a), \tag{C}$$

where ψ_b also satisfies (B) with χ_a replaced with χ_b. Show that we can express R_{ba} as

$$R_{ba} = \left(\psi_b, (V - VG_0^0 V)\psi_a\right). \tag{D}$$

Show that Eqs. (C) and (D) permit us to set

$$R_{ba} = \frac{(\psi_b, V\chi_a)(\chi_b, V\psi_a)}{(\psi_b, (V - VG_0^0 V)\psi_a)}. \tag{E}$$

Show that it is stationary with respect to small variations of either ψ_a or ψ_b about its correct form:

$$\begin{cases} \delta R_{ba}|_{\delta\psi_a} = 0, \\ \delta R_{ba}|_{\delta\psi_b} = 0. \end{cases} \tag{F}$$

(b) Consider the generalization of this result to the T-matrix:

$$T = V + VG^+ V = V + VG_0^+ T, \tag{G}$$

where the wavefunctions satisfy

$$\psi_a^{\pm} = \chi_a + G_0^{\pm} V\psi_a^{\pm}. \tag{H}$$

Show that

$$T_{ba} = (\chi_b, T\chi_a) = (\chi_b, V\psi_a^{\ +}) = (\psi_b^{\ -}, V\chi_a) = \frac{(\psi_b^{\ -}, V\chi_a)(\chi_b, V\psi_a^{\ +})}{(\psi_b^{\ -}, (V - VG_0^+ V)\psi_a^{\ +})}. \tag{I}$$

Show that, if we perform the variation $\psi_a^{\ +} \to \psi_a^{\ +} + \delta\psi_a^{\ +}$ or $\psi_b^{\ -} \to \psi_b^{\ -} + \delta\psi_b^{\ -}$, respectively, we have to the first order in the variation,

$$\begin{cases} \delta T_{ba}|_{\delta\psi_a^{\ +}} & = & 0, \\ \delta T_{ba}|_{\delta\psi_b^{\ -}} & = & 0. \end{cases} \tag{J}$$

(c) As an illustration, let us replace $\psi_a^{\ +}$ and $\psi_b^{\ -}$ by the trial functions χ_a and χ_b, respectively, in the expression (I) for T_{ba}. If we expand this in power of V, show that

$$T_{ba} = (\chi_b, V\chi_a) + (\chi_b, VG_0^+ V\chi_a) + \cdots. \tag{K}$$

This agrees with the Born series,

$$T = V + VG_0^+ V + VG_0^+ VG_0^+ V + \cdots = T^{(1)} + T^{(2)} + \cdots,$$

through the second order.

(d) In application of (I), consider the case of simple potential scattering with $V = V(r)$ and the choice of trial functions,

$$\begin{cases} \psi_a^{\ +} & = & (2\pi)^{-3/2}\exp[i\vec{\kappa}\cdot\vec{r}], \\ \psi_b^{\ -} & = & (2\pi)^{-3/2}\exp\left[i\vec{\kappa}_f\cdot\vec{r}\right]. \end{cases}$$

Show that

$$\begin{aligned} T(\vec{\kappa}_f\cdot\vec{\kappa}, \kappa_f, \kappa) = (2\pi)^{-3}&\left[\int \exp\left[-i(\vec{\kappa}_f - \vec{\kappa})\cdot\vec{r}\right] V(r) d^3\vec{r}\right]^2 \\ &\times\left[\int \exp\left[-i(\vec{\kappa}_f - \vec{\kappa})\cdot\vec{r}\right] V(r) d^3\vec{r}\right. \\ &+ \int d^3\vec{r} d^3\vec{r}\,' \exp[-i\vec{\kappa}_f\cdot\vec{r}\,'] V(r') \left(\frac{2M_r \exp\left[i\kappa\left|\vec{r} - \vec{r}\,'\right|\right]}{4\pi\left|\vec{r} - \vec{r}\,'\right|}\right) V(r) \\ &\left.\exp[i\vec{\kappa}\cdot\vec{r}]\right]^{-1}. \end{aligned} \tag{L}$$

(e) Show that Eq. (E) provides also a variational principle for the partial wave scattering amplitudes. One needs only identify R_l with $-(2/\pi)\kappa \tan\delta_l$, V with the reduced potential v, and G_0^0 with G_{0l}. Then, on setting $\psi_a = \psi_b = w_l^{(0)}$, show that

$$-\frac{2}{\pi}\kappa \tan\delta_l = \frac{(\gamma_l, v w_l^{(0)})(w_l^{(0)}, v\gamma_l)}{(w_l^{(0)}, (v - vG_{0l}v)w_l^{(0)})}. \tag{M}$$

Show that the expression (M) is stationary with respect to small variations of $w_l^{(0)}$ about its correct value.

Hint for Problem 10.44: We cite the following textbook for variational principle due to Schwinger for phase shift and variational principle due to Kohn for atomic scattering.
Goldberger, M.L. and Watson, K.M.: *Collision Theory*, John Wiley & Sons, New York, 1964, Chapters 6 and 11.

10.45. Consider the case of a particle in a central potential and look for a stationary expression for the coefficient T_l of the expansion of the transition amplitude in spherical harmonics,

$$T_{b\leftarrow a} = 16\pi^2 \sum_{lm} (2l+1) T_l Y_{lm}^*(\hat{k}_b) Y_{lm}(\hat{k}_a). \tag{A}$$

(a) Show that, if $\psi_a^{(+)}$ is the complete stationary scattering wave, the partial wave ψ_l satisfies the integral equation

$$\psi_l(r) = j_l(kr) + G_l^{(+)} V \psi_l(r), \tag{B}$$

where $G_l^{(+)}$ is the integral operator whose kernel is Green's function,

$$G_l^{(+)}(r, r') \equiv -(2mk/\hbar^2) j_l(kr_<) h_l^{(+)}(kr_>).$$

In other words,

$$G_l^{(+)} V\psi_l(r) \equiv \int_0^\infty G_l^{(+)}(r, r') V(r') \psi_l(r') r'^2 dr'.$$

Let $\langle \varphi_1, \varphi_2 \rangle$ denote the scalar product of two radial functions φ_1, φ_2:

$$\langle \varphi_1, \varphi_2 \rangle \equiv \int_0^\infty \varphi_1^*(r) \varphi_2(r) r^2 dr.$$

The integral form for T_l can therefore be written as

$$T_l = \langle j_l, V\psi_l \rangle.$$

(b) Let us now put

$$A[\psi] \equiv \langle j_l, V\psi \rangle = \langle \psi^*, V j_l \rangle, \quad B[\psi] \equiv \left\langle \psi^*, (V - V G_l^{(+)} V)\psi \right\rangle.$$

Consider the functional,

$$\mathcal{T}_l[\psi] \equiv \frac{A^2}{B}.$$

Show that

$$\mathcal{T}_l[\psi_l] = T_l.$$

(c) Show that $\mathcal{T}_l$ takes the same value T_l for any function ψ_l satisfying the following less restrictive condition than (B):

$$\psi_l(r) = C j_l(kr) + G_l^{(+)} V \psi_l(r) \quad \text{if} \quad V(r) \neq 0; \quad C, \text{ arbitrary constant.} \tag{C}$$

Calculating the variation of $\mathcal{T}_l$ as a function of $\delta\psi$, show that

$$\delta\mathcal{T}_l = \frac{2A}{B}\delta A - \frac{A^2}{B^2}\delta B,$$
$$\delta A[\psi] = \langle \delta\psi^*, V j_l \rangle,$$
$$\delta B[\psi] = 2\left\langle \delta\psi^*, (V - V G_l^{(+)} V)\psi \right\rangle.$$

Write

$$\delta\mathcal{T}_l = \frac{2A}{B^2}\langle \delta\psi^*, F \rangle,$$

where $F(r)$ is given by

$$F(r) = B[\psi]V(r) j_l(kr) - A[\psi]V(r)(\psi - G_l^{(+)} V\psi).$$

In order to have $\delta\mathcal{T}_l = 0$ for any $\delta\psi$, it is necessary and sufficient that $F(r)$ vanish. For this, it is necessary that $j_l(kr)$ and $\psi - G_l^{(+)} V\psi$ be proportional at any point where $V(r)$ is not zero, that is, that ψ be one of the functions ψ_l obeying equation (C). Thus, by adjusting the value of C in Eq. (C), we obtain

$$T_l = \mathcal{T}_l|_{\text{st}}. \tag{D}$$

(d) Separate the real and imaginary parts of $G_l^{(+)}(r, r')$. The real part is Green's function,

$$G_l^{(1)}(r, r') \equiv -(2mk/\hbar^2) j_l(kr_<) n_l(kr_>).$$

and

$$G_l^{(+)}(r, r') = G_l^{(1)}(r, r') - i(2mk/\hbar^2) j_l(kr) j_l(kr').$$

Substituting this expression in the definition of $\mathcal{T}_l$, show that

$$T_l^{-1} = \mathcal{T}_l^{-1}\Big|_{\text{st}} = \left.\frac{\left\langle \psi^*, \left(V - VG_l^{(1)} V\right) \psi\right\rangle}{\left\langle j_l, V\psi_l\right\rangle^2}\right|_{\text{st}} + i\frac{2mk}{\hbar^2}.$$

From

$$T_l = -\hbar^2 \exp[i\delta_l] \sin \delta_l / 2mk,$$

show that

$$T_l^{-1} - i(2mk/\hbar^2) = -2mk \cot \delta_l / \hbar^2.$$

Obtain the following stationary expression for $k \cot \delta_l$:

$$k \cot \delta_l = -\frac{\hbar^2}{2m} \left.\frac{\left\langle \psi^*, \left(V - VG_l^{(1)} V\right) \psi\right\rangle}{\left\langle j_l, V\psi_l\right\rangle^2}\right|_{\text{st}}. \tag{E}$$

(e) Substituting the free wave $j_l(kr)$ for ψ in the right-hand side of (E), show that

$$k \cot \delta_l = -\frac{\hbar^2}{2m} \frac{1 - \Delta_l}{\langle j_l, Vj_l\rangle} \quad \text{with} \quad \Delta_l \equiv \frac{\left\langle j_l, VG_l^{(1)} Vj_l\right\rangle}{\langle j_l, Vj_l\rangle}. \tag{F}$$

Consider the S-wave scattering by a square well in the low energy limit. Take

$$V(r) = \begin{cases} -V_0, & r < r_0, \\ 0, & r > r_0. \end{cases}$$

Calculate the scattering length,

$$a = -\lim_{k \to 0} (k \cot \delta)^{-1},$$

by the different methods and compare the results with that given by the exact calculation. Setting

$$b = \left(2mV_0 r_0^2/\hbar^2\right)^{1/2},$$

show that

exact calculation:

$$a = -\left(\frac{\tan b}{b} - 1\right) r_0.$$

variational calculation [formula (F)]:

$$a_{\text{var}} = \frac{(1/3)b^2}{1 - (2/5)b^2} r_0.$$

first-order Born approximation:

$$a_{\text{B}}^{(1)} = -\frac{1}{3} b^2 r_0.$$

second-order Born approximation:

$$a_{\text{B}}^{(2)} = -\left(\frac{1}{3} b^2 + \frac{2}{15} b^4\right) r_0.$$

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We cite the following books for applications of integral equations to the scattering problem in nonrelativistic quantum mechanics, namely, the Lippmann–Schwinger equation.

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Index

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